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Convolutions with the Bernoulli and Euler Numbers

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Abstract

We use exponential generating functions to study the relationship between Bernoulli and Euler numbers with k-Jacobsthal numbers, k-Jacobsthal-Lucas numbers, and bivariate Fibonacci, Lucas, Pell and Pell-Lucas polynomials.

1 Introduction and preliminaries

Recently, there has been some study of special numbers and polynomials, especially the Bernoulli numbers and Euler numbers, by virtue of their applications in almost all branches of mathematics, computer algorithms, engineering, and other areas. (For other numbers see [5, 6, 8, 14, 15]).

Bernoulli and Euler polynomials are famous mathematical objects and are fairly well understood. They are, respectively, defined by the exponential generating functions as follows:

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{\exp(z)-1}, \ |z| < 2\pi, \tag{1}$$

$$\sum_{n\geqslant 0} E_n \frac{z^n}{n!} = \frac{2\exp(z)}{\exp(2z)+1}, \ |z| < \pi. \tag{2}$$

The Euler and Bernoulli numbers have appeared in many important results. There has been a growing interest in deriving new relations for these two pairs of sequences. In 1975, Byrd [2] derived the following identity relating Lucas numbers to Euler numbers:

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} \left(\frac{5}{4}\right)^l L_{n-2l} E_{2l} = 2^{1-n}. \tag{3}$$

Zhang and Ma [16] proved a relation between Fibonacci polynomials and Bernoulli numbers. The following identity is a special case of their result:

$$\sum_{l=0}^{n} \binom{n}{l} 5^{\frac{n-k}{2}} F_k B_{n-k} = n\left(\frac{1-\sqrt{5}}{2}\right)^{n-1},$$

or equivalently

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} 5^l F_{n-2l} B_{2l} = \frac{nL_{n-1}}{2}.$$

Patel et al. [13] obtained similar identities when Fibonacci and Lucas numbers are replaced, respectively, by the Pell and Lucas polynomials (defined below in Eqs. (6) and (7)):

$$\sum_{l=0}^{n} \binom{n}{l} (2\sqrt{x^2+1})^l P_{n-l}(x) B_l = n(x-\sqrt{x^2+1})^{n-1}.$$

This can be stated as

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} 4^l (x^2+1)^l P_{n-2l}(x) B_{2l} = \frac{nQ_{n-1}(x)}{2}.$$

Still other relations are contained in the many articles (see [7, 8, 9]). The present paper is devoted to developing further relations between Bernoulli and Euler numbers with famous number sequences. To achieve this goal, we use elementary methods, including exponential generating functions, to study the relationship between Bernoulli and Euler numbers and between the k-Jacobsthal numbers k-Jacobsthal-Lucas numbers and bivariate Fibonacci, Lucas, Pell, and Pell-Lucas polynomials.

For every positive real number k, the k-Jacobsthal numbers $(J_{n,k})_{n\in\mathbb{N}}$ are defined recursively by the relation [11]

$$J_{k,n} = kJ_{k,n-1} + 2J_{k,n-1}, \text{ for } n \geq 2,$$

with initial values $J_{k,0} = 0$ and $J_{k,1} = 1$.

Jhala et al. [12] defined the k-Jacobsthal-Lucas numbers as follows:

$$j_{k,n} = kj_{k,n-1} + 2j_{k,n-1}, \text{ for } n \geq 2, \text{ with } j_{k,0} = 2, \ j_{k,1} = k.$$

The well-known Binet formulas for k-Jacobsthal $(J_{n,k})_{n\in\mathbb{N}}$ and k-Jacobsthal-Lucas numbers $(j_{n,k})_{n\in\mathbb{N}}$ are given as follows:

$$J_{n,k} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$
$$j_{n,k} = \lambda_1^n + \lambda_2^n,$$

where $\lambda_1 = \frac{k+\sqrt{k^2+8}}{2}$ and $\lambda_2 = \frac{k-\sqrt{k^2+8}}{2}$ are the roots of the characteristic equation $\lambda^2 - k\lambda - 2 = 0$.

If $k = 1$, the classical Jacobsthal and Jacobsthal-Lucas numbers are

$$J_n = J_{n-1} + 2J_{n-1}, \text{ for } n \geq 2, \text{ with } J_0 = 0, \ J_1 = 1,$$
$$j_n = j_{n-1} + 2j_{n-1}, \text{ for } n \geq 2, \text{ with } j_0 = 2, \ j_1 = 1.$$

The bivariate Fibonacci $(F_{n,k}(x,y))_{n\geq 0}$ and bivariate Lucas $(L_{n,k}(x,y))_{n\geq 0}$ polynomial sequences are, respectively, defined by the following recurrence relations [3, 4]:

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y), \text{ for } n \geq 2, \tag{4}$$

with initial values $F_0(x,y) = 0$ and $F_1(x,y) = 1$, and

$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y), \text{ for } n \geq 2, \tag{5}$$

with initial values $L_0(x,y) = 2$ and $L_1(x,y) = x$.

The bivariate Pell and Pell-Lucas polynomials are, respectively, as follows [10]:

$$P_n(x,y) = 2xyP_{n-1}(x,y) + yP_{n-2}(x,y), \ n \geq 2, \tag{6}$$

with initial values $P_0(x,y) = 0$ and $P_1(x,y) = 1$, and

$$Q_n(x,y) = 2xyQ_{n-1}(x,y) + yQ_{n-2}(x,y), \ n \geq 2, \tag{7}$$

with initial values $Q_0(x,y) = 2$ and $Q_1(x,y) = 2xy$.

