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Edge Covers of Caterpillars, Cycles with Pendants, and Spider Graphs

Feryal Alayont and Evan Henning
Department of Mathematics
Grand Valley State University
Allendale, MI 49401
USA
alayontf@gvsu.edu henninge@mail.gvsu.edu

Abstract

An edge cover of a simple graph is a subset of the edges so that each vertex is incident with at least one edge in the subset. The edge cover polynomial of a graph is the generating polynomial of the number of edge covers of the graph with k edges. The number of edge covers of path graphs form the Fibonacci sequence, while those of cycle graphs form the Lucas numbers. In this paper, we first provide some known and new general results on edge covers and edge cover polynomials. We then apply these results to find the number of edge covers and the edge cover polynomials of caterpillar graphs, cycles with pendants, and spider graphs, which are generalizations of star graphs. The number of edge covers of all these families can be expressed in terms of Fibonacci numbers. We generate new sequences not in the *On-Line Encyclopedia of Integer Sequences* using these edge covers and provide new combinatorial interpretations of some known sequences.

1 Introduction

A *(simple) graph* G is an ordered pair (V, E) consisting of a finite set of vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$ and a finite set of edges $E(G) = \{e_1, e_2, \ldots, e_m\}$, where each edge is a set of two distinct vertices. All the graphs we consider are simple, and we will refer to them as graphs for brevity. The *degree* of a vertex is the number of edges that are *incident* with that vertex, i.e., the number of edges that meet at that vertex. An *isolated vertex* has degree 0, and a *pendant* (vertex) has degree 1. An *edge cover* of a graph G is a subset of $E(G)$ such that each vertex of G is incident with at least one edge in this subset. In Figure 1, the edges e_3, e_5, e_6, e_7 (highlighted in red and dashed) form an edge cover. The remaining edges, e_1, e_2, e_4, e_8 also form an edge cover. No graph with at least one isolated vertex has an edge cover. In a graph with no isolated vertices, the set of all edges is an edge cover.

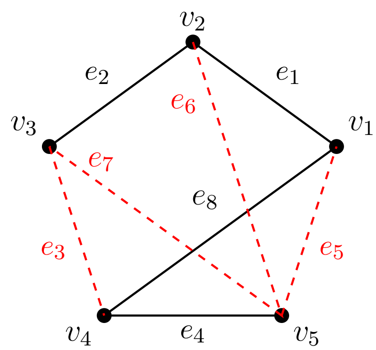


Figure 1: A graph and two edge covers (red edges and black edges).

Although research on edge covers often considers minimum edge covers in optimization applications, in this paper, we focus on counting the edge covers. This problem has applications in counting elements at a given location between two given elements in Hausdorff metric geometry [3]. Honigs [6] showed that the number of edge covers of a bipartite graph representing adjacencies in a finite configuration $[A, B]$ (i.e., two finite sets A, B in Hausdorff metric geometry with specific Hausdorff metric conditions) is equal to the number of elements at each location between A and B. Counting edge covers can also be used to estimate the importance of a line in communication networks [5].

Let $\#G$ denote the total number of edge covers of G. It can be shown [6] by induction that for path graphs P_n with n vertices, $\#P_n = F_{n-1}$, where F_n are the Fibonacci numbers with initial values $F_0 = 0, F_1 = 1$, which is the sequence A000045. By considering cases based on whether there is a fixed edge or not [6], one can also show that for cycle graphs C_n with n vertices $\#C_n = L_n$ for $n \geq 3$, where L_n are the Lucas numbers with initial values $L_0 = 2, L_1 = 1$. The edge cover polynomial, $E(G, x)$, of a graph G is defined as

$$E(G, x) = \sum_{k=1}^{|E(G)|} e(G, k)x^k \,,$$

where $e(G, k)$ is the number of edge covers of G with k edges. For example, P_6 has one edge cover with five edges, three with four edges, and one with three edges, as shown in Figure 2. Hence, the edge cover polynomial of P_6 is

$$E(P_6, x) = x^3 + 3x^4 + x^5 \,.$$

More generally, we have [1]

$$E(P_n, x) = \sum_{i=1}^{n-1} \binom{i-1}{n-i-1} x^i \,.$$

The coefficients of the edge cover polynomials (arranged in descending powers of x) of path graphs form the entry A011973 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [9].

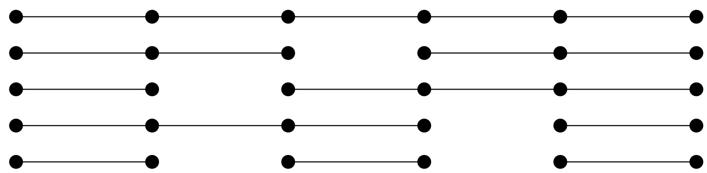


Figure 2: Edge covers of P_6.

This paper focuses on edge covers of path and cycle graphs with pendants and spider graphs. A *spider graph* is a tree (i.e., a connected graph with no cycles) with one vertex of degree greater than or equal to 3 and the rest of degree less than or equal to 2. We calculate the number of edge covers of these graph families and present formulas for their edge cover polynomials. We obtain new sequences not found in the OEIS [9] and provide new combinatorial interpretations of some known sequences.

2 Some general properties of edge covers

Some known results about the total number of edge covers and edge cover polynomials are

Theorem 1.

(i) Given a graph G with connected components $G_1, G_2, \ldots, G_k$, we have [1]

$$\#G = \prod_{i=1}^{k} \#G_i \quad \text{and} \quad E(G,x) = \prod_{i=1}^{k} E(G_i, x)\,.$$

(ii) Given a graph G with edge uv, we have

$$E(G,x) = (x+1)E(G-uv,x) + x\big(E(G-u,x) + E(G-v,x) + E(G-\{u,v\},x)\big),$$

where $G-uv$ represents the graph with edge uv removed, and $G-u, G-v$, and $G-\{u,v\}$ represent the graphs with the vertex u; vertex v; and both u,v removed, respectively [1, 6].

