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Probabilistic Proof for the Generalization of Some Well-Known Binomial Identities

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Abstract

We provide probabilistic proofs for the generalized version of some celebrated identities involving binomial coefficients. The results are based on the properties of the Beta distribution.

1 Introduction

The probabilistic proof of classic mathematical formulas has received considerable attention for decades. For example, Chin [3] proved the Wallis formula and Holst [5] proved some Euler identities related to the zeta function by this approach. Spivey [10] devoted an entire chapter of his recent monograph to the probabilistic proof of binomial identities. In this setting, the present note focuses on generalizing some well-known binomial identities such as

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n, \tag{1}$$

or its alternating version

$$\sum_{k=0}^{n} (-1)^k \binom{2k}{k} \binom{2n-2k}{n-k} = \begin{cases} 2^n \binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd,} \end{cases} \tag{2}$$

where $n \in \mathbb{N}_0$. In a probabilistic approach, Chang and Xu [2] and Pathak [8] proved identities (1) and (2), respectively, on the basis of some properties of the Normal distribution. Vellaisamy and Zeleke [12] also provided proofs of (1) and (2) using identities involving the Gamma and Beta functions. In the same vein, we aim to introduce some general binomial identities by means of certain features of the Beta distribution.

2 Main results

Let Z be a random variable (r.v.) distributed according to the Beta law. The corresponding probability density function is

$$f_Z(z) = \frac{1}{B(a,b)}\, z^{a-1}(1-z)^{b-1}\mathbf{1}_{[0,1]}(z),$$

where $a, b > 0$ are the shape parameters, while $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the usual Euler Beta function. In the following, the Beta law with parameters a and b is denoted by $\mathcal{B}(a,b)$. In addition, if $r, s > 0$, then

$$\mathbf{E}[Z^r(1-Z)^s] = \frac{B(r+a, s+b)}{B(a,b)}. \tag{3}$$

For further details on the Beta distribution see, e.g., Johnson et al. [6].

Let us consider the random vector (X_1, X_2). For $n \in \mathbb{N}_0$, a unifying formula for obtaining many binomial identities is

$$\sum_{k=0}^{n} \binom{n}{k} \mathbf{E}[X_1^k X_2^{n-k}] = \mathbf{E}[(X_1+X_2)^n], \tag{4}$$

which can be easily assessed on the basis of the Binomial theorem. By a suitable choice of (X_1, X_2), remarkable formulas may generally be obtained from (4) (see, e.g., Vignat and Moll [11]). The results described here provide relevant applications of (4), concerning r.v.'s with the Beta distribution. The first result presents a further (and simpler) probabilistic proof of an identity proposed by Vellaisamy and Zeleke [12].

Proposition 1. *If $a, b > 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} B(k+a, n-k+b) = B(a,b). \tag{5}$$

Proof. Let us assume that $X_1 = Z$ and $X_2 = 1 - Z$ in expression (4), where the r.v. Z is distributed according to the Beta law $\mathcal{B}(a,b)$. By means of (3), we have

$$\mathbf{E}[X_1^k X_2^{n-k}] = \mathbf{E}[Z^k(1-Z)^{n-k}] = \frac{B(k+a, n-k+b)}{B(a,b)}.$$

Since the r.v. $(X_1 + X_2)$ is degenerate in such a way that $P(X_1 + X_2 = 1) = 1$, we also have $\mathbf{E}[(X_1+X_2)^n] = 1$. Thus, identity (5) follows from (4) on substitution. $\square$

The application of the Legendre duplication formula for the Gamma function yields

$$B\left(k+\frac{1}{2}, l+\frac{1}{2}\right) = \frac{\Gamma(k+\frac{1}{2})\Gamma(l+\frac{1}{2})}{\Gamma(k+l+1)} = \frac{\pi\binom{2k}{k}\binom{2l}{l}}{4^{k+l}\binom{k+l}{k}}, \tag{6}$$

for $k, l \in \mathbb{N}_0$. Hence, Proposition 1 provides the following corollary, which introduces a generalization of identity (1).