We present the Binet formulas of the bivariate Fibonacci, Lucas, Pell, and Pell-Lucas polynomials in Table 1.

Bivariate polynomials	Roots (λ_1, λ_1)	Binet formula
Bivariate Fibonacci polynomials	$\lambda_{1,2} = \frac{x\pm\sqrt{x^2+4y}}{2}$	$F_n(x, y) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$
Bivariate Lucas polynomials	$\lambda_{1,2} = \frac{x\pm\sqrt{x^2+4y}}{2}$	$Q_n(x, y) = \lambda_1^n + \lambda_2^n$
Bivariate Pell polynomials	$\lambda_{1,2} = xy \pm \sqrt{x^2y^2 + y}$	$P_n(x, y) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$
Bivariate Pell-Lucas polynomials	$\lambda_{1,2} = xy \pm \sqrt{x^2y^2 + y}$	$Q_n(x, y) = \lambda_1^n + \lambda_2^n$

Table 1: Binet's formulas of the bivariate polynomials.

Putting $y = 1$ in the Eqs (4), (5), (6) and (7) yields, respectively, the following four polynomials:

- the Fibonacci polynomials defined by $F_0(x) = 0$, $F_1(x) = 1$, $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for $n \geq 2$,
- the Lucas polynomials defined by $L_0(x) = 2$, $L_1(x) = x$, $L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$, for $n \geq 2$,
- the Pell polynomials defined by $P_0(x) = 0$, $P_1(x) = 1$, $P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$, for $n \geq 2$,
- the Pell-Lucas polynomials defined by $Q_0(x) = 2$, $Q_1(x) = 2x$, $Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)$, for $n \geq 2$.

Putting $x = y = 1$ in Eqs. (4), (5), (6) and (7), we respectively get the Fibonacci, Lucas, Pell, and Pell-Lucas numbers, which are given as follows:

- $F_n = F_{n-1} + F_{n-2},\ n \geq 2$, with initial values $F_0 = 0$, $F_1 = 1$,
- $L_n = L_{n-1} + L_{n-2},\ n \geq 2$, with initial values $L_0 = 2$, $L_1 = 1$,
- $P_n = 2P_{n-1} + P_{n-2},\ n \geq 2$, with initial values $P_0 = 0$, $P_1 = 1$,
- $Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2$, with initial values $Q_0(x) = 2$, $Q_1 = 2$.

2 New k-Jacobsthal-Bernoulli and k-Jacobsthal-Euler relations

In this section, we present our first findings in three theorems, which provide some relations involving Bernoulli and Euler numbers with k-Jacobsthal and k-Jacobsthal-Lucas numbers.

Theorem 1. *Let n be a positive integer. We have*

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{k^2+8}\right)^l J_{k,n-l}B_l = n\Big(\frac{k-\sqrt{k^2+8}}{2}\Big)^{n-1},$$

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{k^2+8}\right)^l (2^l-1)j_{k,n-l}B_l = -n\sqrt{k^2+8}\Big(\frac{k-\sqrt{k^2+8}}{2}\Big)^{n-1}. \tag{8}$$

Proof. Using the change of variables $z=\sqrt{k^2+8}z$ in (1), we obtain

$$\sum_{n=0}^{\infty}\left(\sqrt{k^2+8}\right)^n B_n\frac{z^n}{n!} = \frac{\sqrt{k^2+8}z}{\exp(\sqrt{k^2+8}z)-1}. \tag{9}$$

Multiplying (9) by the exponential generating function for $(J_{k,n})_{n\in\mathbb{N}}$, we have

$$\begin{aligned}\Big(\sum_{n=0}^{\infty}J_{k,n}\frac{z^n}{n!}\Big)\Big(\sum_{l=0}^{\infty}\left(\sqrt{k^2+8}\right)^l B_l\frac{z^l}{l!}\Big) &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\left(\sqrt{k^2+8}\right)^l B_l\frac{z^l}{l!}\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{k^2+8}\right)^l B_l J_{k,n-l}\frac{z^n}{n!}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\exp\Big(\frac{k+\sqrt{k^2+8}}{2}z\Big)-\exp\Big(\frac{k-\sqrt{k^2+8}}{2}z\Big)}{\sqrt{k^2+8}}\times\frac{\sqrt{k^2+8}z}{\exp(\sqrt{k^2+8}z)-1}\\ &=\exp(\frac{k-\sqrt{k^2+8}}{2}z)(\exp(\sqrt{k^2+8}z)-1)\times\frac{z}{\exp(\sqrt{k^2+8}z)-1}\\ &= z\exp\Big(\frac{k-\sqrt{k^2+8}}{2}z\Big)\\ &= \sum_{n=0}^{\infty}n\Big(\frac{k-\sqrt{k^2+8}}{2}z\Big)^{n-1}\frac{z^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{z^n}{n!}$, we obtain the desired result.

