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Golden Ratio Base Expansions of the Logarithm and Inverse Tangent of Fibonacci and Lucas Numbers

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Abstract

Let $\alpha = (1+\sqrt{5})/2$, the golden ratio. Let F_n and L_n be the Fibonacci and Lucas numbers. We derive base-α expansions of $\log F_n$, $\log L_n$, $\arctan \dfrac{1}{F_n}$, and $\arctan \dfrac{1}{L_n}$ for all positive integers n.

1 Introduction

Let α denote the golden ratio; that is, $\alpha = (1+\sqrt{5})/2$. Let $\beta = -1/\alpha = (1-\sqrt{5})/2$. Thus $\alpha\beta = -1$ and $\alpha + \beta = 1$. Let F_n and L_n be the Fibonacci and Lucas numbers, defined by $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$, for all non-negative integers n.

Let b be any non-zero number whose magnitude is greater than unity. Let n and s be positive integers. A convergent series of the form

$$C = \sum_{k=0}^{\infty} \frac{1}{b^k} \left(\frac{a_1}{(kn+1)^s} + \frac{a_2}{(kn+2)^s} + \cdots + \frac{a_n}{(kn+n)^s} \right), \tag{1}$$

where a_1, a_2, ..., a_n are certain numbers, defines a base-b, length-n and degree-s expansion of the mathematical constant C.

If b is an integer and a_k are rational numbers, then (1) is referred to as a BBP-type formula, after the initials of the authors of the paper [4] in which such an expansion was first presented for π and some other mathematical constants. Any mathematical constant that possesses a base-b BBP-type formula has the property that one can compute its n-th digit, in base b, using only $O((\log n)^c)$ space for some constant c, and time $O(n)$.

Our goal in this paper is to derive base-α expansion formulas for the logarithm and the inverse tangent of all Fibonacci and Lucas numbers. We will often give the expansion using the compact P-notation for BBP-type formulas, introduced by Bailey and Crandall [5], namely,

$$C = P(s, b, n, A) = \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{n} \frac{a_j}{(kn+j)^s},$$

where s and n are integers and, in this present paper, b is an integer power of α and $A = (a_1, a_2, \ldots, a_n)$ is a vector of rational multiples of powers of β. For example, we will show (see (19)) that

$$\log F_3 = \log 2 = \sum_{k=0}^{\infty} \frac{1}{\alpha^{12k}} \left(\frac{\beta^2}{6k+1} + \frac{3\beta^4}{6k+2} + \frac{4\beta^6}{6k+3} + \frac{3\beta^8}{6k+4} + \frac{\beta^{10}}{6k+5} \right),$$

which, in the P-notation, can be written as

$$\log F_3 = \log 2 = P(1, \alpha^{12}, 6, (\beta^2, 3\beta^4, 4\beta^6, 3\beta^8, \beta^{10}, 0)).$$

To conclude this section we provide a brief summary of some previous studies on base-α expansions.

Bailey and Crandall [5] derived a base-$2/\alpha$ formula for $\pi\sqrt{\alpha}$. Similar formulas were found by Chan [7] who also later proved [8] several formulas expressing π in terms of α. Borwein and Chamberland [6] found a base-α expansion for π^2.

Zhang [11] gave base-α expansions for $\pi\sqrt{\alpha\sqrt{5}}$ and some other constants.

In a previous paper [2], we established base-α formulas for π, $\log\alpha$, $\log 2$ and several other mathematical constants.

Recently, Kristensen and Mathiasen [9], using an algorithm implemented in `SAGE`, found a base-α formula for π. They also obtained several base-α zero relations; that is, series that sum to zero.

2 Base-α expansions of logarithms

The base-α expansions of the logarithms of Fibonacci and Lucas numbers are presented in Theorems 3 and 5 but first we state a couple of lemmata upon which the results are based.

Let

$$\mathrm{Li}_1(x) = -\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} = \sum_{k=0}^{\infty} x^k \frac{x}{k+1}, \quad -1 \le x < 1.$$

Lemma 1. *If $|b| > 1$, $t > 0$ and m and n are arbitrary positive integers, then*

$$\mathrm{Li}_1\left(\frac{1}{b^{\,t}}\right) = \sum_{k=0}^{\infty} \frac{1}{b^{tmk}} \sum_{j=1}^{m} \frac{1/b^{\,tj}}{(mk+j)}, \tag{2}$$

$$\mathrm{Li}_1\left(-\frac{1}{b^{\,t}}\right) = \sum_{k=0}^{\infty} \frac{1}{b^{2tnk}} \sum_{j=1}^{2n} \frac{(-1)^j/b^{\,tj}}{(2nk+j)}. \tag{3}$$

Proof. We have

$$\mathrm{Li}_1\left(\frac{1}{b^{\,t}}\right) = \sum_{k=0}^{\infty} \frac{1}{b^{\,tk}} \frac{1/b^{\,t}}{k+1},$$

from which (2) follows upon using the identity

$$\sum_{k=0}^{\infty} f_k = \sum_{k=0}^{\infty} \sum_{j=1}^{m} f_{mk+j-1}, \tag{4}$$

with

$$f_k = \frac{1}{b^{tk+t}} \frac{1}{k+1}.$$

The proof of (3) is similar, with $m = 2n$ in (4). □

Lemma 2. *If r is an integer, then*

$$\log L_r = r\,\mathrm{Li}_1\left(\frac{1}{\alpha^2}\right) - \mathrm{Li}_1\left(\frac{(-1)^{r+1}}{\alpha^{2r}}\right), \tag{5}$$

$$\log F_r = (r-2)\,\mathrm{Li}_1\left(\frac{1}{\alpha^2}\right) + \mathrm{Li}_1\left(\frac{1}{\alpha^4}\right) - \mathrm{Li}_1\left(\frac{(-1)^{r}}{\alpha^{2r}}\right), \quad r \ne 0. \tag{6}$$

Proof. We have

$$\mathrm{Li}_1\left(-\frac{\beta^r}{\alpha^r}\right) = -\log\left(\frac{\alpha^r+\beta^r}{\alpha^r}\right) = -\log\left(\frac{L_r}{\alpha^r}\right) = -\log L_r + r\log\alpha, \tag{7}$$

in which setting $r = 1$ gives

$$\log\alpha = \mathrm{Li}_1\left(\frac{1}{\alpha^2}\right). \tag{8}$$

Using (8) in (7) gives (5).

