Journal of Integer Sequences, Vol. 26 (2023),
Article 23.3.6

On Sándor's Inequality for the Riemann Zeta Function

Horst Alzer
Morsbacher Straße 10
51545 Waldbröl
Germany
h.alzer@gmx.de

Man Kam Kwong
Department of Applied Mathematics
The Hong Kong Polytechnic University
Hunghom, Hong Kong
mankwong@connect.polyu.hk

Abstract

Let $\omega(n)$ denote the number of distinct prime factors of n. We prove an analogue of a recently published inequality of Sándor that relates a series involving $\omega(n)$ to a quotient of zeta functions.

1 Introduction and statement of the main results

The classical Riemann zeta function is defined for real numbers $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

There are several Dirichlet series whose coefficients are arithmetical functions which can be expressed in terms of the zeta function. As an example we have the elegant identity

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s}, \quad s > 1, \tag{1}$$

where $\omega(n)$ denotes the number of distinct prime factors of n; see Apostol [1, p. 247], Hardy and Wright [2, p. 265]. The additive analogues of (1),

$$\zeta^2(s) + \zeta(2s) = 2\sum_{n=1}^{\infty} \frac{\zeta_n(s)}{n^s}, \quad s > 1,$$

and

$$\zeta^2(s) - \zeta(2s) = 2\sum_{n=2}^{\infty} \frac{\zeta_{n-1}(s)}{n^s}, \quad s > 1,$$

where $\zeta_m(s) = \sum_{k=1}^{m} 1/k^s$, are due to Hassani and Rahimpour [3].

In 2018, Sándor [5] provided an interesting counterpart of (1),

$$\sum_{n=1}^{\infty} \frac{\lambda^{\omega(n)}}{n^s} < \frac{\zeta^\lambda(s)}{\zeta(\lambda s)}, \quad \lambda > 2,\ s > 1. \tag{2}$$

Two questions arise naturally. Is there a similar inequality which offers an upper bound for $\zeta^\lambda(s)/\zeta(\lambda s)$? Do there exist related results for real parameters λ with $1 < \lambda < 2$? The aim of this note is to give affirmative answers to both questions.

Our first theorem presents a complement of (2).

Theorem 1. *For all real numbers $\lambda > 2$ and $s > 1$, we have*

$$\frac{\zeta^\lambda(s)}{\zeta(\lambda s)} < \sum_{n=1}^{\infty} \frac{(2^\lambda - 2)^{\omega(n)}}{n^s}. \tag{3}$$

Next, we provide upper and lower bounds for $\zeta^\lambda(s)/\zeta(\lambda s)$ which are valid for all real numbers λ with $1 < \lambda < 2$.

Theorem 2. *Let $1 < \lambda < 2$ and $s > 1$. Then*

$$\sum_{n=1}^{\infty} \frac{(2^\lambda - 2)^{\omega(n)}}{n^s} < \frac{\zeta^\lambda(s)}{\zeta(\lambda s)} < \sum_{n=1}^{\infty} \frac{\lambda^{\omega(n)}}{n^s}. \tag{4}$$

In the next section, we collect three lemmas. The proofs of the two theorems are given in Section 3.

2 Lemmas

A proof of the first lemma can be found in Hardy and Wright [2, p. 249].

Lemma 3. *Let f be a multiplicative arithmetical function with $f(1) = 1$ such that $\sum_{n=1}^{\infty} f(n)/n^s$ is absolutely convergent. Then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p\,\text{prime}} \Big(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\Big).$$

The next lemma might be known. Since we cannot give a reference, we include a proof.

Lemma 4. *Let $a > 0$ and $s > 1$ be real numbers. The series*

$$\sum_{n=1}^{\infty} \frac{a^{\omega(n)}}{n^s} \tag{5}$$

is convergent.

Proof. Let $s > 1$. From (1) we conclude that if $0 < a \le 2$, then the series is convergent. Next, let $a > 2$. A result of Robin [4] states that there exists a number $b \approx 1.384$ such that

$$\omega(n) \le b \frac{\log n}{\log \log n}, \quad n \ge 3.$$

This leads to

$$\frac{a^{\omega(n)}}{n^s} \le \frac{a^{b \log n / \log \log n}}{n^s}, \quad n \ge 3. \tag{6}$$

Moreover, there exists a positive integer m such that

$$\frac{a^{b \log n / \log \log n}}{n^s} \le \frac{1}{n^{(s+1)/2}}, \quad n \ge m. \tag{7}$$

From (6) and (7) we conclude that the series in (5) is convergent. □

Lemma 5. *Let c and x be real numbers.*

(i) *If $1 < c < 2$ and $0 < x < 1/2$, then*

$$\frac{1 + (2^c - 3)x}{1 - x} < \frac{1 - x^c}{(1 - x)^c} < \frac{1 + (c - 1)x}{1 - x}. \tag{8}$$