Similarly, we use the change of variables $z=2\sqrt{k^2+8}z$ in (1), we obtain

$$\sum_{n=0}^{\infty}\left(2\sqrt{k^2+8}\right)^n B_n\frac{z^n}{n!} = \frac{2\sqrt{k^2+8}z}{\exp(2\sqrt{k^2+8}z)-1}, \tag{10}$$

and multiplying (10), by the exponential generating function for $(j_{n,k})_{n\in\mathbb{N}}$, we get

$$\begin{aligned}\Big(\sum_{n=0}^{\infty}j_{k,n}\frac{z^n}{n!}\Big)\Big(\sum_{l=0}^{\infty}\left(2\sqrt{k^2+8}\right)^l B_l\frac{z^l}{l!}\Big) &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\left(2\sqrt{k^2+8}\right)^l B_l j_{k,n}\frac{z^l}{l!}\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2^l\left(\sqrt{k^2+8}\right)^l j_{k,n-l}B_l\frac{z^n}{n!}.\end{aligned}$$

Then we obtain

$$
\begin{aligned}
&\Big(\exp\Big(\frac{(k+\sqrt{k^2+8})z}{2}\Big)+\exp\Big(\frac{(k-\sqrt{k^2+8})z}{2}\Big)\Big)\frac{2\sqrt{k^2+8}z}{\exp(2\sqrt{k^2+8}z)-1}\\
&=\exp\Big(\frac{(k-\sqrt{k^2+8})z}{2}\Big)\frac{2\sqrt{k^2+8}z}{\exp(\sqrt{k^2+8}z)-1}\\
&=\Big(\sum_{n=0}^{\infty}\Big(\frac{k-\sqrt{k^2+8}}{2}\Big)^n\frac{z^n}{n!}\Big)\Big(\sum_{n=0}^{\infty}2\Big(\sqrt{k^2+8}\Big)^l B_l\frac{z^l}{l!}\Big)\\
&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2\Big(\sqrt{k^2+8}\Big)^l\Big(\frac{k-\sqrt{k^2+8}}{2}\Big)^{n-l}B_l\frac{z^n}{n!}.
\end{aligned}
$$

By comparing the coefficients of $\frac{z^n}{n!}$, we get

$$
\sum_{l=0}^{n}\binom{n}{k}2^l\Big(\sqrt{k^2+8}\Big)^l j_{k,n-l}B_l=\sum_{l=0}^{n}\binom{n}{l}2\Big(\sqrt{k^2+8}\Big)^l\Big(\frac{k-\sqrt{k^2+8}}{2}\Big)^{n-l}B_l,
$$

which is equivalent to

$$
\sum_{l=0}^{n}\binom{n}{k}2^l\Big(\sqrt{k^2+8}\Big)^l j_{k,n-l}B_l=\sum_{l=0}^{n}\binom{n}{l}2\Big(\sqrt{k^2+8}\Big)^l\Big(\frac{j_{k,n-1}-\sqrt{k^2+8}J_{k,n-1}}{2}\Big)B_l.
$$

Therefore

$$
\begin{aligned}
\sum_{l=0}^{n}\binom{n}{l}\Big(\sqrt{k^2+8}\Big)^l(2^l-1)j_{k,n-l}B_l&=-\sqrt{k^2+8}\sum_{l=0}^{n}\binom{n}{l}\Big(\sqrt{k^2+8}\Big)^l J_{k,n-l}B_l,\\
&=-n\sqrt{k^2+8}\Big(\frac{k-\sqrt{k^2+8}}{2}\Big)^{n-1}.
\end{aligned}
$$

Hence the desired result. □

Theorem 2. *Let n be a positive integer. The following results hold:*

$$
\begin{aligned}
\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\Big(k^2+8\Big)^l J_{k,n-2l}B_{2l}&=\frac{nj_{k,n-1}}{2},\\
\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\Big(k^2+8\Big)^l(2^{2l}-1)j_{k,n-2l}B_{2l}&=\frac{(k^2+8)nJ_{k,n-1}}{2}.
\end{aligned}
$$

Proof. We have

$$
\sum_{l=0}^{n}\binom{n}{l}\Big(\sqrt{k^2+8}\Big)^l J_{k,n-l}B_l=n\Big(\frac{k-\sqrt{k^2+8}}{2}\Big)^{n-1}.
$$

This gives

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{k^2+8}\right)^{n-l}J_{k,n-l}B_l=\frac{n(j_{k,n-1}-\sqrt{k^2+8}J_{k,n-1})}{2},$$

with the conditions $B_0=1$ and $B_1=-\frac{1}{2}$.

For $l\geq 0$, we have $B_{2l+1}=0$ and then

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(k^2+8)^l J_{k,n-2l}B_{2l}=\frac{nj_{k,n-1}}{2}.$$

Similarly, from (8), we get

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{k^2+8}\right)^{l}(2^l-1)j_{k,n-l}B_l=\frac{-\sqrt{k^2+8}n(j_{k,n-1}-\sqrt{k^2+8}J_{k,n-1})}{2},$$

with $B_0=1$ and $B_1=-\frac{1}{2}$.