Also,

$$\mathrm{Li}_1\left(\frac{\beta^r}{\alpha^r}\right) = -\log\left(\frac{\alpha^r - \beta^r}{\alpha^r}\right) = -\log\left(\frac{F_r\sqrt{5}}{\alpha^r}\right) = -\log F_r + r\log\alpha - \log\sqrt{5}, \tag{9}$$

in which setting $r = 2$ gives

$$\log\sqrt{5} = 2\log\alpha - \mathrm{Li}_1\left(\frac{1}{\alpha^4}\right) = 2\,\mathrm{Li}_1\left(\frac{1}{\alpha^2}\right) - \mathrm{Li}_1\left(\frac{1}{\alpha^4}\right), \tag{10}$$

where we used (8). Identity (6) follows from (9) and (10).

□

Theorem 3. *If r is an integer, then*

$$\log F_r = \sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{r}\frac{\beta^{4j-2}(r-2+r\delta_{j,(r+1)/2})}{2rk+2j-1} + \sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{r-1}\frac{\beta^{4j}r}{2rk+2j}, \quad r \text{ odd}, \tag{11}$$

$$\log F_r = \sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{r}\frac{(r-2)\beta^{4j-2}}{2rk+2j-1} + \sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{r-1}\frac{\beta^{4j}r(1-\delta_{j,r/2})}{2rk+2j}, \quad r \text{ even}. \tag{12}$$

Here and throughout this paper, δ_{mn} denotes the Kronecker delta symbol whose value is unity when m equals n and zero otherwise.

Proof. We prove (12). When r is even, (6) reads

$$\log F_r = (r-2)\,\mathrm{Li}_1\left(\frac{1}{\alpha^2}\right) + \mathrm{Li}_1\left(\frac{1}{\alpha^4}\right) - \mathrm{Li}_1\left(\frac{1}{\alpha^{2r}}\right), \quad r \neq 0. \tag{13}$$

We proceed to write the three Li_1 terms in a common base α^{4r}, using (2) with appropriate t and m choices. Thus,

$$\mathrm{Li}_1\left(\frac{1}{\alpha^2}\right) = \sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{2r}\frac{1/\alpha^{2j}}{2rk+j}, \tag{14}$$

$$\mathrm{Li}_1\left(\frac{1}{\alpha^4}\right) = \sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{r}\frac{1/\alpha^{4j}}{rk+j}, \tag{15}$$

$$\mathrm{Li}_1\left(\frac{1}{\alpha^{2r}}\right) = \sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\left(\frac{1/\alpha^{2r}}{2k+1} + \frac{1/\alpha^{4r}}{2k+2}\right). \tag{16}$$

Using (14), (15) and (16) in (13) gives

$$\begin{aligned}\log F_r &= \sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{2r}\frac{\beta^{2j}(r-2)}{2rk+j}+\sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{r}\frac{2\beta^{4j}}{2rk+2j}\\&\quad-\sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\left(\frac{r\beta^{2r}}{2rk+r}+\frac{r\beta^{4r}}{2rk+2r}\right).\end{aligned}\tag{17}$$

Using the summation identity

$$\sum_{j=1}^{2r}f_j=\sum_{j=1}^{r}f_{2j}+\sum_{j=1}^{r}f_{2j-1}$$

to write its inner sum, the first term on the right hand side of (17) can be written as

$$\sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{2r}\frac{\beta^{2j}(r-2)}{2rk+j}=\sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{r}\frac{\beta^{4j}(r-2)}{2rk+2j}+\sum_{k=0}^{\infty}\frac{1}{\alpha^{4rk}}\sum_{j=1}^{r}\frac{\beta^{4j-2}(r-2)}{2rk+2j-1}.\tag{18}$$

Using (18) in (17) yields (12).

□

Identities (11) and (12) written in the P-notation are

$$\log F_r = P(1,\alpha^{4r},2r,(a_1,a_2,\ldots,a_{2r})),$$

where for $1\le j\le r$,

$$a_{2j-1}=\beta^{4j-2}(r-2+r\delta_{j,(r+1)/2}),\quad a_{2j}=\beta^{4j}r(1-\delta_{rj}),\quad r\text{ odd};$$

and

$$a_{2j-1}=(r-2)\beta^{4j-2},\quad a_{2j}=\beta^{4j}r(1-\delta_{j,r/2}-\delta_{j,r}),\quad r\text{ even}.$$

Example 4.

$$\log F_3=\log 2=P(1,\alpha^{12},6,(\beta^2,3\beta^4,4\beta^6,3\beta^8,\beta^{10},0)),\tag{19}$$

$$\begin{aligned}\log F_5=\log 5=P(1,\alpha^{20},10,(&3\beta^2,5\beta^4,3\beta^6,5\beta^8,8\beta^{10},\\&5\beta^{12},3\beta^{14},5\beta^{16},3\beta^{18},0)),\end{aligned}\tag{20}$$

$$\log F_4=\log 3=P(1,\alpha^{16},8,(2\beta^2,4\beta^4,2\beta^6,0,2\beta^{10},4\beta^{12},2\beta^{14},0)),\tag{21}$$

$$\begin{aligned}\log F_8=\log 21=P(1,\alpha^{32},16,(&6\beta^2,8\beta^4,6\beta^6,8\beta^8,6\beta^{10},8\beta^{12},6\beta^{14},\\&0,6\beta^{18},8\beta^{20},6\beta^{22},8\beta^{24},6\beta^{26},8\beta^{28},6\beta^{30},0)),\end{aligned}$$

$$\begin{aligned}\log F_{12}=\log 144=P(1,\alpha^{48},24,(&10\,\beta^2,12\,\beta^4,10\,\beta^6,12\,\beta^8,10\,\beta^{10},\\&12\,\beta^{12},10\,\beta^{14},12\,\beta^{16},10\,\beta^{18},12\,\beta^{20},10\,\beta^{22},0,\\&10\,\beta^{26},12\,\beta^{28},10\,\beta^{30},12\,\beta^{32},10\,\beta^{34},12\,\beta^{36},\\&10\,\beta^{38},12\,\beta^{40},10\,\beta^{42},12\,\beta^{44},10\,\beta^{46},0)).\end{aligned}\tag{22}$$