Corollary 2. *If $r, s \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\binom{2k+2r}{k+r}\binom{2n-2k+2s}{n-k+s}}{\binom{n+r+s}{k+r}} = 4^n \frac{\binom{2r}{r}\binom{2s}{s}}{\binom{r+s}{r}}. \tag{7}$$

Proof. By assuming $a = r + \frac{1}{2}$ and $b = s + \frac{1}{2}$ in expression (5), we have identity (7) by means of (6). □

Remark 3. Identity (1) is obtained from (7) for $r = s = 0$. In addition, for $r = n$ and $s = 0$, we also obtain the interesting identity

$$\sum_{k=0}^{n} \binom{n+k}{n}\binom{2n+2k}{n+k}\binom{2n-2k}{n-k} = 4^n \binom{2n}{n}^2.$$

Furthermore, after some algebra, identity (7) may be written as

$$\sum_{k=0}^{n} \binom{k+r}{r}\binom{2k+2r}{k+r}\binom{n-k+s}{s}\binom{2n-2k+2s}{n-k+s} = 4^n \binom{2r}{r}\binom{2s}{s}\binom{n+r+s}{n}.$$

The following proposition provides the alternating version of identity (5) for $a = b$. Vellaisamy and Zeleke [12] proposed a different probabilistic proof of this proposition based on the properties of the Gamma distribution, though if the present proof is simpler.

Proposition 4. *If $a > 0$, we have*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} B(k+a, n-k+a) = 2^{1-2a} B\left(\frac{n+1}{2}, a\right), \tag{8}$$

for n even, while the sum vanishes for n odd.

Proof. Let us assume that $X_1 = -Z$ and $X_2 = 1 - Z$ in expression (4), where the r.v. Z is distributed according to the Beta law $\mathcal{B}(a, a)$. Using expression (3), we have

$$\mathbf{E}[X_1^k X_2^{n-k}] = (-1)^k \mathbf{E}[Z^k (1-Z)^{n-k}] = (-1)^k \frac{B(k+a, n-k+a)}{B(a, a)}.$$

In addition, the r.v. $Y = (1-2Z)^2$ is distributed according to the Beta law $\mathcal{B}(\frac{1}{2}, a)$. For n even we have

$$\mathbf{E}[(X_1+X_2)^n] = \mathbf{E}[(1-2Z)^n] = \mathbf{E}[Y^{\frac{n}{2}}] = \frac{B(\frac{n+1}{2}, a)}{B(\frac{1}{2}, a)} = \frac{B(\frac{n+1}{2}, a)}{2^{2a-1}B(a,a)},$$

where the last equality follows from the Legendre duplication formula for the Gamma function. Finally, since the distribution of Z is symmetric with respect to $\mathbf{E}[Z] = \frac{1}{2}$, we have $\mathbf{E}[(1-2Z)^n] = 0$ for n odd. Thus, (8) follows from (4). □

Proposition 4 gives rise to the following corollary, where a generalization of identity (2) is obtained.

Corollary 5. *If $r \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{2k+2r}{k+r}\binom{2n-2k+2r}{n-k+r}}{\binom{n+2r}{k+r}} = 2^n \frac{\binom{n}{\frac{n}{2}}\binom{2r}{r}}{\binom{\frac{n}{2}+r}{r}}, \tag{9}$$

for n even, while the sum vanishes for n odd.

Proof. Assuming $a = r + \frac{1}{2}$ in expression (8) and using (6), we obtain (9). □

Remark 6. Identity (2) is promptly obtained from (9) for $r = 0$. After some algebra, for n even identity (9) may also be written as follows

$$\begin{aligned}\sum_{k=0}^{n} (-1)^k \binom{k+r}{r}\binom{2k+2r}{k+r}\binom{n-k+r}{r}\binom{2n-2k+2r}{n-k+r} \\ = 2^n \binom{2r}{r}\binom{n+2r}{\frac{n}{2}+2r}\binom{\frac{n}{2}+2r}{\frac{n}{2}+r}.\end{aligned}$$