For $l\geq 0$, we have $B_{2l+1}=0$. Then we obtain

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(k^2+8)^l(2^{2l}-1)j_{k,n-2l}B_{2l}=\frac{(k^2+8)^2nJ_{k,n-1}}{2}.$$

Which complete the proof. □

- Putting $k=1$ in Theorem 1 and Theorem 2 we have the following Jacobsthal-Bernoulli and Jacobsthal-Lucas-Bernoulli identities

$$\sum_{l=0}^{n}\binom{n}{l}3^lJ_{n-l}B_l=(-1)^{n-1}n,\text{ or }\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}9^lJ_{n-2l}B_{2l}=\frac{nj_{n-1}}{2}.$$

$$\sum_{l=0}^{n}\binom{n}{2l}3^l(2^l-1)j_{n-l}B_l=(-1)^{n}n,\text{ or }\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}9^l(2^l-1)j_{n-2l}B_{2l}=\frac{9nJ_{n-1}}{2}.$$

Theorem 3. *For every positive integer n, we have*

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2(4)^{n-2l-1}\left(\sqrt{k^2+8}\right)^{2l+1}J_{k,n-2l}E_{2l}=(2k+\sqrt{k^2+8})^n-(2k+\sqrt{k^2+8})^n,$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(\frac{k^2+8}{4}\right)^{l}j_{k,n-2l}E_{2l}=2\left(\frac{k}{2}\right)^{n}.$$

Proof. Using the change of variables $z = \sqrt{k^2+8}z$ in (2) and $z = 4z$ in the exponential generating function for $(J_{k,n})_{n\in\mathbb{N}}$, we obtain

$$\sum_{n=0}^{\infty}\left(\sqrt{k^2+8}\right)^l E_l\frac{z^l}{l!} = \frac{2\exp(\sqrt{k^2+8}z)}{\exp(2\sqrt{k^2+8}z)+1}, \tag{11}$$

$$\sum_{n=0}^{\infty}4^n J_{k,n}\frac{z^n}{n!} = \frac{\exp(k-\sqrt{k^2+8})z)}{\sqrt{k^2+8}}(\exp(2\sqrt{k^2+8}z)-1)(\exp(2\sqrt{k^2+8}z)+1). \tag{12}$$

Multiplying (12) by (11), this gives

$$\Big(\sum_{n=0}^{\infty}4^n J_{k,n}\frac{z^n}{n!}\Big)\Big(\sum_{l=0}^{\infty}\left(\sqrt{k^2+8}\right)^l E_l\frac{z^l}{l!}\Big) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}4^{n-l}\left(\sqrt{k^2+8}\right)^l J_{k,n-l}E_l\frac{z^n}{n!}.$$

Then we get

$$\begin{aligned}
&\frac{\exp(k-\sqrt{k^2+8})z)}{\sqrt{k^2+8}}(\exp(2\sqrt{k^2+8}z)-1)(\exp(2\sqrt{k^2+8}z)+1)\times\frac{2\exp(\sqrt{k^2+8}z)}{(\exp 2\sqrt{k^2+8}z)+1}\\
&=\frac{2}{\sqrt{k^2+8}}\Big(\exp\Big((2k+\sqrt{k^2+8})z\Big)-\exp\Big((2k-\sqrt{k^2+8})z\Big)\Big)\\
&=\frac{2}{\sqrt{k^2+8}}\sum_{n=0}^{\infty}\Big(2k+\sqrt{k^2+8}\Big)^n-\Big(2k-\sqrt{k^2+8}\Big)^n\frac{z^n}{n!}.
\end{aligned}$$

By comparing the coefficients of $\frac{z^n}{n!}$, we obtain

$$\sum_{l=0}^{n}\binom{n}{l}2(4)^{n-l-1}\left(\sqrt{k^2+8}\right)^{l+1}J_{k,n-l}E_l = \Big(2k+\sqrt{k^2+8}\Big)^n-\Big(2k-\sqrt{k^2+8}\Big)^n.$$

For $l\geq 0$, we have $E_{2l+1}=0$, which gives the desired result.

Similarly, we use the change of variables $z=\frac{\sqrt{k^2+8}}{2}z$ in (2). We have

$$\sum_{n=0}^{\infty}\Big(\frac{\sqrt{k^2+8}}{2}\Big)^n E_n\frac{z^n}{n!} = \frac{2\exp\Big(\frac{\sqrt{k^2+8}}{2}z\Big)}{\exp(\sqrt{k^2+8}z)+1}, \tag{13}$$

and multiplying (13) by the exponential generating function for $(j_{k,n})_{n\in\mathbb{N}}$, we get

$$\Big(\sum_{n=0}^{\infty}j_{k,n}\frac{z^n}{n!}\Big)\Big(\sum_{l=0}^{\infty}\Big(\frac{\sqrt{k^2+8}}{2}\Big)^l E_l\frac{z^l}{l!}\Big) = \sum_{l=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\Big(\frac{\sqrt{k^2+8}}{2}\Big)^l j_{k,n-l}E_l\frac{z^n}{n!},$$

and

$$\frac{2\exp(\frac{\sqrt{k^2+8}}{2}z)}{\exp(\sqrt{k^2+8}z)+1}\Big(\exp\Big(\frac{k+\sqrt{k^2+8}}{2}z\Big)+\exp\Big(\frac{k-\sqrt{k^2+8}}{2}z\Big)\Big) = 2\sum_{k=0}^{\infty}\Big(\frac{k}{2}\Big)^n\frac{z^n}{n!}.$$

By comparing the coefficients of $\frac{z^n}{n!}$, we get

$$\sum_{l=0}^{n}\binom{n}{l}\Big(\frac{\sqrt{k^2+8}}{2}\Big)^l j_{k,n-l}E_l = 2\Big(\frac{k}{2}\Big)^n.$$

For $k \geq 0$, we obtain $E_{2k+1} = 0$; hence the desired result. □

- By taking $k = 1$ in the Theorem 3, we can state the following identities involving Euler numbers with Jacobsthal and Jacobsthal-Lucas numbers

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2(4)^{n-2l-1}3^{2l+1}J_{n-2l}E_{2l} = 5^n-(-1)^n,$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\Big(\frac{9}{4}\Big)^l j_{n-2l}E_{2l} = 2^{1-n}.$$

3 Bivariate polynomial identities

The aim of this section is to establish some new identities involving two variables x and y. We start with three theorems involving Bernoulli and Euler numbers with bivariate Fibonacci and Lucas polynomials.