Theorem 5. *If r is an integer, then*

$$\log L_r = \sum_{k=0}^{\infty} \frac{1}{\alpha^{2rk}} \sum_{j=1}^{r-1} \frac{\beta^{2j} r}{rk+j}, \;\; r \;\; odd, \tag{23}$$

$$\log L_r = \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{2r-1} \frac{\beta^{2j} r(1+\delta_{rj})}{2rk+j}, \;\; r \;\; even. \tag{24}$$

Proof. We prove (23). If r is an odd integer, (5) gives

$$\log L_r = r \operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) - \operatorname{Li}_1\left(\frac{1}{\alpha^{2r}}\right). \tag{25}$$

With

$$\operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) = \sum_{k=0}^{\infty} \frac{1}{\alpha^{2rk}} \sum_{j=1}^{r} \frac{1/\alpha^{2j}}{rk+j}$$

and

$$\operatorname{Li}_1\left(\frac{1}{\alpha^{2r}}\right) = \sum_{k=0}^{\infty} \frac{1}{\alpha^{2rk}} \frac{r/\alpha^{2r}}{rk+r}$$

in (25); identity (23) follows. □

Identities (23) and (24) in the P-notation are

$$\log L_r = P(1, \alpha^{2r}, r, (a_1, a_2, \ldots, a_r)), \quad r \text{ odd},$$

with

$$a_j = r\beta^{2j}(1-\delta_{rj}), \quad 1 \le j \le r,$$

and

$$\log L_r = P(1, \alpha^{4r}, 2r, (a_1, a_2, \ldots, a_{2r})), \quad r \text{ even},$$

with

$$a_j = r\beta^{2j}(1+\delta_{jr}-\delta_{j,2r}), \quad 1 \le j \le 2r.$$

Example 6.

$$\log L_2 = \log 3 = \sum_{k=0}^{\infty} \frac{1}{\alpha^{8k}} \left(\frac{2\beta^2}{4k+1} + \frac{4\beta^4}{4k+2} + \frac{2\beta^6}{4k+3}\right);$$

that is,

$$\log 3 = P(1, \alpha^8, 4, (2\beta^2, 4\beta^4, 2\beta^6, 0)). \tag{26}$$

$$\log L_3 = \log 4 = P(1, \alpha^6, 3, (3\beta^2, 3\beta^4, 0)), \tag{27}$$

$$\log L_4 = \log 7 = 4\beta^2 P(1, \alpha^{16}, 8, (1, \beta^2, \beta^4, 2\beta^6, \beta^8, \beta^{10}, \beta^{12}, 0)),$$

$$\begin{aligned} \log L_6 = \log 18 = P(1, \alpha^{24}, 12, (&6\beta^2, 6\beta^4, 6\beta^6, 6\beta^8, 6\beta^{10}, \\ &12\beta^{12}, 6\beta^{14}, 6\beta^{16}, 6\beta^{18}, 6\beta^{20}, 6\beta^{22}, 0)). \end{aligned} \tag{28}$$

3 Base-α expansions of inverse tangents

The base-α expansions of the inverse tangent of Fibonacci and Lucas numbers are stated in Theorems 10–19 but first we collect some required identities in Lemmata 7–9.

Lemma 7. *If r is an integer, then*

$$\alpha^r - \alpha^{-r} = \begin{cases} F_r\sqrt{5}, & r \text{ even;} \\ L_r, & r \text{ odd;} \end{cases}$$

$$\alpha^r + \alpha^{-r} = \begin{cases} L_r, & r \text{ even;} \\ F_r\sqrt{5}, & r \text{ odd.} \end{cases}$$

Lemma 8. *If r and m are integers, then*

$$\arctan\frac{1}{\alpha^{r-m}} - \arctan\frac{1}{\alpha^{r+m}} = \begin{cases} \arctan\left(L_m/(F_r\sqrt{5})\right), & m \text{ odd, } r \text{ odd;} \\ \arctan(L_m/L_r), & m \text{ odd, } r \text{ even;} \\ \arctan(F_m/F_r), & m \text{ even, } r \text{ odd;} \\ \arctan(F_m\sqrt{5}/L_r), & m \text{ even, } r \text{ even;} \end{cases}$$

$$\arctan\frac{1}{\alpha^{r-m}} + \arctan\frac{1}{\alpha^{r+m}} = \begin{cases} \arctan(F_m\sqrt{5}/L_r), & m \text{ odd, } r \text{ odd;} \\ \arctan(F_m/F_r), & m \text{ odd, } r \text{ even;} \\ \arctan(L_m/L_r), & m \text{ even, } r \text{ odd;} \\ \arctan\left(L_m/(F_r\sqrt{5})\right), & m \text{ even, } r \text{ even.} \end{cases}$$

Proof. The arctangent subtraction and addition formulas give

$$\arctan\frac{1}{\alpha^{r-m}} - \arctan\frac{1}{\alpha^{r+m}} = \arctan\left(\frac{\alpha^r(\alpha^m - \alpha^{-m})}{\alpha^{2r}+1}\right),$$
$$\arctan\frac{1}{\alpha^{r-m}} + \arctan\frac{1}{\alpha^{r+m}} = \arctan\left(\frac{\alpha^r(\alpha^m + \alpha^{-m})}{\alpha^{2r}-1}\right);$$

and hence the stated identities upon the use of Lemma 7. □

Lemma 9. *If r is an integer, then*

$$\alpha^{2r} - 1 = \alpha^r L_r, \quad \beta^{2r} - 1 = \beta^r L_r, \quad r \text{ odd,} \tag{29}$$
$$\alpha^{2r} - 1 = \alpha^r F_r\sqrt{5}, \quad \beta^{2r} - 1 = -\beta^r F_r\sqrt{5}, \quad r \text{ even,} \tag{30}$$
$$\alpha^{2r} + 1 = \alpha^r F_r\sqrt{5}, \quad \beta^{2r} + 1 = -\beta^r F_r\sqrt{5}, \quad r \text{ odd,}$$
$$\alpha^{2r} + 1 = \alpha^r L_r, \quad \beta^{2r} + 1 = \beta^r L_r, \quad r \text{ even.}$$