(iii) Given a graph G with a pendant vertex v whose neighbor is u, we have [1]

$$E(G, x) = xE(G - v, x) + xE(G - \{u, v\}, x)\,.$$

(iv) Suppose $G = G_1 \cdot uv \cdot G_2$ is formed by combining two graphs G_1 and G_2 by identifying the vertex $u \in G_1$ with the vertex $v \in G_2$. Then [4]

$$E(G, x) = E(G_1, x)E(G_2, x) + E(G_1 - u, x)E(G_2, x) + E(G_1, x)E(G_2 - v, x)\,.$$

We now prove two new results, the first of which is about 2-sums of graphs, a particular case of k-sums [2]. Given two graphs, G_1 and G_2, and an edge from each graph, the *2-sum* $G_1 \oplus_2 G_2$ is obtained by identifying the two edges. The 2-sum depends on which edges in the graphs are chosen and how they are identified. An example of the 2-sum of two graphs is shown in Figure 3.

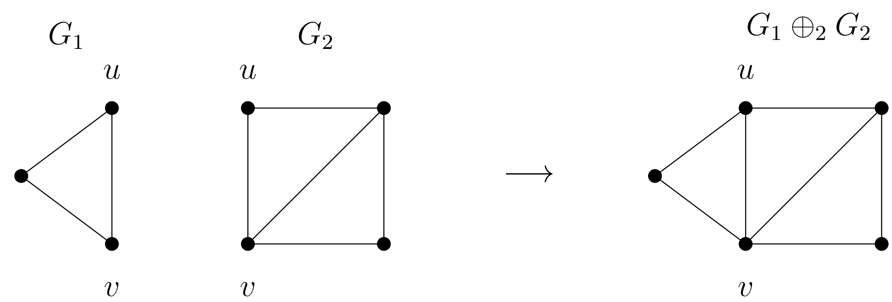


Figure 3: The 2-sum of two graphs.

Lemma 2 (2-sum Lemma)**.** *Let $G = G_1 \oplus_2 G_2$ with the common identified edge labeled uv in G and G_i. Then the set of edge covers of G that include the edge uv is in bijection with the cross product of the sets of edge covers of $G_1' = (G_1 \cup \{uw_1, vw_2\}) - uv$ and $G_2' = (G_2 \cup \{uy_1, vy_2\}) - uv$, where w_1, w_2 (respectively y_1, y_2) are pendant vertices added to G_1 (respectively G_2).*

Proof. To clarify the definitions, Figure 4 shows how G_1' and G_2' would look like using the example of $G = G_1 \oplus_2 G_2$ shown in Figure 3.

To prove the lemma, we define a map $S \to (S_1, S_2)$ where S is an edge cover of G that includes uv, and S_i is an edge cover of G_i' for $i = 1, 2$, and show that the map is a bijection. Let S be an edge cover of G that includes uv. Because the edges of G besides uv belong either to G_1 or G_2 (but not both), we can label the elements of S as

$$S = \{uv, e_1, \ldots, e_r, e_{r+1}, \ldots, e_s\},$$

where $e_1, \ldots, e_r \in G_1$ and $e_{r+1}, \ldots, e_s \in G_2$. Let $S_1 = \{uw_1, vw_2, e_1, \ldots, e_r\}$ and $S_2 = \{uy_1, vy_2, e_{r+1}, \ldots, e_s\}$. We will show that each S_i is an edge cover of G_i'.

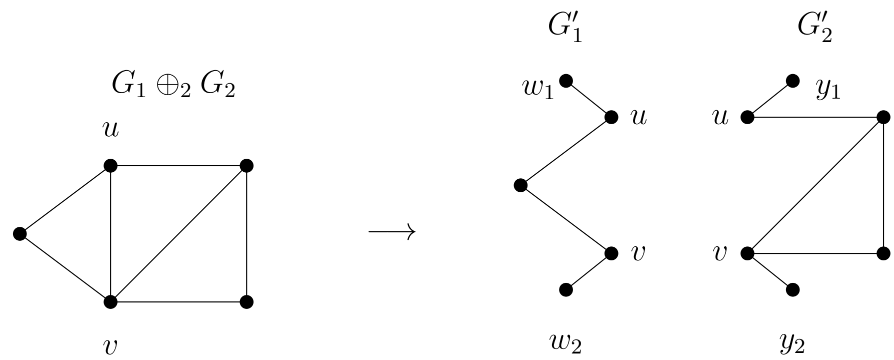


Figure 4: An example of constructing G_1' and G_2' from $G = G_1 \oplus_2 G_2$.

If $x \in G_i$ and $x \neq u, v, w_1, w_2, y_1, y_2$, then x is also in G and S has an edge that is incident with x, which is also in the respective S_i. The pendant vertices w_1, w_2, y_1, and y_2, and vertices u, v are endpoints of the newly added edges uw_1, vw_2, uy_1, and vy_2. Therefore, S_i is an edge cover of G_i.

The map $S \to (S_1, S_2)$ is one-to-one because if two edge covers agree on the edges restricted to G_1' and G_2', they must agree on all but uv. However, the map is defined only on edge covers containing uv. Therefore, the edge covers must agree on all of G.

The map is also onto. Let S_1 and S_2 be edge covers of G_1' and G_2', respectively. Because uw_1 and uw_2 are both pendant edges in G_1', we know they will be in S_1. The remaining edges in S_1 are edges that are in G_1. Therefore, we can write $S_1 = \{uw_1, vw_2, e_1, \ldots, e_r\}$ for some e_i in G_1. Similarly, $S_2 = \{uy_1, vy_2, e_{r+1}, \ldots, e_s\}$ for some e_i in G_2. Define

$$S = S_1 \cup S_2 \cup \{uv\} - \{uw_1, vw_2, uy_1, vy_2\} = \{uv, e_1, \ldots, e_r, e_{r+1}, \ldots, e_s\}.$$

It is easy to show that S is an edge cover of G and maps to (S_1, S_2).