Theorem 4. *For every positive integer n, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\Big(\sqrt{x^2+4y}\Big)^l F_{n-l}(x,y)B_l = n\Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^{n-1}, \tag{14}$$

$$\sum_{l=0}^{n}\binom{n}{l}\Big(\sqrt{x^2+4y}\Big)^l(2^l-1)L_{n-l}(x,y)B_l = -2\sqrt{x^2y^2+y}\,n\Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^{n-1}.$$

Proof. Using the change of variables $z = \sqrt{x^2+4y}z$ in (1), we obtain

$$\sum_{n=0}^{\infty}\Big(\sqrt{x^2+4y}\Big)^n B_n\frac{z^n}{n!} = \frac{\sqrt{x^2+4y}z}{\exp(\sqrt{x^2+4y}z)-1}. \tag{15}$$

Multiplying (15) by the exponential generating function for $(F_n(x,y))_{n\in\mathbb{N}}$, we get

$$\begin{aligned}\Big(\sum_{n=0}^{\infty}F_n(x,y)\frac{z^n}{n!}\Big)\Big(\sum_{l=0}^{\infty}\Big(\sqrt{x^2+4y}\Big)^l B_l\frac{z^l}{l!}\Big) &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\Big(\sqrt{x^2+4y}\Big)^l B_lF_n(x,y)\frac{z^l}{l!}\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\Big(\sqrt{x^2+4y}\Big)^l B_lF_n(x,y)\frac{z^n}{n!}.\end{aligned}$$

Then

$$\frac{\exp\Big(\frac{x-\sqrt{x^2+4y}}{2}z\Big)-\exp\Big(\frac{x-\sqrt{x^2+4y}}{2}z\Big)}{\sqrt{x^2+4y}}\times\frac{\sqrt{x^2+4y}z}{\exp(\sqrt{x^2+4y}z)-1}$$
$$=z\exp\Big(\frac{x-\sqrt{x^2+4y}}{2}z\Big)=\sum_{n=0}^{\infty}n\Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^{n-1}\frac{z^n}{n!}.$$

By comparing the coefficients of $\frac{z^n}{n!}$, we obtain the desired result.

Similarly, by using the change of variables $z=2\sqrt{x^2+4y}z$ in (1), we obtain

$$\sum_{n=0}^{\infty}\Big(2\sqrt{x^2+4y}\Big)^nB_n\frac{z^n}{n!}=\frac{2\sqrt{x^2+4y}z}{\exp(2\sqrt{x^2+4y}z)-1},\tag{16}$$

and multiplying (16) by the exponential generating function for $(L_n(x,y))_{n\in\mathbb{N}}$, we get

$$\Big(\sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!}\Big)\Big(\sum_{l=0}^{\infty}\Big(2\sqrt{x^2+4y}\Big)^lB_l\frac{z^l}{l!}\Big)=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\Big(2\sqrt{x^2+4y}\Big)^lB_lL_n(x,y)\frac{z^l}{l!}\frac{z^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2^l\Big(\sqrt{x^2+4y}\Big)^lL_{n-l}(x,y)B_l\frac{z^n}{n!}.$$

Therefore

$$\Big(\exp\Big(\frac{x+\sqrt{x^2+4y}}{2}z\Big)+\exp\Big(\frac{x-\sqrt{x^2+4y}}{2}z\Big)\Big)\frac{2\sqrt{x^2+4y}z}{\exp(2\sqrt{x^2+4y}z)-1}$$
$$=\exp\Big(\frac{x-\sqrt{x^2+4y}}{2}z\Big)\frac{2\sqrt{x^2+4y}z}{\exp(\sqrt{x^2+4y}z)-1}$$
$$=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}2\Big(\sqrt{x^2+4y}\Big)^l\Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^{n-l}B_l\frac{z^n}{n!}.$$

By comparing the coefficients of $\frac{z^n}{n!}$, we obtain

$$\sum_{l=0}^{n}\binom{n}{k}2^l\Big(\sqrt{x^2+4y}\Big)^lL_{n-l}(x,y)B_l=\sum_{l=0}^{n}\binom{n}{l}2\Big(\sqrt{x^2+4y}\Big)^l\Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^{n-l}B_l.$$

We know that

$$\Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^{n-l}=\frac{L_{n-l}(x,y)-\sqrt{x^2+4y}F_{n-l}(x,y)}{2},$$

so we get

$$\sum_{l=0}^{n}\binom{n}{l}\Big(\sqrt{x^2+4y}\Big)^l(2^l-1)L_{n-l}(x,y)B_l=-\sqrt{x^2+4y}n\Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^{n-1}.$$

Hence the result. □

Theorem 5. *For every positive integer n, we have*

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(x^2+4y)^l F_{n-2l}(x,y)B_{2l}=\frac{nL_{n-1}(x,y)}{2},$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(x^2+4y)^l(2^{2l}-1)L_{n-2l}(x,y)B_{2l}=\frac{n(x^2+4y)F_{n-1}(x,y)}{2}. \tag{17}$$

Proof. From (14) and on account of the identity

$$\Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^{n-l}=\frac{L_{n-l}(x,y)-\sqrt{x^2+4y}F_{n-l}(x,y)}{2},$$

we get

$$\sum_{l=0}^{n}\binom{n}{l}\big(\sqrt{x^2+4y}\big)^{n-l}F_{n-l}(x,y)B_l=\frac{n(L_{n-1}(x,y)-\sqrt{x^2+4y}F_{n-1}(x,y))}{2},$$

with $B_0=1$ and $B_1=-\frac{1}{2}$.