(ii) *If $c > 2$ and $0 < x < 1/2$, then (8) holds with "$>$" instead of "$<$".*

Proof. Let $c > 0$ and $0 < x < 1$. We define

$$u_c(x) = \frac{1 - x^c}{(1 - x)^{c-1}},$$
$$v_c(x) = \frac{u_c(x) - 1}{x},$$
$$w_c(x) = x u_c'(x) - u_c(x) + 1.$$

Then

$$x^2 v_c'(x) = w_c(x),$$
$$w_c'(x) = x u_c''(x) = \frac{(c - 1)cx(1 - x^{c-2})}{(1 - x)^{c+1}}.$$

(i) Let $1 < c < 2$ and $0 < x < 1/2$. Then $w_c'(x) < 0$. Thus,

$$w_c(x) < w_c(0) = 0.$$

It follows that

$$2^c - 3 = v_c(1/2) < v_c(x) < v_c(0) = u_c'(0) = c - 1.$$

This implies (8).

(ii) Let $c > 2$ and $0 < x < 1/2$. Then $w_c'(x) > 0$ and $w_c(x) > w_c(0) = 0$. This gives

$$v_c(0) < v_c(x) < v_c(1/2)$$

which is equivalent to (8) with "$>$" instead of "$<$".

□

3 Proof of the theorems

Proof. We follow the method of proof given in [5]. Let $a > 0$ and $s > 1$. For positive integers n, we define

$$F_a(n) = a^{\omega(n)}.$$

Since

$$\omega(1) = 0 \quad \text{and} \quad \omega(mn) = \omega(m) + \omega(n), \quad \gcd(m, n) = 1,$$

we obtain

$$F_a(1) = 1 \quad \text{and} \quad F_a(mn) = F_a(m)F_a(n), \quad \gcd(m, n) = 1.$$

From Lemma 3 and Lemma 4 we get

$$\sum_{n=1}^{\infty} \frac{F_a(n)}{n^s} = \prod_{p \,\text{prime}} \Big(1 + \frac{F_a(p)}{p^s} + \frac{F_a(p^2)}{p^{2s}} + \cdots\Big). \tag{9}$$

We have

$$\sum_{n=1}^{\infty} \frac{F_a(p^n)}{p^{ns}} = \sum_{n=1}^{\infty} \frac{a^{\omega(p^n)}}{p^{ns}} = \sum_{n=1}^{\infty} \frac{a}{p^{ns}} = a\frac{p^{-s}}{1 - p^{-s}}. \tag{10}$$

(i) Let $1 < \lambda < 2$. Using (9), (10) and Lemma 5, part (i) with $x = p^{-s}$ yields

$$\sum_{n=1}^{\infty} \frac{(2^\lambda - 2)^{\omega(n)}}{n^s} = \prod_{p \,\text{prime}} \frac{1 + (2^\lambda - 3)p^{-s}}{1 - p^{-s}} < \prod_{p \,\text{prime}} \frac{1 - p^{-s\lambda}}{(1 - p^{-s})^\lambda} \tag{11}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda^{\omega(n)}}{n^s} = \prod_{p\,\text{prime}} \frac{1+(\lambda-1)p^{-s}}{(1-p^{-s})^{\lambda}} > \prod_{p\,\text{prime}} \frac{1-p^{-s\lambda}}{(1-p^{-s})^{\lambda}}. \tag{12}$$

From (11), (12) and

$$\zeta^{\lambda}(s) = \prod_{p\,\text{prime}} \frac{1}{(1-p^{-s})^{\lambda}}, \quad \zeta(\lambda s) = \prod_{p\,\text{prime}} \frac{1}{1-p^{-\lambda s}} \tag{13}$$

we conclude that (4) is valid.

(ii) Let $\lambda > 2$. We apply Lemma 5, part (ii) with $x = p^{-s}$, then we obtain (11) with "$>$" instead of "$<$". Using this result and (13) leads to (3).

□

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th Ed., Oxford University Press, 1959.

[3] M. Hassani and S. Rahimpour, L-Summing method, *RGMIA Research Report Collection* **7**(4) (2004), article 10.

[4] G. Robin, Estimation de la fonction de Tchebychef θ sur le k-ième nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n, *Acta Arith.* **42** (1983), 367–389.

[5] J. Sándor, An inequality involving a ratio of zeta functions, *Notes Number Th. Disc. Math.* **24** (2018), 92–94.

2020 *Mathematics Subject Classification:* Primary 11M06; Secondary 11B83, 26D15.
Keywords: Riemann zeta function, prime omega function, inequality.

(Concerned with sequence A129251.)

Received January 8 2023; revised versions received March 3 2023; March 5 2023. Published in *Journal of Integer Sequences*, March 16 2023.

Return to Journal of Integer Sequences home page.