We recall that the bivariate Selberg integral is given by

$$\begin{aligned}S_2(\alpha,\beta,\gamma) &= \int_0^1\int_0^1 t_1^{\alpha-1}(1-t_1)^{\beta-1}t_2^{\alpha-1}(1-t_2)^{\beta-1}|t_1-t_2|^{2\gamma}\,dt_1dt_2 \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)\Gamma(1+2\gamma)}{\Gamma(\alpha+\beta+\gamma)\Gamma(\alpha+\beta+2\gamma)\Gamma(1+\gamma)},\end{aligned}$$

for $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\Re(\gamma) > -\min(\frac{1}{2}, \Re(\alpha), \Re(\beta))$ (see, e.g., Forrester and Warnaar [4]). By means of the Legendre duplication formula, we also have

$$S_2(\alpha,\beta,\gamma) = 2^{2\gamma}\frac{B(\alpha,\beta+\gamma)B(\alpha+\gamma,\beta)B(\alpha+\beta+\gamma,\gamma)}{B(\gamma,\frac{1}{2})}, \tag{10}$$

and

$$S_2(\alpha,\alpha,\gamma) = 2^{1-2\alpha}\, B\left(\alpha\,,\gamma+\frac{1}{2}\right) B(\alpha,\alpha+\gamma). \tag{11}$$

Moreover, if Z_1 and Z_2 are independent r.v.'s distributed according to the Beta law $\mathcal{B}(a,b)$, for r even it is apparent that

$$\mathbf{E}[(Z_1-Z_2)^r] = \frac{S_2(a,b,\frac{r}{2})}{B(a,b)^2}. \tag{12}$$

Proposition 7. *If $a>0$, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}B(k+a,a)B(n-k+a,a) = 2^{1-2a}B\left(\frac{n+1}{2},a\right)B\left(\frac{n}{2}+a,a\right), \tag{13}$$

for n even, while the sum vanishes for n odd.

Proof. Let us assume that $X_1=-Z_1$ and $X_2=Z_2$ in expression (4), where Z_1 and Z_2 are independent r.v.'s distributed according to the Beta law $\mathcal{B}(a,a)$. From (3) it follows that

$$\mathbf{E}[X_1^kX_2^{n-k}] = (-1)^k\mathbf{E}[Z_1^k]\mathbf{E}[Z_2^{n-k}] = (-1)^k\,\frac{B(k+a,a)B(n-k+a,a)}{B(a,a)^2}.$$

In addition, from (11) and (12), for n even we have

$$\mathbf{E}[(X_1+X_2)^n] = \mathbf{E}[(Z_1-Z_2)^n] = \frac{S_2(a,a,\frac{n}{2})}{B(a,a)^2} = 2^{1-2a}\,\frac{B(\frac{n+1}{2},a)B(\frac{n}{2}+a,a)}{B(a,a)^2}.$$

Since Z_1 and Z_2 are independent r.v.'s distributed according to the same law, the distribution of the r.v. (Z_1-Z_2) is symmetric with respect to zero and we have $\mathbf{E}[(Z_1-Z_2)^n]=0$ for n odd. Expression (13) therefore follows from (4) on substitution. □

Corollary 8. *If $r\in\mathbb{N}_0$, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{2k+2r}{k+r}\binom{2n-2k+2r}{n-k+r}}{\binom{k+2r}{r}\binom{n-k+2r}{r}} = \frac{\binom{n}{\frac{n}{2}}\binom{n+2r}{\frac{n}{2}+r}}{\binom{\frac{n}{2}+r}{r}\binom{\frac{n}{2}+2r}{r}}, \tag{14}$$

for n even, while the sum vanishes for n odd.

Proof. If we assume $a=r+\frac{1}{2}$ in expression (13), the result follows from (6). □

Remark 9. By assuming that $r=0$ in identity (14), we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k} = \begin{cases}\binom{n}{\frac{n}{2}}^2, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd,}\end{cases}$$

an identity considered by Petkovšek et al. [9, Example 3.6.2], and Mikíc [7], among others, although its probabilistic proof does not seem available in the literature. Moreover, identity (14) may also be written

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \binom{k+r}{r}\binom{2k+2r}{k+r}\binom{n-k+r}{r}\binom{2n-2k+2r}{n-k+r}\binom{n+4r}{k+2r}\\ = \binom{\frac{n}{2}+r}{r}\binom{\frac{n}{2}+2r}{r}\binom{n+2r}{\frac{n}{2}+r}\binom{n+4r}{\frac{n}{2}+2r}.\end{aligned}$$

A more general identity involving Beta functions is provided in the following proposition. A suitable choice of parameters in this formula may in turn produce new binomial identities, as in the cases of Corollaries 2, 5 and 8.