Theorem 10. *If r is an odd integer greater than unity, then*

$$\arctan\frac{1}{F_r} = P(1, \alpha^{4(r^2-4)}, 4(r^2-4), (a_1, a_2, \ldots, a_{4(r^2-4)})), \tag{31}$$

where the only non-zero constants a_j are given by

$$\begin{aligned}
a_{(r-2)(4j-3)} &= -\beta^{(r-2)(4j-3)}(r-2), \quad j = 1, 2, \ldots, r+2,\\
a_{(r-2)(4j-1)} &= \beta^{(r-2)(4j-1)}(r-2), \quad j = 1, 2, \ldots, r+2,\\
a_{(r+2)(4j-3)} &= \beta^{(r+2)(4j-3)}(r+2), \quad j = 1, 2, \ldots, r-2,\\
a_{(r+2)(4j-1)} &= -\beta^{(r+2)(4j-1)}(r+2), \quad j = 1, 2, \ldots, r-2,\\
a_{(r-2)(r+2)} &= (-1)^{(r+1)/2} 4\beta^{r^2-4}\\
a_{3(r-2)(r+2)} &= (-1)^{(r-1)/2} 4\beta^{3(r^2-4)}.
\end{aligned}$$

Proof. With r an odd number, setting $m = 2$ in (8) gives

$$\arctan\frac{1}{F_r} = \arctan\frac{1}{\alpha^{r-2}} - \arctan\frac{1}{\alpha^{r+2}}.$$

The following identity, proved in a previous work [1, Identity (10)]:

$$n\sqrt{n}\arctan\left(\frac{1}{\sqrt{n}}\right) = \sum_{k=0}^{\infty}\frac{1}{n^{2k}}\left(\frac{n}{4k+1} - \frac{1}{4k+3}\right) \tag{32}$$

gives

$$\arctan\frac{1}{\alpha^{r-2}} = \sum_{k=0}^{\infty}\frac{1}{\alpha^{(r-2)(4k+3)}}\left(\frac{\alpha^{2r-4}}{4k+1} - \frac{1}{4k+3}\right)$$

and

$$\arctan\frac{1}{\alpha^{r+2}} = \sum_{k=0}^{\infty}\frac{1}{\alpha^{(r+2)(4k+3)}}\left(\frac{\alpha^{2r+4}}{4k+1} - \frac{1}{4k+3}\right),$$

or, by (4),

$$\arctan\frac{1}{\alpha^{r-2}} = \sum_{k=0}^{\infty}\frac{1}{\alpha^{(4r^2-16)k}}\sum_{j=1}^{r+2}\left(\frac{\alpha^{-(r-2)(4j-3)}}{4(r+2)k+4j-3} - \frac{\alpha^{-(r-2)(4j-1)}}{4(r+2)k+4j-1}\right)$$

and

$$\arctan\frac{1}{\alpha^{r+2}} = \sum_{k=0}^{\infty}\frac{1}{\alpha^{(4r^2-16)k}}\sum_{j=1}^{r-2}\left(\frac{\alpha^{-(r+2)(4j-3)}}{4(r-2)k+4j-3} - \frac{\alpha^{-(r+2)(4j-1)}}{4(r-2)k+4j-1}\right).$$

Thus,

$$\arctan\frac{1}{F_r}=\sum_{k=0}^{\infty}\frac{1}{\alpha^{(4r^2-16)k}}\times$$
$$\left(\sum_{j=1}^{r+2}\left(\frac{-\beta^{(r-2)(4j-3)}(r-2)}{4(r^2-4)k+(r-2)(4j-3)}+\frac{\beta^{(r-2)(4j-1)}(r-2)}{4(r^2-4)k+(r-2)(4j-1)}\right)\right.$$
$$\left.+\sum_{j=1}^{r-2}\left(\frac{\beta^{(r+2)(4j-3)}(r+2)}{4(r^2-4)k+(r+2)(4j-3)}-\frac{\beta^{(r+2)(4j-1)}(r+2)}{4(r^2-4)k+(r+2)(4j-1)}\right)\right). \tag{33}$$

Identity (31) is (33) expressed in the P-notation. □

Example 11.

$$\begin{aligned}\arctan\frac{1}{F_3}=\arctan\frac{1}{2}=P(1,\alpha^{20},20,(&-\beta,0,\beta^3,0,4\,\beta^5,0,\beta^7,0,-\beta^9,0,\beta^{11},\\&0,-\beta^{13},0,-4\,\beta^{15},0,-\beta^{17},0,\beta^{19},0)),\end{aligned} \tag{34}$$

$$\begin{aligned}\arctan\frac{1}{F_5}=\arctan\frac{1}{5}=P(1,\alpha^{84},84,(&0,0,-3\,\beta^3,0,0,0,7\,\beta^7,0,3\,\beta^9,0,0,0,0,0,-3\,\beta^{15},\\&0,0,0,0,0,-4\,\beta^{21},0,0,0,0,0,-3\,\beta^{27},0,0,0,0,0,3\,\beta^{33},\\&0,7\,\beta^{35},0,0,0,-3\,\beta^{39},0,0,0,0,0,3\,\beta^{45},0,0,0,-7\,\beta^{49},\\&0,-3\,\beta^{51},0,0,0,0,0,3\,\beta^{57},0,0,0,0,0,4\,\beta^{63},0,0,0,0,0,\\&3\,\beta^{69},0,0,0,0,0,-3\,\beta^{75},0,-7\,\beta^{77},0,0,0,3\,\beta^{81},0,0,0)),\end{aligned}$$