Since the map is one-to-one and onto, it is a bijection, as claimed. □

A direct consequence of the 2-sum lemma is

Corollary 3. *Let $\#(G; uv)$ denote the number of edge covers of G that include the edge uv. Suppose $G = G_1 \oplus_2 G_2$, the common identified edge is labeled uv in G, and G_i and G_i' are defined as in the 2-sum lemma. Then*

$$\#(G; uv) = \#G_1' \cdot \#G_2'.$$

The 2-sum lemma implies another direct result regarding the edge cover polynomials. Let $E(G; uv, x)$ be the edge cover polynomial of the edge covers of G that include the edge uv. By the 2-sum lemma, since every edge cover of G that includes uv corresponds to an edge cover of the graph with components G_1' and G_2', the numbers of the edge covers will be the same. However, when defining the correspondence of edge covers, the edge uv turns into four edges: uw_1, vw_2, uy_1, and vy_2. Therefore, in the polynomial, we have an extra x^3 factor on the G_i' side. Thus, we have the following edge cover polynomial result:

Corollary 4. *Suppose $G = G_1 \oplus_2 G_2$, the common identified edge is labeled uv in G, and G_i and G_i' are defined as in the 2-sum lemma. Then*

$$E(G; uv, x) = \frac{E(G_1') \cdot E(G_2')}{x^3}\,.$$

Another similar result, the pendant lemma, applies to graphs with a pendant vertex.

Lemma 5 (Pendant Lemma)**.** *Let G be a graph with a pendant vertex v. Let u be the neighbor of v, and w_i, $1 \le i \le k$, be the neighbors of u besides v. Then the set of edge covers of G is in bijection with the set of edge covers of*

$$G' = (G - \{u, v\}) \cup \{w_1u_1, u_1v_1, w_2u_2, u_2v_2, \ldots, w_ku_k, u_kv_k\},$$

where the u_i, v_i are added new vertices to G.

Proof. A representative example of the G and G' pair is shown in Figure 5.

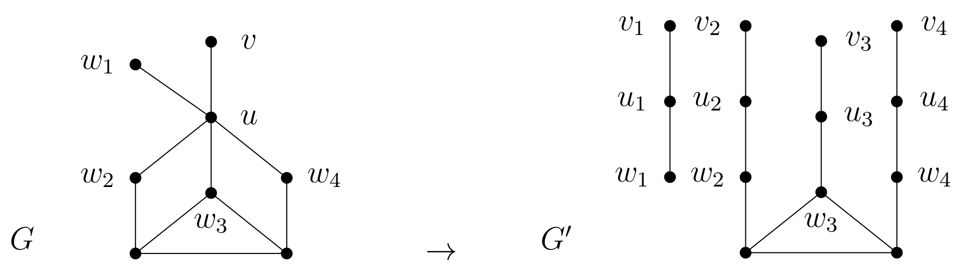


Figure 5: An example of G' construction.

The graph G' is obtained by replacing the vertices u, v with multiple copies, one for each edge incident with u besides uv. Every pendant edge must be included in every edge cover of G. Therefore, the vertices u and v are already covered, allowing the remaining edges incident with u to be flexible. Hence, we can represent the pendant instead as an added edge to each edge incident with u. More precisely, an edge cover $\{e_1, e_2, \ldots, e_r, uv\}$ of G corresponds to the edge cover $\{e_1, e_2, \ldots, e_r, u_1v_1, u_2v_2, \ldots, u_kv_k\}$ of G'. In this correspondence, by abuse of notation, for an edge $e_i = w_ju$ that might be in the edge cover of G, we still use e_i to represent the corresponding edge w_ju_j in G'. □

Corollary 6. *Let G be a graph with pendant v, whose neighbor is u. Suppose* $\deg u = k$*. Let G' be as in the pendant lemma. Then $\#G = \#G'$ and*

$$E(G, x) = \frac{1}{x^{k-2}} E(G', x)\,.$$

Proof. The first claim directly follows from the pendant lemma. In the correspondence between edge covers of G and G', we replace all of the k edges incident with u with $k-1$ non-pendant and $k-1$ pendant edges. Hence, the edge cover polynomial of G' has an extra x^{k-2} factor. □

3 Paths and cycles with pendants

The two lemmas from the previous section can be applied to path and cycle graphs with one or more pendants to express their edge cover numbers in terms of Fibonacci numbers. By repeatedly splitting the graph at the pendant edge using the pendant lemma, each path or cycle graph with pendants becomes a union of disconnected path graphs.

3.1 Caterpillars

A *caterpillar graph* is a tree in which every vertex is either on the longest path or exactly one graph edge away from this path. We will refer to this longest path as the *central stalk* of the caterpillar.

Theorem 7. *A caterpillar graph G with central stalk P_n and $k_1, k_2, \ldots, k_r$ pendants at inner vertices (counting from one pendant vertex of P_n) located at vertices $m_1, m_2, \ldots, m_r$, respectively, has a total of*

$$\#G = F_{m_1} F_{m_2-m_1+2} \cdots F_{m_r-m_{r-1}+2} F_{n-m_r+1}$$

edge covers. Its edge cover polynomial is

$$E(G,x) = x^{k_1+k_2+\cdots+k_r-2r} E(P_{m_1+1},x) E(P_{n-m_r+2},x) \prod_{i=2}^{r} E(P_{m_i-m_{i-1}+3},x)\,.$$

Proof. Let G be a caterpillar graph with a central stalk P_n and additional pendants attached to vertices (counting from one end) $m_1, m_2, \ldots, m_r$. Without loss of generality, we can assume that exactly one pendant is attached to each spot since an edge cover includes all pendants. Additional pendants only change the edge cover polynomial by a factor of $x^{k_1+k_2+\cdots+k_r-r}$.

The proof proceeds by induction on the number of added pendants. We demonstrate the idea using the example caterpillar graph shown in Figure 6.

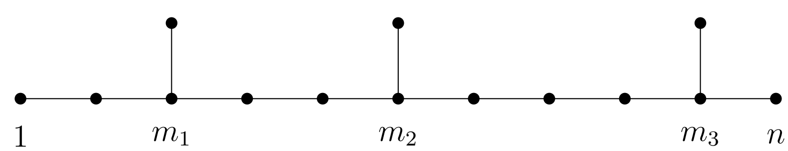


Figure 6: A caterpillar graph with three pendants attached to the central stalk.

By applying the pendant lemma at vertex m_1, we obtain a disjoint union of P_{m_1+1} and a caterpillar with one fewer pendant. In this specific example, after applying the pendant lemma at $m_1 = 3$, we obtain the path P_4 and a caterpillar with two added pendants attached at $m_2 - m_1 + 2 = 5$ and $m_3 - m_1 + 2 = 9$ on a central stalk $P_{n-m_1+2} = P_{10}$. This new disconnected graph is shown in Figure 7.