For $l\geq 0$, we have $B_{2l+1}=0$, and then

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(x^2+4y)^l F_{n-2l}(x,y)B_{2l}=\frac{nL_{n-1}(x,y)}{2}.$$

Similarly, from (17), we get

$$\sum_{l=0}^{n}\binom{n}{l}\big(\sqrt{x^2+4y}\big)^{l}(2^l-1)L_{n-l}(x,y)B_l=\frac{-\sqrt{x^2+4y}n(L_{n-1}(x,y)-\sqrt{x^2+4y}F_{n-1}(x,y))}{2},$$

with $B_0=1$, $B_1=-\frac{1}{2}$.

For $l\geq 0$, we have $B_{2l+1}=0$. Hence we obtain

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(x^2+4y)^l(2^{2l}-1)L_{n-2l}(x,y)B_{2l}=\frac{(x^2+4y)^2nF_{n-1}(x,y)}{2}.$$

Hence the desired result is proved. □

- By putting $y=1$ in Theorem 4 and Theorem 5, we obtain the following identities

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{x^2+4}\right)^l F_{n-l}(x)B_l = n\Big(\frac{x-\sqrt{x^2+4}}{2}\Big)^{n-1}, \tag{18}$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(x^2+4)^l F_{n-2l}(x)B_{2l} = \frac{nL_{n-1}(x)}{2}, \tag{19}$$

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{x^2+4}\right)^l (2^l-1)L_{n-l}(x)B_l = -2(\sqrt{x^2+4})n\Big(\frac{x-\sqrt{x^2+4}}{2}\Big)^{n-1}, \tag{20}$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(x^2+4)^l(2^{2l}-1)L_{n-2l}(x)B_{2l} = \frac{n(x^2+4)F_{n-1}(x)}{2}. \tag{21}$$

- By setting $x=y=1$ in Theorem 4 and Theorem 5 or $x=1$ in the relationships (18), (19), (20) and (21) we obtain the following identities involving Bernoulli numbers with Fibonacci and Lucas numbers (see [1, 2]):

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{5}\right)^l F_{n-l}B_l = n\Big(\frac{1-\sqrt{5}}{2}\Big)^{n-1},$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}5^l F_{n-2l}B_{2l} = \frac{nL_{n-1}}{2},$$

$$\sum_{l=0}^{n}\binom{n}{l}\left(\sqrt{5}\right)^l (2^l-1)L_{n-l}B_l = -2\sqrt{5}n\Big(\frac{1-\sqrt{5}}{2}\Big)^{n-1},$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}5^l(2^{2l}-1)L_{n-2l}B_{2l} = \frac{n5F_{n-1}}{2}.$$

Theorem 6. *For every positive integer n, we have*

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2(4)^{n-2k-1}\left(\sqrt{x^2+4y}\right)^{2l+1}F_{n-2l}(x,y)E_{2l} = \left(2x+\sqrt{x^2+4y}\right)^n - \left(2x-\sqrt{x^2+4y}\right)^n,$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\Big(\frac{x^2+4y}{4}\Big)^{2l}L_{n-2l}(x,y)E_{2l} = 2\Big(\frac{x}{2}\Big)^n.$$

Proof. Using the change of variables $z=\sqrt{x^2+4y}z$ in (2), we obtain

$$\sum_{l=0}^{\infty}\left(\sqrt{x^2+4y}\right)^l E_l\frac{z^l}{l!} = \frac{2\exp(\sqrt{x^2+4y}z)}{\exp(2\sqrt{x^2+4y}z)+1}. \tag{22}$$

By multiplying (22) by the exponential generating function for the sequence $(4^n F_n(x,y))_{n\in\mathbb{N}}$, we obtain

$$\Big(\sum_{k=0}^{\infty} 4^n F_n(x,y)\frac{z^n}{n!}\Big)\Big(\sum_{l=0}^{\infty}\Big(\sqrt{x^2+4y}\Big)^l E_l\frac{z^l}{l!}\Big) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}4^{n-l}\Big(\sqrt{x^2+4y}\Big)^l F_{n-l}(x,y)E_l\frac{z^n}{n!}.$$