$$\begin{aligned}\arctan\frac{1}{F_7}=\arctan\frac{1}{13}=P(1,\alpha^{180},180,(&0,0,0,0,-5\,\beta^5,0,0,0,9\,\beta^9,0,0,0,0,0,5\,\beta^{15},\\&0,0,0,0,0,0,0,0,0,-5\,\beta^{25},0,-9\,\beta^{27},0,0,0,0,0,0,0,\\&5\,\beta^{35},0,0,0,0,0,0,0,0,0,4\,\beta^{45},0,0,0,0,0,0,0,0,0,\\&5\,\beta^{55},0,0,0,0,0,0,0,-9\,\beta^{63},0,-5\,\beta^{65},0,0,0,0,0,\\&0,0,0,0,5\,\beta^{75},0,0,0,0,0,9\,\beta^{81},0,0,0,-5\,\beta^{85},0,0,\\&0,0,0,0,0,0,0,5\,\beta^{95},0,0,0,-9\,\beta^{99},0,0,0,0,0,\\&-5\,\beta^{105},0,0,0,0,0,0,0,0,0,5\,\beta^{115},0,9\,\beta^{117},0,0,0,\\&0,0,0,0,-5\,\beta^{125},0,0,0,0,0,0,0,0,0,-4\,\beta^{135},\\&0,0,0,0,0,0,0,0,0,-5\,\beta^{145},0,0,0,0,0,0,0,9\,\beta^{153},0,\\&5\,\beta^{155},0,0,0,0,0,0,0,0,0,-5\,\beta^{165},0,0,0,0,0,-9\,\beta^{171},\\&0,0,0,5\,\beta^{175},0,0,0,0,0)).\end{aligned}$$

Theorem 12. *If r is a positive even integer, then*

$$\arctan\frac{1}{F_r} = P(1, \alpha^{4(r^2-1)}, 4(r^2-1), (a_1, a_2, \ldots, a_{4(r^2-1)})),$$

where the only non-zero constants a_j are given by

$$\begin{aligned}
a_{(r-1)(4j-3)} &= -\beta^{(r-1)(4j-3)}(r-1), \quad j = 1, 2, \ldots, r+1,\\
a_{(r-1)(4j-1)} &= \beta^{(r-1)(4j-1)}(r-1), \quad j = 1, 2, \ldots, r+1,\\
a_{(r+1)(4j-3)} &= -\beta^{(r+1)(4j-3)}(r+1), \quad j = 1, 2, \ldots, r-1,\\
a_{(r+1)(4j-1)} &= \beta^{(r+1)(4j-1)}(r+1), \quad j = 1, 2, \ldots, r-1,\\
a_{(r-1)(r+1)} &= (-1)^{r/2} 2\beta^{r^2-1}\\
a_{3(r-1)(r+1)} &= (-1)^{(r+2)/2} 2\beta^{3(r^2-1)}.
\end{aligned}$$

Proof. Setting $m = 1$ in (8) gives

$$\arctan\frac{1}{F_r} = \arctan\frac{1}{\alpha^{r-1}} + \arctan\frac{1}{\alpha^{r+1}}, \quad r \text{ even}.$$

The proof now proceeds as in that of Theorem 10. □

Example 13.

$$\arctan\frac{1}{F_2} = \frac{\pi}{4} = P(1, \alpha^{12}, 12, (-\beta, 0, -2\,\beta^3, 0, -\beta^5, 0, \beta^7, 0, 2\,\beta^9, 0, \beta^{11}, 0)),$$

$$\begin{aligned}
\arctan\frac{1}{F_4} = \arctan\frac{1}{3} = P(1, \alpha^{60}, 60, (&0, 0, -3\,\beta^3, 0, -5\,\beta^5, 0, 0, 0, 3\,\beta^9, 0, 0, 0, 0, 0, 2\,\beta^{15},\\
&0, 0, 0, 0, 0, 3\,\beta^{21}, 0, 0, 0, -5\,\beta^{25}, 0, -3\,\beta^{27}, 0, 0, 0, 0, 0, 3\,\beta^{33},\\
&0, 5\,\beta^{35}, 0, 0, 0, -3\,\beta^{39}, 0, 0, 0, 0, 0, -2\,\beta^{45}, 0, 0, 0, 0, 0, -3\,\beta^{51},\\
&0, 0, 0, 5\,\beta^{55}, 0, 3\,\beta^{57}, 0, 0, 0)).
\end{aligned} \tag{35}$$

Theorem 14. *If r is a positive even integer, then*

$$\arctan\frac{1}{L_r} = P(1, \alpha^{4(r^2-1)}, 4(r^2-1), (a_1, a_2, \ldots, a_{4(r^2-1)})),$$

where the only non-zero constants a_j are given by

$$\begin{aligned}
a_{(r-1)(4j-3)} &= -\beta^{(r-1)(4j-3)}(r-1), \quad j = 1, 2, \ldots, r+1,\\
a_{(r-1)(4j-1)} &= \beta^{(r-1)(4j-1)}(r-1), \quad j = 1, 2, \ldots, r+1,\\
a_{(r+1)(4j-3)} &= \beta^{(r+1)(4j-3)}(r+1), \quad j = 1, 2, \ldots, r-1,\\
a_{(r+1)(4j-1)} &= -\beta^{(r+1)(4j-1)}(r+1), \quad j = 1, 2, \ldots, r-1,\\
a_{(r-1)(r+1)} &= (-1)^{(r+2)/2}\, 2r\,\beta^{r^2-1}\\
a_{3(r-1)(r+1)} &= (-1)^{r/2}\, 2r\,\beta^{3(r^2-1)}.
\end{aligned}$$

Proof. Setting $m = 1$ in (8) gives

$$\arctan\frac{1}{L_r} = \arctan\frac{1}{\alpha^{r-1}} - \arctan\frac{1}{\alpha^{r+1}}, \quad r \text{ even}.$$