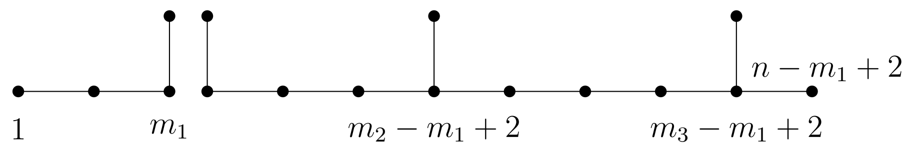


Figure 7: Caterpillar in Figure 6 with one pendant section separated.

Therefore, by induction on the number of pendants attached, the number of edge covers is the product of F_{m_1} and $F_{m_2-m_1+2} \cdot F_{m_3-m_2+2} \cdot F_{n-m_3+1}$. Therefore, the first claim for the total number of edge covers follows.

Regarding the edge cover polynomials, each time we split off a path graph, we need to multiply by $1/x$ due to Corollary 6 since the degree of the pendant vertex is three. Thus, for r pendants, we have an extra x^{-r} factor. □

We can generate number sequences out of caterpillars by considering evenly spaced pendants in caterpillars [8]. For simplicity, at every vertex of the caterpillar, assume there is at most one pendant. Let $\mathrm{Cat}_{k,n}$ denote a caterpillar with n pendants where $k-1$ is the number of edges between consecutive pendants, and before the first and after the last pendants. Using the results above, we have

$$\#\,\mathrm{Cat}_{k,n} = F_k^2 F_{k+1}^{n-1}\,.$$

Fixing k or n generates a sequence of numbers. For example, $k=3$ generates the sequence A003946, while $n=2$ generates A066258 in OEIS [9].

3.2 Cycle graphs with pendants

When pendants are attached to cycle graphs, we can similarly use the pendant lemma at each pendant to split the graph into paths.

Theorem 8. *Suppose G is obtained by attaching pendants to a cycle graph C_n with vertices consecutively labeled $1, 2, \ldots, n$ and $k_0, k_1, k_2, \ldots, k_r$ pendants attached to vertices (counting clockwise starting at 1) $1, m_1, m_2, \ldots, m_r$, respectively. Then G has a total of*

$$\#G = F_{m_1+1} F_{m_2-m_1+2} \cdots F_{m_r-m_{r-1}+2} F_{n-m_r+3}$$

edge covers. Its edge cover polynomial is

$$E(G,x) = x^{k_0+k_1+k_2+\cdots+k_r-2(r+1)} E(P_{m_1+2},x) E(P_{n-m_r+4},x) \prod_{i=2}^{r} E(P_{m_i-m_{i-1}+3},x)\,.$$

Proof. A representative example labeled cycle with pendants is shown in Figure 8.

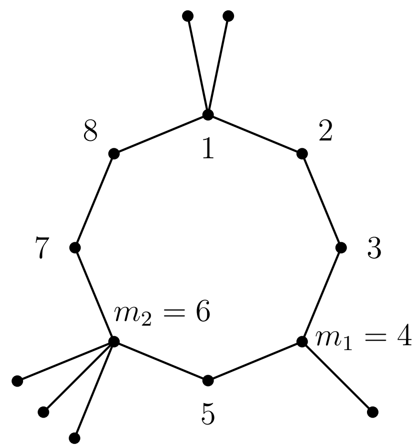


Figure 8: A cycle with $k_0 = 2$ pendants at vertex 1, $k_1 = 1$ pendant at $m_1 = 4$, and $k_2 = 3$ pendants at $m_2 = 6$.

Let G be a cycle with n vertices with k_0 pendants attached to vertex 1, k_1 attached to m_1, …, and k_r attached to m_r. As before, without loss of generality, assume there is one pendant at each pendant location. We first apply the pendant lemma at vertex 1 to turn the cycle C_n with pendants at locations $1, m_1, m_2, \ldots, m_r$ into a path P_{n+3} with pendants at locations $m_1 + 1, m_2 + 1, \ldots, m_r + 1$. The example graph given in Figure 8 will turn into the caterpillar in Figure 9 if we first remove the extra pendants at each vertex and apply the pendant lemma. The two pendants in the figure without labels are the added pendants after using the lemma.

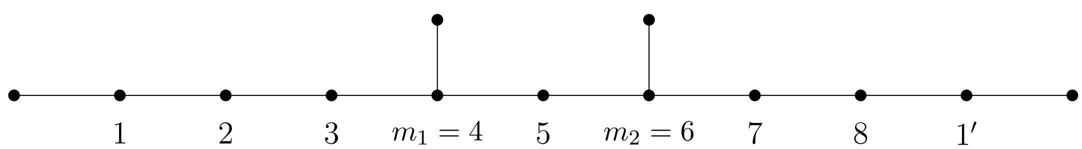


Figure 9: The caterpillar obtained after applying the pendant lemma.

Once the graph becomes a caterpillar, we apply Theorem 7 on caterpillars to obtain the result. □

3.2.1 Evenly spaced pendants in cycle graphs

We now consider evenly spaced single pendants on cycles to generate sequences. We denote a cycle with evenly spaced single pendants as $\mathrm{PC}_{k,n}$, where k is the number of edges between consecutive pendants and n is the number of pendants on the graph. As a result, $\mathrm{PC}_{k,n}$ has kn vertices on the cycle. After applying the Theorem 8 in this case, we obtain the following theorem.

Theorem 9. *Given a cycle* $\mathrm{PC}_{k,n}$ *with evenly spaced pendants, we have*

$$\#\,\mathrm{PC}_{k,n} = (F_{k+2})^n \text{ and } E(\mathrm{PC}_{k,n}, x) = \frac{E(P_{k+3}, x)^n}{x^n}.$$

To generate sequences, we first fix k, the spacing between the consecutive pendants, and change n. First, let $k = 1$. The graphs $\mathrm{PC}_{1,n}$ are cycles with a pendant on each vertex. These turn out to be rather uninteresting. By Theorem 9, we have $\#\,\mathrm{PC}_{1,n} = 2^n$ and $E(\mathrm{PC}_{1,n}, x) = \frac{(x^3+x^2)^n}{x^n} = x^n(x+1)^n$. The triangle formed of coefficients (arranged in descending powers of x) results in Pascal's triangle.