Then we get

$$\begin{aligned}
&\frac{\exp\Big(2(x+\sqrt{x^2+4y})z\Big)-\exp\Big(2(x-\sqrt{x^2+4y})z\Big)}{\sqrt{x^2+4y}}\times\frac{2\exp(\sqrt{x^2+4y}z)}{\exp(2\sqrt{x^2+4y}z)+1}\\
&=\frac{2\exp\Big(2(x-\sqrt{x^2+4y})z\Big)\exp(\sqrt{x^2+4y}z)}{\sqrt{x^2+4y}\Big(\exp(2\sqrt{x^2+4y}z)+1\Big)}\Big(\exp(4\sqrt{x^2+4y}z)-1\Big)\\
&=\frac{2}{\sqrt{x^2+4y}}\Big(\exp\Big((2x+\sqrt{x^2+4y})z\Big)+\exp\Big((2x-\sqrt{x^2+4y})z\Big)\Big)\\
&=\frac{2}{\sqrt{x^2+4y}}\sum_{n=0}^{\infty}\Big((2x+\sqrt{x^2+4y})^n-(2x-\sqrt{x^2+4y})^n\Big)\frac{z^n}{n!}.
\end{aligned}$$

By comparing the coefficients of $\frac{z^n}{n!}$, we get

$$\sum_{l=0}^{n}\binom{n}{l}2(4)^{n-l-1}\Big(\sqrt{x^2+4y}\Big)^{l+1}F_{n-l}(x,y)E_l = \Big(2x+\sqrt{x^2+4y}\Big)^n-\Big(2x-\sqrt{x^2+4y}\Big)^n.$$

For $l \geq 0$, we have $E_{2l+1}=0$, which gives the result.

Similarly, using the change of variables $z=\frac{\sqrt{x^2+4y}}{2}z$ in (2), we obtain

$$\sum_{n=0}^{\infty}\Big(\frac{\sqrt{x^2+4y}}{2}\Big)^n E_n\frac{z^n}{n!} = \frac{2\exp(\frac{\sqrt{x^2+4y}}{2}z)}{\exp(\sqrt{x^2+4y}z)+1}, \tag{23}$$

and multiplying (23) by the exponential generating function for $(L_n(x,y))_{n\in\mathbb{N}}$, we get

$$\Big(\sum_{n=0}^{\infty}L_n(x,y)\frac{z^n}{n!}\Big)\Big(\sum_{l=0}^{\infty}\Big(\frac{\sqrt{x^2+4y}}{2}\Big)^l E_l\frac{z^l}{l!}\Big) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\Big(\frac{\sqrt{x^2+4y}}{2}\Big)^l L_{n-l}(x,y)E_l\frac{z^n}{n!},$$

and

$$\begin{aligned}
\Big(\exp\Big(\frac{x+\sqrt{x^2+4y}}{2}z\Big)+\exp\Big(\frac{x-\sqrt{x^2+4y}}{2}z\Big)\Big)\times\frac{2\exp\Big(\frac{\sqrt{x^2+4y}}{2}z\Big)}{\exp(\sqrt{x^2+4y}z)+1} &= 2\exp(\frac{x}{2}z)\\
&= 2\sum_{n=0}^{\infty}\Big(\frac{x}{2}\Big)^n\frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain

$$\sum_{l=0}^{n}\binom{n}{l}\Big(\frac{\sqrt{x^2+4y}}{2}\Big)^{l}L_{n-l}(x,y)E_l = 2\Big(\frac{x}{2}\Big)^{n},$$

For $k \geq 0$, we have $E_{2k+1} = 0$, and the result follows. □

- Putting $y = 1$ in Theorem 6, we obtain the following identities linking Euler numbers to Fibonacci and Lucas polynomials

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2(4)^{n-2k-1}(\sqrt{x^2+4})^{2l+1}F_{n-2l}(x)E_{2l} = \Big(2x+\sqrt{x^2+4}\Big)^{n} - \Big(2x-\sqrt{x^2+4}\Big)^{n}, \tag{24}$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\Big(\frac{x^2+4}{4}\Big)^{l}L_{n-2l}(x)E_{2l} = 2\Big(\frac{x}{2}\Big)^{n}. \tag{25}$$

- Putting $y = x = 1$ in Theorem 6 or $x = 1$ in the relationships (24) and (25), we obtain the following identities linking Euler numbers to Fibonacci and Lucas numbers

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2(4)^{n-2l-1}(\sqrt{5})^{2l+1}F_{n-2l}E_{2l} = (2+\sqrt{5})^{n} - (2-\sqrt{5})^{n},$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\Big(\frac{5}{4}\Big)^{l}L_{n-2l}E_{2l} = 2^{1-n}. \tag{26}$$

Note that Eq. (26) is Byrd's result (3).

Now we present the analogue results for the relation between Bernoulli numbers with bivariate Pell and bivariate Pell-Lucas polynomials.

Theorem 7. *For every positive integer n, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\Big(2\sqrt{x^2y^2+y}\Big)^{l}P_{n-l}(x,y)B_l = n\Big(xy-\sqrt{x^2y^2+y}\Big)^{n-1}, \tag{27}$$

$$\sum_{l=0}^{n}\binom{n}{l}\Big(2\sqrt{x^2y^2+y}\Big)^{l}(2^l-1)Q_{n-l}(x,y)B_l = -2\sqrt{x^2y^2+y}\,n\Big(xy-\sqrt{x^2y^2+y}\Big)^{n-1}. \tag{28}$$

Proof. The proof is similar to the proof of Theorem 4, by using the change of variables $z = 2\sqrt{x^2y^2+y}z$ in (1) to prove Eq. (27) and $z = 4\sqrt{x^2y^2+y}z$ in (1) to prove the relationship (28). □

Theorem 8. *For every positive integer n, we have*

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} 4^l (x^2y^2+y)^l P_{n-2l}(x,y) B_{2l} = \frac{nQ_{n-1}(x,y)}{2},$$