The proof now proceeds as in that of Theorem 10. □

Example 15.

$$\arctan\frac{1}{L_2} = \arctan\frac{1}{3} = (1, \alpha^{12}, 12, (-\beta, 0, 4\,\beta^3, 0, -\beta^5, 0, \beta^7, 0, -4\,\beta^9, 0, \beta^{11}, 0)), \tag{36}$$

$$\begin{aligned}\arctan\frac{1}{L_4} = \arctan\frac{1}{7} = (1, \alpha^{60}, 60, (&0, 0, -3\,\beta^3, 0, 5\,\beta^5, 0, 0, 0, 3\,\beta^9, 0, 0, 0, 0, 0, -8\,\beta^{15},\\ &0, 0, 0, 0, 0, 3\,\beta^{21}, 0, 0, 0, 5\,\beta^{25}, 0, -3\,\beta^{27}, 0, 0, 0, 0, 0, 3\,\beta^{33},\\ &0, -5\,\beta^{35}, 0, 0, 0, -3\,\beta^{39}, 0, 0, 0, 0, 0, 8\,\beta^{45}, 0, 0, 0, 0, 0,\\ &-3\,\beta^{51}, 0, 0, 0, -5\,\beta^{55}, 0, 3\,\beta^{57}, 0, 0, 0)).\end{aligned}$$

Theorem 16. *If r is an integer, then*

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{\alpha^{12rk}}\left(\frac{\beta^r}{12k+1} + \frac{2\beta^{3r}}{12k+3} + \frac{\beta^{5r}}{12k+5} - \frac{\beta^{7r}}{12k+7} - \frac{2\beta^{9r}}{12k+9} - \frac{\beta^{11r}}{12k+11}\right)\\ &\quad = \begin{cases} -\arctan\left(\frac{1}{L_r}\right), & r \text{ odd};\\ \arctan\left(\frac{1}{F_r\sqrt{5}}\right), & r \text{ even};\end{cases}\end{aligned} \tag{37}$$

that is,

$$\begin{aligned}&P(1, \alpha^{12r}, 12, (\beta^r, 0, 2\beta^{3r}, 0, \beta^{5r}, 0, -\beta^{7r}, 0, -2\beta^{9r}, 0, -\beta^{11r}, 0))\\ &\quad = \begin{cases} -\arctan\left(\frac{1}{L_r}\right), & r \text{ odd};\\ \arctan\left(\frac{1}{F_r\sqrt{5}}\right), & r \text{ even}.\end{cases}\end{aligned}$$

Proof. In an earlier work [1, Identity (27)], we showed that

$$n^2\sqrt{n}\arctan\left(\frac{\sqrt{n}}{n-1}\right) = \sum_{k=0}^{\infty}\frac{1}{(-n^3)^k}\left(\frac{n^2}{6k+1} + \frac{2n}{6k+3} + \frac{1}{6k+5}\right).$$

In base n^6, length 12, this is

$$\begin{aligned}&n^2\sqrt{n}\arctan\left(\frac{\sqrt{n}}{n-1}\right)\\ &\quad = \sum_{k=0}^{\infty}\frac{1}{n^{6k}}\left(\frac{n^2}{12k+1} + \frac{2n}{12k+3} + \frac{1}{12k+5} - \frac{1/n}{12k+7} - \frac{2/n^2}{12k+9} - \frac{1/n^3}{12k+11}\right).\end{aligned} \tag{38}$$

Identity (37) follows upon setting $n = \alpha^{2r}$ in (38) and making use of (29) and (30). □

Example 17.

$$\frac{\pi}{4} = P(1, \alpha^{12}, 12, (-\beta, 0, -2\beta^3, 0, -\beta^5, 0, \beta^7, 0, 2\beta^9, 0, \beta^{11}, 0)), \tag{39}$$

$$\arctan\left(\frac{1}{4}\right) = P(1, \alpha^{12}, 12, (-\beta^3, 0, -2\beta^9, 0, -\beta^{15}, 0, \beta^{21}, 0, 2\beta^{27}, 0, \beta^{33}, 0)),$$

$$\arctan\left(\frac{1}{\sqrt{5}}\right) = P(1, \alpha^{12}, 12, (\beta^2, 0, 2\beta^6, 0, \beta^{10}, 0, -\beta^{14}, 0, -2\beta^{18}, 0, -\beta^{22}, 0)),$$

$$\arctan\left(\frac{1}{3\sqrt{5}}\right) = P(1, \alpha^{12}, 12, (\beta^4, 0, 2\beta^{12}, 0, \beta^{20}, 0, -\beta^{28}, 0, -2\beta^{36}, 0, -\beta^{44}, 0)).$$

Remark 18. Identity (39) is the same golden ratio base expansion of π that was obtained in Theorem 12.

Theorem 19. *If r is an integer, then*

$$\sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \left(\frac{2\beta^r}{4k+1} - \frac{2\beta^{3r}}{4k+3} \right) = \begin{cases} -\arctan\left(\frac{2}{L_r}\right), & r \text{ odd;} \\ \arctan\left(\frac{2}{F_r\sqrt{5}}\right), & r \text{ even;} \end{cases} \tag{40}$$

that is

$$P(1, \alpha^{4r}, 4, (2\beta^r, 0, -2\beta^{3r}, 0)) = \begin{cases} -\arctan\left(\frac{2}{L_r}\right), & r \text{ odd;} \\ \arctan\left(\frac{2}{F_r\sqrt{5}}\right), & r \text{ even.} \end{cases}$$

Proof. Setting $x = \beta^r$ in the identity

$$2\arctan x = \arctan\left(\frac{2x}{1-x^2}\right)$$

and using (29) and (30), we have

$$2\arctan\frac{1}{\alpha^r} = \begin{cases} \arctan\left(\frac{2}{L_r}\right), & r \text{ odd;} \\ \arctan\left(\frac{2}{F_r\sqrt{5}}\right), & r \text{ even.} \end{cases} \tag{41}$$

Setting $n = \alpha^{2r}$ in (32) and comparing with (41), we obtain (40). □

Example 20.

$$\arctan\frac{2}{L_3} = \arctan\frac{1}{2} = P(1, \alpha^{12}, 4, (-2\beta^3, 0, 2\beta^9, 0)), \tag{42}$$

$$\arctan\left(\frac{2}{F_2\sqrt{5}}\right) = \arctan\frac{2}{\sqrt{5}} = P(1, \alpha^8, 4, (2\beta^2, 0, -2\beta^6, 0)).$$

4 Zero relations

Zero relations are expansion formulas that evaluate to zero. They are useful in the determination and classification of new expansion formulas. A base-α expansion is not considered new if it can be written as a linear combination of existing formulas and known zero relations.