For $k = 2$, the graphs $\mathrm{PC}_{2,n}$ are cycles with pendants on every other vertex. The first few examples of these graphs are shown in Figure 10.

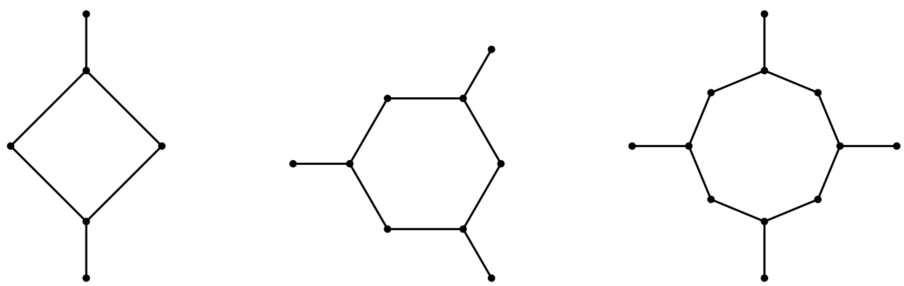


Figure 10: From left to right, $\mathrm{PC}_{2,2}, \mathrm{PC}_{2,3}$, and $\mathrm{PC}_{2,4}$.

By Theorem 9, we have $\#\,\mathrm{PC}_{2,n} = 3^n$ and $E(\mathrm{PC}_{2,n}, x) = \frac{(x^4+2x^3)^n}{x^n} = x^{2n}(x+2)^n$. The coefficients of the polynomials (arranged in descending powers of x) form the sequence A013609, described as the sequence of coefficients of $(1+2x)^n$.

When $k = 3$, the first few graphs are shown in Figure 11.

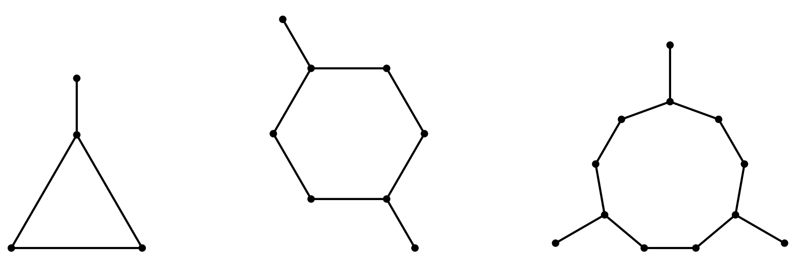


Figure 11: From left to right, $\mathrm{PC}_{3,1}, \mathrm{PC}_{3,2}$, and $\mathrm{PC}_{3,3}$.

By Theorem 9, we have $\#\,\mathrm{PC}_{3,n} = 5^n$ and

$$E(\mathrm{PC}_{3,n}, x) = \frac{(x^5+3x^4+x^3)^n}{x^n} = x^{2n}(x^2+3x+1)^n.$$

The sequence formed by these polynomial coefficients (arranged in descending powers of x) is the sequence A272866, a triangle made up of Gegenbauer polynomial values, specifically that of $C_k^{(-n)}\left(-\frac{3}{2}\right)$ for row n. The correspondence follows from the generating function definition of the Gegenbauer polynomials [7], which is

$$(1-2xt+t^2)^{-\nu} = \sum_{k=0}^{\infty} C_k^{(\nu)}(x)\, t^k\,.$$

Now let us consider fixing n, the number of pendants. The first case is when $n = 1$, $\mathrm{PC}_{k,1}$. These are cycles with one single pendant. We find $\#\,\mathrm{PC}_{k,1} = F_{k+2}$ and $E(\mathrm{PC}_{k,1}, x) = \frac{E(P_{k+3},x)}{x}$. This is again an uninteresting case since we obtain the same edge cover numbers and edge cover polynomial coefficients as path graphs.

If we consider two evenly spaced pendants, $n = 2$, things become more interesting. The first few cases of these graphs are shown in Figure 12.

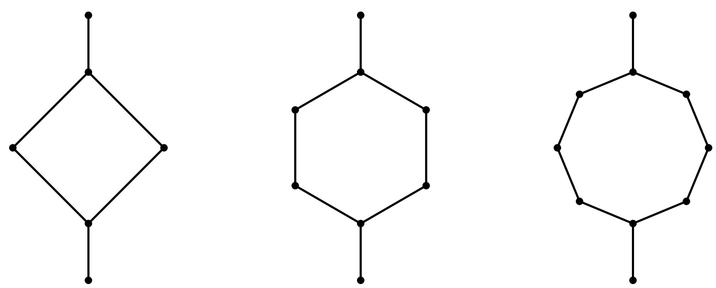


Figure 12: From left to right, $\mathrm{PC}_{2,2}, \mathrm{PC}_{3,2}$, and $\mathrm{PC}_{4,2}$.

We obtain $\#\,\mathrm{PC}_{k,2} = (F_{k+2})^2$ and the edge cover polynomials satisfy

$$E(\mathrm{PC}_{k,2}, x) = \frac{E(P_{k+3}, x)^2}{x^2}\,.$$

The triangle formed by these coefficients (arranged in descending powers of x) matches the sequence A123521 (starting at the row $n = 3$), which counts the tilings of a $2 \times n$ grid with some pieces being horizontal domino tiles and the remaining pieces being square tiles. Since the dominoes are allowed to be only horizontal, the top row and bottom row act independently of each other. We can show a direct correspondence between these tilings and the edge covers of $\mathrm{PC}_{k,2}$. Having two pendants k apart from each other allows each side of the cycle to turn into a path P_{k+3} behaving independently. For each side, an edge cover of the path corresponds to tiling the $1 \times (k+1)$ board with dominoes and squares in the following way. For each of the $k+1$ non-pendant vertices of P_{k+3}, we place a square if two edges on both sides cover that vertex and place a domino if only one edge covers it. Considering both sides of the cycle together, we obtain a $2 \times (k+1)$ board to cover. Missing an edge in this edge cover means we have a corresponding domino in the tiling. Therefore, an edge cover of $\mathrm{PC}_{k,2}$ with j missing edges corresponds to a tiling of $2 \times (k+1)$ with j dominoes total.

