The Arithmetic Partial Derivative

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Abstract

The arithmetic partial derivative (with respect to a prime p) is a function from the set of integers that sends p to 1 and satisfies the Leibniz rule. In this paper, we prove that the p-adic valuation of the sequence of higher order partial derivatives is eventually periodic. We also prove a criterion to determine when an integer has integral anti-partial derivatives. As an application, we show that there are infinitely many integers with exactly n integral anti-partial derivatives for every nonnegative integer n.

1 Introduction

Let p be a prime number. We recall the p-adic valuation $\mathrm{ord}_p : \mathbb{Z} \to \mathbb{N} \cup \{+\infty\}$. For $x \in \mathbb{Z}$, we have

$$\mathrm{ord}_p(x) = \begin{cases} \max\{\ell \in \mathbb{N} : p^\ell \mid x\}, & \text{if } x \neq 0; \\ +\infty, & \text{if } x = 0. \end{cases}$$

In other words, if $x \neq 0$, then $\mathrm{ord}_p(x)$ is the largest integer ℓ such that $p^\ell \mid x$. We sometimes denote this by $p^\ell \mid\mid x$.

The arithmetic derivative of a nonzero integer x is a function $D : \mathbb{Z}\backslash\{0\} \to \mathbb{Z}$ defined as

$$D(x) = x \sum_{p\mid x} \frac{\mathrm{ord}_p(x)}{p}.$$

If $x = p^k$ is a prime power, then

$$D(p^k) = p^k \cdot \frac{k}{p} = kp^{k-1}.$$

This shows that D mimics the power rule for differentiable functions on prime powers. One can check that D satisfies the Leibniz rule $D(xy) = xD(y) + yD(x)$, and sends every prime to 1. By the Leibniz rule, we are able to extend D uniquely to $\mathbb{Z}$ by defining $D(0) = 0$.

Ufnarovski and Åhlander [6] made several conjectures related to this arithmetic derivative, two of which we will investigate here. The first conjecture relates to the dynamical system by considering higher order arithmetic derivatives of an integer. We denote by $D^i(x)$ the i-th arithmetic derivative of x for $i \geq 1$.

We note that the first conjecture that we would like to address is not correct as stated. The original conjecture [6, Conjecture 2] states that for every positive integer x, exactly one of the following can happen: either $D^i(x) = 0$ for sufficiently large i, $\lim_{i \to +\infty} D^i(x) = +\infty$, or $x = p^p$ for some prime p. The fact that $D(1647082) = 823543 = 7^7$ means that it is not quite true. We modify it slightly here to the following conjecture.

Conjecture 1. For every positive integer x, exactly one of the following can happen: either $D^i(x) = 0$ or p^p for some prime p for sufficiently large i, or $\lim_{i \to +\infty} D^i(x) = +\infty$.

The second conjecture relates to the number of anti-derivatives of an integer. An integer y is said to be an (integral) anti-derivative of x if $D(y) = x$. Unlike anti-derivatives of functions, x has only finitely many anti-derivatives [6, Corollary 3] for every integer $x > 1$. We denote by $i(x)$ the number of anti-derivatives of an integer x.

Conjecture 2. For each nonnegative integer n, the number of solutions to the equation $i(x) = n$ is infinite.

Part of what makes investigating the function $D(x)$ challenging is that the derivative of an integer x will often have prime factors that are not factors of x itself. For instance, if $x = 72 = 2^3 \, 3^2$, then $D(x) = 156 = 2^2 \, 3^1 \, 13$, and we see that while the multiplicities of the prime factors 2 and 3 each decreased by 1, the derivative $D(x)$ has a factor of 13. In particular, when investigating the dynamical system $D^i(x)$, we would have to keep track of new prime factors that may arise following each successive differentiation. If x has more than one prime factor, predicting new prime factors of $D(x)$ relies on the ability of predicting prime factors of $a + b$ when knowing the prime factors of a and b. A difficult conjecture that tries to tackle a question in this direction is the famous abc conjecture. There is still ongoing debate in the mathematics community about whether abc conjecture has been proved [5]. Loosely speaking, the abc conjecture implies that when two numbers a and b are divisible by large powers of small primes, then $a + b$ tends to be divisible by small powers of large primes. Because of difficulties like these, we start by investigating these conjectures in the case of arithmetic partial derivatives with respect to a prime p, which was formally introduced by Kovič [3].