4.1 Zero relations arising from the logarithm formulas

4.1.1 Zero relation from $\log(F_3^2/L_3) = 0$

Theorem 21. *We have*

$$\sum_{k=0}^{\infty} \frac{1}{\alpha^{12k}} \left(\frac{1}{6k+1} - \frac{3\beta^2}{6k+2} - \frac{8\beta^4}{6k+3} - \frac{3\beta^6}{6k+4} + \frac{\beta^8}{6k+5} \right) = 0;$$

that is,

$$0 = P(1, \alpha^{12}, 6, (1, -3\beta^2, -8\beta^4, -3\beta^6, \beta^8, 0)).$$

Proof. We have

$$2 \log F_3 - \log L_3 = 0. \tag{43}$$

The expansion of $\log L_3$ given in (27) has the following base-α^{12}, length-6 version:

$$\log L_3 = P(1, \alpha^{12}, 6, (3\beta^2, 3\beta^4, 0, 3\beta^8, 3\beta^{10}, 0)). \tag{44}$$

Use of (19) and (44) in (43) yields the zero relation stated in Theorem 21.

□

4.1.2 Zero relation from $\log(L_6/(F_4^2 F_3)) = 0$

Theorem 22. *We have*

$$\begin{aligned} 0 = P(1, \alpha^{48}, 24, (&1, -5\,\beta^2, -2\,\beta^4, 3\,\beta^6, \beta^8, 4\,\beta^{10}, \beta^{12}, 3\,\beta^{14}, \\ &- 2\,\beta^{16}, -5\,\beta^{18}, \beta^{20}, 0, \beta^{24}, -5\,\beta^{26}, -2\,\beta^{28}, 3\,\beta^{30}, \beta^{32}, \\ &4\,\beta^{34}, \beta^{36}, 3\,\beta^{38}, -2\,\beta^{40}, -5\,\beta^{42}, \beta^{44}, 0)). \end{aligned}$$

Proof. Write $\log F_3$, $\log F_4$ and $\log L_6$, that is, identities (19), (21) and (28), respectively, in the common base α^{48} and common length 24 and use

$$\log L_6 - 2 \log F_4 - \log F_3 = 0.$$

□

4.1.3 Zero relation from $\log(F_{12}/(F_3^4 L_2^2)) = 0$

Theorem 23. *We have*

$$0 = P(1, \alpha^{48}, 24, (1, -4\,\beta^2, -5\,\beta^4, 0, \beta^8, 2\,\beta^{10}, \beta^{12}, 0, -5\,\beta^{16}, -4\,\beta^{18}, \beta^{20}, 0, \beta^{24}, -4\,\beta^{26}, -5\,\beta^{28}, 0, \beta^{32}, 2\,\beta^{34}, \beta^{36}, 0, -5\,\beta^{40}, -4\,\beta^{42}, \beta^{44}, 0)).$$

Proof. Write $\log F_3$, $\log F_{12}$ and $\log L_2$ from (19), (22) and (26), in common base α^{48} and consider

$$\log F_{12} - 4\,\log F_3 - 2\,\log L_2 = 0.$$

□

4.2 Zero relations arising from the inverse tangent formulas

4.2.1 Zero relation from $2\arctan(2/L_3) + \arctan(2/L_5) - \arctan(2/L_1) = 0$

Theorem 24. *We have*

$$0 = P(1, \alpha^{60}, 60, (1, 0, -7\,\beta^2, 0, -4\,\beta^4, 0, -\beta^6, 0, 7\,\beta^8, 0, -\beta^{10}, 0, \beta^{12}, 0, -2\,\beta^{14}, 0, \beta^{16}, 0, -\beta^{18}, 0, 7\,\beta^{20}, 0, -\beta^{22}, 0, -4\,\beta^{24}, 0, -7\,\beta^{26}, 0, \beta^{28}, 0, -\beta^{30}, 0, 7\,\beta^{32}, 0, 4\,\beta^{34}, 0, \beta^{36}, 0, -7\,\beta^{38}, 0, \beta^{40}, 0, -\beta^{42}, 0, 2\,\beta^{44}, 0, -\beta^{46}, 0, \beta^{48}, 0, -7\,\beta^{50}, 0, \beta^{52}, 0, 4\,\beta^{54}, 0, 7\,\beta^{56}, 0, -\beta^{58}, 0)).$$

Proof. Using the addition and subtraction formulas for inverse tangents, it is easy to verify that

$$\arctan\left(\frac{2}{L_1}\right) - \arctan\left(\frac{2}{L_3}\right) = \arctan\left(\frac{3}{4}\right)$$

and

$$\arctan\left(\frac{2}{L_5}\right) + \arctan\left(\frac{2}{L_3}\right) = \arctan\left(\frac{3}{4}\right);$$

so that

$$2\arctan\left(\frac{2}{L_3}\right) + \arctan\left(\frac{2}{L_5}\right) - \arctan\left(\frac{2}{L_1}\right) = 0,$$

from which the zero relation follows upon use of (40). □

Remark 25. The zero relation stated in Theorem 24 can also be obtained directly from

$$\arctan\frac{1}{F_3} = \arctan\frac{2}{L_3},$$

by writing (34) and (42) in the common base α^{60} and common length 60; or from

$$\arctan\frac{1}{F_4} = \arctan\frac{1}{L_2},$$

using (35) and (36).