Proposition 10. *If $a, b > 0$, we have*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \binom{n}{k} B(k+a, n-k+b)B(n-k+a, k+b)\\ = 2^n \frac{B(a, \frac{n}{2}+b)B(\frac{n}{2}+a, b)B(\frac{n}{2}+a+b, \frac{n}{2})}{B(\frac{n}{2}, \frac{1}{2})},\end{aligned} \tag{15}$$

for n even, while the sum vanishes for n odd.

Proof. Let us assume that $X_1 = -Z_1(1-Z_2)$ and $X_2 = (1-Z_1)Z_2$ in expression (4), where Z_1 and Z_2 are independent r.v.'s distributed according to the Beta law $\mathcal{B}(a, b)$. Using (3) it follows that

$$\begin{aligned}\mathbf{E}[X_1^k X_2^{n-k}] &= (-1)^k \mathbf{E}[Z_1^k(1-Z_1)^{n-k}]\,\mathbf{E}[Z_2^{n-k}(1-Z_2)^k]\\ &= (-1)^k \frac{B(k+a, n-k+b)B(n-k+a, k+b)}{B(a,b)^2}.\end{aligned}$$

Using (10) and (12), for n even we have

$$\begin{aligned}\mathbf{E}[(X_1+X_2)^n] = \mathbf{E}[(Z_1-Z_2)^n] &= \frac{S_2(a, b, \frac{n}{2})}{B(a,b)^2}\\ &= 2^n \frac{B(a, \frac{n}{2}+b)B(\frac{n}{2}+a, b)B(\frac{n}{2}+a+b, \frac{n}{2})}{B(a,b)^2 B(\frac{n}{2}, \frac{1}{2})}.\end{aligned}$$

In turn, the distribution of the r.v. $(Z_1 - Z_2)$ is symmetric with respect to zero and we have $\mathbf{E}[(Z_1-Z_2)^n] = 0$ for n odd. Expression (15) therefore follows from (4) on substitution. ☐

Remark 11. Assuming $a = b$ in identity (15), since we have

$$2^n \frac{B(a, \frac{n}{2}+a)^2 B(\frac{n}{2}+2a, \frac{n}{2})}{B(\frac{n}{2}, \frac{1}{2})} = 2^{1-2a} B\left(\frac{n+1}{2}, a\right) B\left(\frac{n}{2}+a, a\right),$$

on the basis of Propositions 7 and 10, it also follows that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}B(k+a,n-k+a)^2=\sum_{k=0}^{n}(-1)^k\binom{n}{k}B(k+a,a)B(n-k+a,a).$$

Corollary 12. *If $r,s\in\mathbb{N}_0$, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{2k+2r}{k+r}\binom{2k+2s}{k+s}\binom{2n-2k+2r}{n-k+r}\binom{2n-2k+2s}{n-k+s}}{\binom{n+r+s}{k+r}\binom{n+r+s}{k+s}}=4^n\frac{\binom{n}{\frac{n}{2}}\binom{2r}{r}\binom{2s}{s}\binom{n+2r}{\frac{n}{2}+r}\binom{n+2s}{\frac{n}{2}+s}}{\binom{\frac{n}{2}+r+s}{r}\binom{\frac{n}{2}+r+s}{s}\binom{n+r+s}{\frac{n}{2}}},\tag{16}$$

for n even, while the sum vanishes for n odd.

Proof. Assuming $a=r+\frac{1}{2}$ and $b=s+\frac{1}{2}$ in Proposition 10 and applying (6), the result follows. □

Remark 13. For $r=s=0$, expression (16) provides the following nice binomial identity

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{2k}{k}^2\binom{2n-2k}{n-k}^2}{\binom{n}{k}}=\begin{cases}4^n\binom{n}{\frac{n}{2}}^2, & \text{if } n \text{ is even;}\\ 0, & \text{if } n \text{ is odd.}\end{cases}$$

Remark 14. We emphasize that the results given by Adell and Lekuona [1] and Chang and Xu [2] can be obtained in an alternative way, using the properties of the Beta distribution and the Dirichlet distribution (the well-known multivariate generalization of the Beta distribution), in a manner similar to that followed in the present note.

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