The first few cases of $\mathrm{PC}_{k,3}$ are shown in Figure 13.

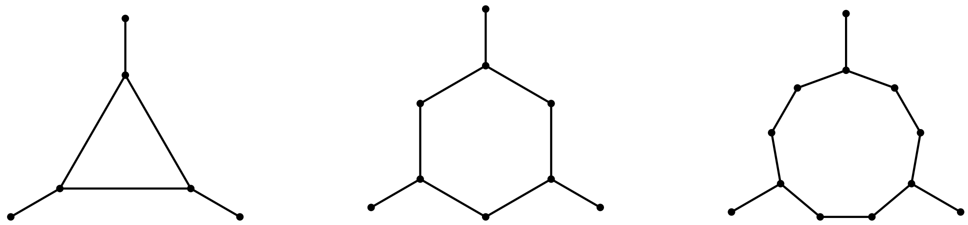


Figure 13: From left to right, $\mathrm{PC}_{1,3}, \mathrm{PC}_{2,3}$, and $\mathrm{PC}_{3,3}$.

We have $\# \mathrm{PC}_{k,3} = (F_{k+2})^3$, which is the sequence A056570. The edge cover polynomials satisfy

$$E(\mathrm{PC}_{k,3}, x) = \frac{E(P_{k+3}, x)^3}{x^3},$$

which form the triangle of coefficients in Table 1.

$k \setminus i$	0	1	2	3	4	5	6	7	8	9
1	1	3	3	1						
2	1	6	12	8						
3	1	9	30	45	30	9	1			
4	1	12	57	136	171	108	27			
5	1	15	93	308	588	651	399	123	18	1

Table 1: Coefficients of the edge cover polynomials of $\mathrm{PC}_{k,3}$.

In Table 1, the entry in row k and column i represents the number of edge covers of $\mathrm{PC}_{k,3}$ missing i edges from the total. In other words, the coefficients of the edge cover polynomials are arranged in descending powers of x.

4 Spider graphs

A *spider graph* is a tree with one vertex v with $\deg v \geq 3$ and the rest of the vertices with degree at most 2. Therefore, the vertex v is the center vertex, and all other vertices are on paths that start from vertex v. We refer to these paths as *branches*. We denote the non-center vertices by v_{ij}, where i is the branch number and j is the vertex number on that branch, with $j = 1$ being the closest to v and increasing outward. Let $S_{n_1,n_2,\ldots,n_k}$ denote a spider graph with k branches where the i-th branch has n_i vertices (not including v), shown in Figure 14.

Theorem 10. *Given a spider graph $S_{n_1,n_2,\ldots,n_k}$, we have*

$$\#S_{n_1,n_2,\ldots,n_k} = \prod_{i=1}^{k} F_{n_i+1} - \prod_{i=1}^{k} F_{n_i-1}$$

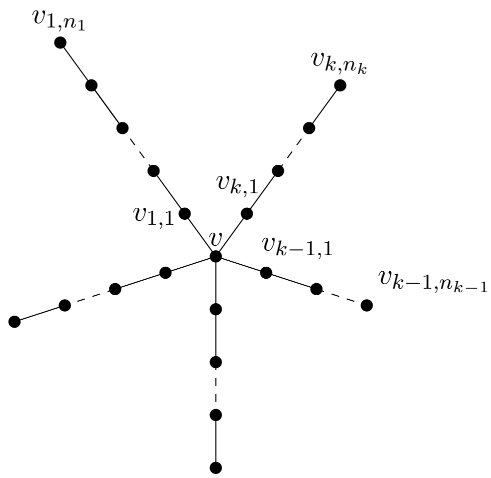


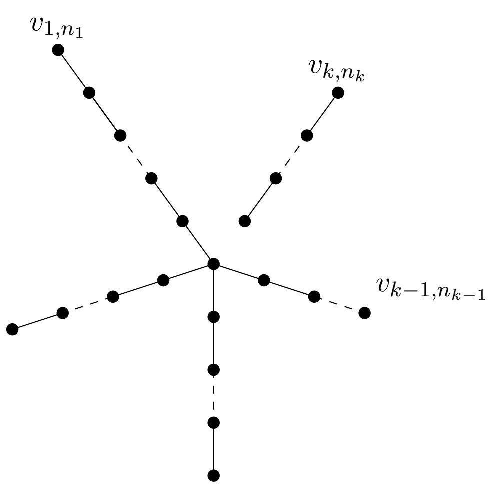
Figure 14: A spider graph $S_{n_1,n_2,\ldots,n_k}$.

and

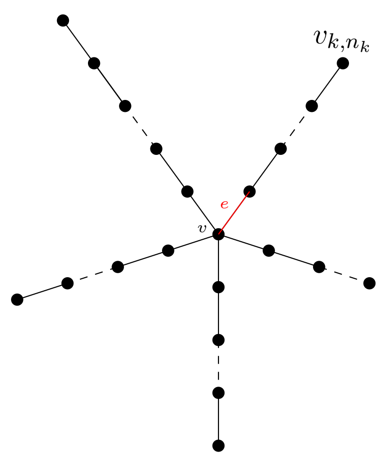
$$E(S_{n_1,n_2,\ldots,n_k},x) = E(P_{n_k},x)E(S_{n_1,n_2,\ldots,n_{k-1}},x) + \frac{1}{x^{k-1}}E(P_{1+n_k},x)\prod_{i=1}^{k-1}E(P_{2+n_i},x)\,.$$

Proof. We prove the first claim by induction on k, the number of branches. First, note that for $k=1$, $S_{n_1} = P_{n_1+1}$. This means that $\#S_{n_1} = F_{n_1} = F_{n_1+1} - F_{n_1-1}$. Therefore, the claim is true for $k=1$.

Consider now a spider with k branches. Let $e = vv_{k,1}$ be the edge connecting the last branch to the center. We count the edge covers of $\#S_{n_1,n_2,\ldots,n_k}$ using two disjoint cases: those that include e and those that do not, as in Figure 15.



(a) $S_{n_1,n_2,\ldots,n_k}$ without e.