4.2.2 Zero relation from $2\arctan(1/L_1) - 2\arctan(2/(F_2\sqrt{5})) - \arctan(2/(F_6\sqrt{5})) = 0$

Theorem 26. *We have*

$$0 = P(1, \alpha^{24}, 24, (1, 4\,\beta, 2\,\beta^2, 0, \beta^4, 2\,\beta^5, -\beta^6, 0, -2\,\beta^8, 4\,\beta^9, -\beta^{10}, 0, \beta^{12}, -4\,\beta^{13}, 2\,\beta^{14}, 0, \beta^{16}, -2\,\beta^{17}, -\beta^{18}, 0, -2\,\beta^{20}, -4\,\beta^{21}, -\beta^{22}, 0)).$$

Proof. The identity

$$\begin{aligned}\frac{\pi}{2} - \arctan\left(\frac{2}{\sqrt{5}}\right) &= \arctan\left(\frac{1}{4\sqrt{5}}\right) + \arctan\left(\frac{2}{\sqrt{5}}\right)\\ &= \arctan\left(\frac{\sqrt{5}}{2}\right)\end{aligned}$$

can be arranged as

$$2\arctan\left(\frac{1}{L_1}\right) - 2\arctan\left(\frac{2}{F_2\sqrt{5}}\right) - \arctan\left(\frac{2}{F_6\sqrt{5}}\right) = 0$$

which, on account of (40), gives the stated zero relation. □

5 Other degree 1 base-α expansions and zero relations

5.1 Base-α expansions of $\log\alpha$

Theorem 27.

$$\log\alpha = P(1, \alpha, 2, (0, -\beta)). \tag{45}$$

Proof. We have

$$\log\alpha = \frac{1}{2}\,\mathrm{Li}_1\left(\frac{1}{\alpha}\right) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{\alpha^k}\,\frac{1/\alpha}{k+1} = \sum_{k=0}^{\infty}\frac{1}{\alpha^k}\,\frac{-\beta}{2k+2}.$$

□

Theorem 28.

$$\log\alpha = P(1, \alpha^2, 2, (0, 2\beta^2)). \tag{46}$$

Proof. We have

$$\log\alpha = \mathrm{Li}_1\left(\frac{1}{\alpha^2}\right) = \sum_{k=0}^{\infty}\frac{1}{\alpha^{2k}}\frac{1/\alpha^2}{k+1} = \sum_{k=0}^{\infty}\frac{1}{\alpha^{2k}}\frac{2\beta^2}{2k+2}.$$

□

5.2 Another base-α expansion of $\log 2$

Theorem 29.

$$\log 2 = P(1, \alpha^3, 3, (-\beta, \beta^2, 2\beta^3)). \tag{47}$$

Proof. A straightforward consequence of the identity

$$\log 2 = \mathrm{Li}_1\left(\frac{1}{\alpha}\right) - \mathrm{Li}_1\left(\frac{1}{\alpha^3}\right).$$

□

5.3 Another base-α expansion of $\log 5$

Theorem 30.

$$\log 5 = P(1, \alpha^4, 2, (4\,\beta^2, 0)). \tag{48}$$

Proof. A consequence of the identity

$$\log 5 = 2\ \mathrm{Li}_1\left(\frac{1}{\alpha^2}\right) - 2\ \mathrm{Li}_1\left(-\frac{1}{\alpha^2}\right).$$

□

5.4 A length 2, base-α zero relation

Theorem 31.

$$P(1, \alpha^2, 2, (1, 3\,\beta)) = 0. \tag{49}$$

Proof. Follows from

$$\mathrm{Li}_1\left(\frac{1}{\alpha^2}\right) + \mathrm{Li}_1\left(-\frac{1}{\alpha}\right) = 0.$$

□

Remark 32. Relation (49) also follows from (45) and (46).

5.5 A length 12, base-α zero relation

Theorem 33.

$$P(1, \alpha^{12}, 12, (1, \beta, -2\,\beta^2, 5\,\beta^3, \beta^4, 10\,\beta^5, \beta^6, 5\,\beta^7, -2\,\beta^8, \beta^9, \beta^{10}, 2\,\beta^{11})) = 0.$$

Proof. Follows fron (19) and (47). □

5.6 A length 10, base-α zero relation

Theorem 34.

$$P(1, \alpha^{20}, 10, (1, -5\,\beta^2, \beta^4, -5\,\beta^6, -4\,\beta^8, -5\,\beta^{10}, \beta^{12}, -5\,\beta^{14}, \beta^{16}, 0)) = 0.$$

Proof. Ensues from (20) and (48). □

5.7 A length 5, base-α zero relation

Theorem 35.

$$P(1, \alpha^5, 5, (\beta, 1, -\beta, -\beta^4, -2\,\beta^4)) = 0. \tag{50}$$

Proof. Setting $p = 2\cos x$ in the identity

$$\sum_{k=1}^{\infty} \frac{p^k \cos(kx)}{k} = -\frac{1}{2}\log(1 - 2p\cos x + p^2)$$

produces

$$\sum_{k=1}^{\infty} \frac{(2\cos x)^k \cos(kx)}{k} = 0. \tag{51}$$

Now $2\cos(2\pi/5) = -\beta$. □

Thus, setting $x = 2\pi/5$ in (51) gives

$$\sum_{k=0}^{\infty} \frac{1}{\alpha^{5k}} \left(\frac{\beta}{5k+1} + \frac{1}{5k+2} - \frac{\beta}{5k+3} - \frac{\beta^4}{5k+4} - \frac{2\beta^4}{5k+5} \right) = 0;$$

since

$$\cos\left(\frac{2\pi}{5}(5j-4)\right) = \frac{-\beta}{2} = \cos\left(\frac{2\pi}{5}(5j-1)\right), \quad j = 1, 2, \ldots$$

and

$$\cos\left(\frac{2\pi}{5}(5j-2)\right) = \frac{1}{2\beta} = \cos\left(\frac{2\pi}{5}(5j-3)\right), \quad j = 1, 2, \ldots$$

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