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} 4^l (x^2y^2+y)^l (2^{2l}-1) Q_{n-2l}(x,y) B_{2l} = 2n(x^2y^2+y)P_{n-1}(x,y).$$

Proof. We have

$$\sum_{l=0}^{n} \binom{n}{l} \Big(2\sqrt{x^2y^2+y}\Big)^l P_{n-l}(x,y) B_l = n\Big(xy-\sqrt{x^2y^2+y}\Big)^{n-1},$$

which gives

$$\sum_{l=0}^{n} \binom{n}{l} \Big(2\sqrt{x^2y^2+y}\Big)^{n-l} P_{n-l}(x,y) B_l = \frac{n\Big(Q_{n-1}(x,y) - 2\sqrt{x^2y^2+y}P_{n-1}(x,y)\Big)}{2},$$

with $B_0 = 1$ and $B_1 = -\frac{1}{2}$.

For $l \geq 0$, we have $B_{2l+1} = 0$. Then we get

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} 4^l (x^2y^2+y)^l P_{n-2l}(x,y) B_{2l} = \frac{nQ_{n-1}(x,y)}{2}.$$

Similarly, from (28), we get

$$\sum_{l=0}^{n} \binom{n}{l} 4^l (x^2y^2+y)^l (2^l-1) Q_{n-l}(x,y) B_l = \frac{-2\sqrt{x^2y^2+y}n\Big(Q_{n-1}(x,y) - 2\sqrt{x^2y^2+y}P_{n-1}(x,y)\Big)}{2},$$

with $B_0 = 1$ and $B_1 = -\frac{1}{2}$.

For $l \geq 0$, we have $B_{2l+1} = 0$. Then we obtain

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} 4^l (x^2y^2+y)^l (2^{2l}-1) Q_{n-2l}(x,y) B_{2l} = \frac{(x^2y^2+y)nF_{n-1}(x,y)}{2},$$

and the result follows. □

- By putting $y = 1$ in Theorem 7 and Theorem 8, we start with the following results involving Bernoulli numbers and Pell, Pell-Lucas polynomials:

$$\sum_{l=0}^{n}\binom{n}{l}\left(2\sqrt{x^2+1}\right)^{l}P_{n-l}(x)B_l = n(x-\sqrt{x^2+1})^{n-1}, \tag{29}$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}4^l(x^2+1)^lP_{n-2l}(x)B_{2l} = \frac{nQ_{n-1}(x)}{2}, \tag{30}$$

$$\sum_{l=0}^{n}\binom{n}{l}\left(2\sqrt{x^2+1}\right)^{l}(2^l-1)Q_{n-l}(x)B_l = -2n\sqrt{x^2+1}(x-\sqrt{x^2+1})^{n-1}, \tag{31}$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}4^l(x^2+1)^l(2^{2l}-1)Q_{n-2l}(x)B_{2l} = 2n(x^2+1)P_{n-1}(x). \tag{32}$$

- By setting $x=y=1$ in Theorem 7 and Theorem 8 or $x=1$ in the relationships (29), (30), (31) and (32) we deduce the following Bernoulli-Pell and the Bernoulli-Pell-Lucas identities:

$$\sum_{l=0}^{n}\binom{n}{l}(2\sqrt{2})^lP_{n-l}B_l = n(1-\sqrt{2})^{n-1},$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}8^lP_{n-2l}B_{2l} = \frac{nQ_{n-1}}{2},$$

$$\sum_{l=0}^{n}\binom{n}{l}\left(2\sqrt{2}\right)^{l}(2^l-1)Q_{n-l}B_l = -2n\sqrt{2}(1-\sqrt{2})^{n-1},$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}8^l(2^{2l}-1)Q_{n-2l}B_{2l} = 2n(1+1)P_{n-1}.$$

Theorem 9. *For every positive integer n, we have*

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2^{n-2l}\left(\sqrt{x^2y^2+y}\right)^{2l+1}P_{n-2l}(x,y)E_{2l} = \left(2xy+\sqrt{x^2y^2+y}\right)^{n} - \left(2xy-\sqrt{x^2y^2+y}\right)^{n}, \tag{33}$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(x^2y^2+y\right)^{l}Q_{n-2l}(x,y)E_{2l} = 2(xy)^n. \tag{34}$$

Proof. The proof is similar to the proof of Theorem 6, by using the change of variables $z=2\sqrt{x^2y^2+y}z$ in (2) to prove (33) and $z=4\sqrt{x^2y^2+y}z$ in (2) to prove (34). □

- By putting $y=1$ in Theorem 9, we obtain some relations between the Euler numbers and the Pell and Pell-Lucas polynomials:

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2^{n-2l}\left(\sqrt{x^2+1}\right)^{2l+1}P_{n-2l}(x)E_{2l}=\left(2x+\sqrt{x^2+1}\right)^n-\left(2x-\sqrt{x^2+1}\right)^n, \tag{35}$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\left(x^2+1\right)^{l}Q_{n-2l}(x)E_{2l}=2x^n. \tag{36}$$

- By putting $x=y=1$ in Theorem 9 or $x=1$ in the relationships (35) and (36), we can state the following Pell-Euler and Pell-Lucas-Euler identities:

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}(\sqrt{2})2^{n-l}P_{n-2l}E_{2l}=\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n,$$

$$\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}2^{l}P_{n-2l}E_{2l}=2.$$

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