(b) $S_{n_1,n_2,\ldots,n_k}$ with e.

Figure 15: Two cases of $S_{n_1,n_2,\ldots,n_k}$ edge covers, with or without e.

The edge covers without e are the same as those of two disconnected graphs, P_{n_k} and the spider graph $S_{n_1,n_2,\ldots,n_{k-1}}$. This gives us $\#P_{n_k} \cdot \#S_{n_1,n_2,\ldots,n_{k-1}}$ edge covers.

To find the number of edge covers with e, we first use the 2-sum lemma. We let the last branch be G_1, one of the graphs in the 2-sum, and the edge connecting that branch to the center vertex v be the identified edge in the 2-sum. Then, the spider $S_{n_1,n_2,\ldots,n_{k-1}}$ with one pendant edge at the center vertex is G_2, and the pendant edge is the identified edge in the 2-sum. By the 2-sum lemma, we find that the set of edge covers with e is in bijection with the edge covers of the disconnected graph consisting of P_{1+n_k}, a P_2, $S_{n_1,n_2,\ldots,n_{k-1}}$ with a pendant at the center vertex, and another P_2. We then apply the pendant lemma to this smaller spider graph with a pendant and split it into $k-1$ paths. This gives

$$\#(S_{n_1,n_2,\ldots,n_k}; e) = \#P_{1+n_k} \cdot \#P_2 \cdot \prod_{i=1}^{k-1} \#P_{2+n_i} \cdot \#P_2 \,.$$

Adding the two cases and using $\#P_n = F_{n-1}$, we have

$$\#S_{n_1,n_2,\ldots,n_k} = F_{n_k-1} \cdot \#S_{n_1,n_2,\ldots,n_{k-1}} + F_{n_k} \cdot \prod_{i=1}^{k-1} F_{n_i+1} \,.$$

If we substitute the inductive hypothesis into the above expression and simplify, we get

$$\begin{aligned}
\#S_{n_1,n_2,\ldots,n_k} &= F_{n_k-1}\left(\prod_{i=1}^{k-1} F_{n_i+1} - \prod_{i=1}^{k-1} F_{n_i-1}\right) + F_{n_k} \cdot \prod_{i=1}^{k-1} F_{n_i+1} \\
&= \left(F_{n_k-1} + F_{n_k}\right) \prod_{i=1}^{k-1} F_{n_i+1} - \prod_{i=1}^{k} F_{n_i-1} \\
&= \prod_{i=1}^{k} F_{n_i+1} - \prod_{i=1}^{k} F_{n_i-1} \,,
\end{aligned}$$

which is the claim. We can also think of this result heuristically as creating an edge cover of the spider by joining edge covers of the branches where the edge to the center vertex may or may not be there. These branch edge covers are in one-to-one correspondence with the edge covers of P_{2+n_i}. Therefore, all these multiplied together gives us $\prod_{i=1}^{k} F_{n_i+1}$ covers. We then remove the cases where each of the branches was missing the edge to the center vertex, which corresponds to each edge cover of the branch corresponding to one smaller path. Hence, we obtain a total of $\prod_{i=1}^{k} F_{n_i-1}$ covers to exclude.

Using the same cases, if we apply the 2-sum lemma and pendant lemma corollaries for the edge cover polynomials, specifically Corollaries 4 and 6, we obtain the edge cover polynomial recurrence relation given in the theorem statement. □

4.1 Spider graphs with equal-length branches

We obtain a particular case of the spider graphs when all branches are equal in length. We let S_{n^k} denote such a spider graph, where k is the number of branches and n is the number of vertices on each branch (except the center vertex). In this case, we have the following result:

Theorem 11. *Given a spider graph with $k \geq 1$ branches each of length $n \geq 1$, we have*

$$\#S_{n^k} = F_{n+1}^k - F_{n-1}^k \ \text{ and } \ E(S_{n^k}, x) = E(P_{n+1}, x) \sum_{i=0}^{k-1} \frac{E(P_n, x)^{k-1-i} E(P_{n+2}, x)^i}{x^i}.$$

Proof. By Theorem 10, the first claim follows directly, and we have

$$E(S_{n^k}, x) = E(P_n, x) \cdot E(S_{n^{k-1}}, x) + \frac{E(P_{n+1}, x) \cdot E(P_{n+2}, x)^{k-1}}{x^{k-1}}. \tag{1}$$

For $k = 1$, we have $S_{n^1} = P_{n+1}$, and the edge cover polynomial expression in the second claim holds in this case.

More generally, if $E(S_{n^k}, x)$ is as claimed, then again, by equation (1),

$$\begin{aligned} E(S_{n^{k+1}}, x) &= E(P_n, x) E(P_{n+1}, x) \sum_{i=0}^{k-1} \frac{E(P_n, x)^{k-1-i} E(P_{n+2}, x)^i}{x^i} + \frac{E(P_{n+1}, x) \cdot E(P_{n+2}, x)^k}{x^k} \\ &= E(P_{n+1}, x) \sum_{i=0}^{k} \frac{E(P_n, x)^{k-i} E(P_{n+2}, x)^i}{x^i}. \end{aligned}$$

Therefore, the claim holds for all k. □

An even tidier expression for the edge cover polynomials of the spider graphs can be obtained using Theorem 1(iv).

Theorem 12. *Given a spider graph with $k \geq 1$ branches each of length $n \geq 1$, we have*

$$E(S_{n^k}, x) = (E(P_{n+1}, x) + E(P_n, x))^k - E(P_n, x)^k.$$

Proof. Since $S_{n^1} = P_{n+1}$, we have

$$E(S_{n^1}, x) = (E(P_{n+1}, x) + E(P_n, x))^1 - E(P_n, x)^1.$$

Now assume the claim is true for k. Recall that $G_1 \cdot uv \cdot G_2$ is formed by combining the graphs G_1 and G_2 by identifying the vertex $u \in G_1$ with the vertex $v \in G_2$. Theorem 1(iv) provides a formula for the edge cover polynomial of $G = G_1 \cdot uv \cdot G_2$ in terms of those of G_1, G_2, and their subgraphs. Spider graph $S_{n^{k+1}}$ is $P_{n+1} \cdot uv \cdot S_{n^k}$, where u is a pendant of P_{n+1} and v is the center of S_{n^k}. By applying Theorem 1(iv), we obtain

$$E(S_{n^{k+1}}, x) = E(P_{n+1}, x) E(S_{n^k}, x) + E(P_{n+1} - u, x) E(S_{n^k}, x) + E(P_{n+1}, x) E(S_{n^k} - v, x).$$

Removing the center vertex in a spider graph S_{n^k} splits it into k disconnected paths P_n, and removing the pendant from P_{n+1} turns it into P_n. Hence, we find

$$\begin{aligned} E(S_{n^{k+1}}, x) &= E(P_{n+1}, x)E(S_{n^k}, x) + E(P_n, x)E(S_{n^k}, x) + E(P_{n+1}, x)E(P_n, x)^k \\ &= (E(P_{n+1}, x) + E(P_n, x))E(S_{n^k}, x) + E(P_{n+1}, x)E(P_n, x)^k\,. \end{aligned}$$

Using the assumption about $E(S_{n^k}, x)$, we then have

$$\begin{aligned} E(S_{n^{k+1}}, x) &= (E(P_{n+1}, x) + E(P_n, x))((E(P_{n+1}, x) + E(P_n, x))^k - E(P_n, x)^k) \\ &\quad + E(P_{n+1}, x)E(P_n, x)^k \\ &= (E(P_{n+1}, x) + E(P_n, x))^{k+1} - E(P_n, x)^{k+1}\,. \end{aligned}$$

Therefore, the claim holds for all k. Note that if we express $E(S_{n^k}, x)$ in expanded form in terms of powers of $E(P_{n+1}, x)$ and $E(P_n, x)$, then the coefficients will form the beheaded Pascal's triangle A074909 since they will be the same as the coefficients of $(x+y)^k - y^k$. The beheaded Pascal's triangle omits the last 1 in each row of Pascal's triangle, which results from subtracting 1 from $(x+1)^k$. □

4.1.1 Spider graph sequences

In these spider graphs with branches of equal length, we generate sequences if we further fix n or k. Let us first consider fixing k, the number of branches. The cases $k = 1, 2$ are boring since these spiders are path graphs. So let $k = 3$. By Theorem 10, we have $\#S_{n^3} = (F_{n+1})^3 - (F_{n-1})^3$, which is the sequence A350473. The edge cover polynomials are

$$E(S_{n^3}, x) = (E(P_{n+1}, x) + E(P_n, x))^3 - E(P_n, x)^3\,.$$

These edge cover polynomials produce the triangle of numbers given in Table 2. In this table, row n corresponds to the coefficients of the edge cover polynomial of S_n^3 arranged in descending powers of x.

$n \setminus i$	0	1	2	3	4	5	6
1	1						
2	1	3	3				
3	1	6	12	7			
4	1	9	30	44	27	6	
5	1	12	57	135	165	96	19

Table 2: Coefficients of $E(S_{n^3}, x)$.

Now let $k = 4$. The total number of edge covers of S_{n^4} is $\#S_{n^4} = (F_{n+1})^4 - (F_{n-1})^4$, generating the sequence A358917, which starts with

$$1, 15, 80, 609, 4015, 27936, 190385, 1307775, \ldots$$

The edge cover polynomials of S_{n^4} are

$$E(S_{n^4}, x) = (E(P_{n+1}, x) + E(P_n, x))^4 - E(P_n, x)^4 ,$$

which generate the triangle in Table 3, again in descending powers of x.

$n \setminus i$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	4	6	4						
3	1	8	24	32	15					
4	1	12	58	144	194	140	52	8		
5	1	16	108	400	885	1192	948	400	68	

Table 3: Coefficients of $E(S_{n^4}, x)$.

For $k = 5$, we have $\#S_{n^5} = (F_{n+1})^5 - (F_{n-1})^5$, generating the sequence A358934, which starts with

$$1, 31, 242, 3093, 32525, 368168, 4051333, 499200274, \ldots$$

The edge cover polynomials are

$$E(S_{n^5}, x) = (E(P_{n+1}, x) + E(P_n, x))^5 - E(P_n, x)^5 ,$$

generating the triangle of numbers in Table 4.

$n \setminus i$	0	1	2	3	4	5	6	7	8	9	10
1	1										
2	1	5	10	10	5						
3	1	10	40	80	80	31					
4	1	15	95	330	872	680	320	85	10		
5	1	20	175	880	2810	5943	8420	7880	4645	1540	211

Table 4: Coefficients of $E(S_{n^5}, x)$.

Instead of fixing k to generate a sequence, we can also fix n to create spider graphs with fixed branch lengths and an increasing number of branches.

If $n = 1$, we obtain spider graphs S_{1^k}, which are star graphs. These are uninteresting since $\#S_{1^k} = 1$ and $E(S_{1^k}, x) = x^k$.

Spider graphs S_{2^k} produce more interesting edge cover numbers than star graphs do. We have $\#S_{2^k} = F_3^k - F_1^k = 2^k - 1$, generating A000225, and

$$E(S_{2^k}, x) = (E(P_3, x) + E(P_2, x))^k - E(P_2, x)^k = (x^2 + x)^k - x^k = x^k((x + 1)^k - 1) .$$

The triangle generated by the coefficients of these polynomials (arranged in descending powers of x) is A074909, the beheaded Pascal's triangle.

For $n = 3$, we have $\#S_{3^k} = F_4^k - F_2^k = 3^k - 1$, generating A024023, and

$$E(S_{3^k}, x) = (E(P_4, x) + E(P_3, x))^k - E(P_3, x)^k = (x^3 + 2x^2)^k - x^{2k} = x^{2k}((x+2)^k - 1)\,.$$

The coefficients of these edge cover polynomials, in descending powers of x, form Table 5.

$k \setminus i$	0	1	2	3	4	5
1	1	1				
2	1	4	3			
3	1	6	12	7		
4	1	8	24	32	15	
5	1	10	40	80	80	31

Table 5: Coefficients of $E(S_{3^k}, x)$.

The edge cover numbers for $n = 4$ generate the sequence A005057, and for $n = 5$ generate the sequence A190543. Larger n values generate sequences not in OEIS.

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