For nonzero $x \in \mathbb{Z}$, the (arithmetic) partial derivative (with respect to p) is defined as

$$D_p(x) = \frac{x}{p} \cdot \text{ord}_p(x).$$

One can check that D_p satisfies the Leibniz rule $D_p(xy) = D_p(x)y + xD_p(y)$. And to extend the definition of D_p to 0, we observe that for every nonzero $x \in \mathbb{Z}$ we have

$$D_p(0) = D_p(0 \cdot x) = D_p(0) \cdot x + 0 \cdot D_p(x) = D_p(0) \cdot x.$$

This implies that $D_p(0) = 0$. If p does not divide x, then $D_p(x) = 0$. For every integer x, the arithmetic derivative of x is a finite sum of arithmetic partial derivative of x, that is,

$$D(x) = \sum_{p|x} D_p(x).$$

The main advantage that we have in exploring the dynamical system $D_p^i(x)$ over $D^i(x)$ is that now we do not need to keep track of new primes that may arise when we differentiate. This is because if $x = ap^\ell$, where $\ell = \text{ord}_p(x)$, then $D_p(x) = aD_p(p^\ell)$, and so upon repeated differentiation, we only need to keep track of $\text{ord}_p(x)$ and its partial derivatives with respect to p. Any new prime factors that may arise when differentiating get absorbed into the coefficient.

Because we do not have to deal with new primes occurring in our dynamical system, we are able to prove a stronger version of Conjecture 1 for the arithmetic partial derivative. Theorem 8 provides an exact prescription of the ord_p sequence of x when $\text{ord}_p(x) \geq p$, and as a corollary we prove Theorem 10, which is analogous to Conjecture 1; see Section 2 for the definition of ord_p sequence. We go on to prove in Theorem 11 that given any possible ord_p sequence as prescribed in Theorem 8, we are able to construct an integer x with this ord_p sequence.

For each integer x, we say that an integer y is an (integral) anti-partial derivative of x if $D_p(y) = x$. Let $i_p(x)$ be the number of anti-partial derivatives of x. Theorem 21 and Example 13 together show that for each nonnegative integer n, the number of solutions to the equation $i_p(x) = n$ is infinite, which is analogous to Conjecture 2. We think the natural next step is to study analogous conjectures for the arithmetic subderivative D_S defined by Haukkanen, Merikoski, and Tossavainen [2] in the following way:

$$D_S(x) := x \sum_{p \in S} \frac{\text{ord}_p(x)}{p} = \sum_{p \in S} D_p(x),$$

where S is a finite set of primes.

The definition of arithmetic derivative and arithmetic partial derivative can be generalized to rational numbers [6, 1]. Haukkanen, Merikoski, and Tossavainen [1, Conjecture 27] conjecture that for every nonzero rational number x, there are at most four rational anti-partial derivatives if $p = 2$, and at most two rational anti-partial derivatives if $p > 2$. As a result of Theorem 21, we prove that this conjecture is false; see Remark 22. This also confirms Pandey and Saxena's earlier observations [4] that [1, Conjecture 27] might be false.

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2 Period of the ord_p sequence

We fix a prime p. For $x \in \mathbb{Z}$, we refer to the sequence $x, D_p(x), D_p^2(x), \ldots$ as the D_p sequence of x, and the sequence $\text{ord}_p(x), \text{ord}_p(D_p(x)), \text{ord}_p(D_p^2(x)), \ldots$ as the ord_p sequence of x. In this section, we will study properties of ord_p sequences and show that they are eventually periodic. For the sake of completeness, we first give the definition of eventually periodic.

Definition 3. Given a sequence $x_1, x_2, \ldots$, we say that it is *eventually periodic* if there exist integers $m, L \geq 1$ such that for all $0 \leq j \leq L - 1$ and $i \geq 1$, we have $x_{m+j+iL} = x_{m+j}$. If a sequence is eventually periodic with $m = 1$, then we say the sequence is *periodic*. The smallest possible L is called the *period* of the sequence.

If the D_p sequence of x is periodic (resp., eventually periodic), then the ord_p sequence of x is periodic (resp., eventually periodic), but not vice versa.

Lemma 4. *Let $x \in \mathbb{Z}$. If $x = 0$, then the D_p sequence of x (and thus the ord_p sequence of x) is periodic with period 1. If $\text{ord}_p(x) < p$, then the D_p sequence of x (and thus the ord_p sequence of x) is eventually periodic with period 1.*

Proof. If $x = 0$, then $D_p(x) = 0$ and thus $D_p^i(x) = 0$ for all $i \geq 0$. Therefore the D_p sequence of x is periodic of period 1.

Suppose $x \neq 0$. We write $x = ap^\ell$ where $p \nmid a$ and $\text{ord}_p(x) = \ell < p$. We can compute that $D_p^\ell(x) = a\ell!$. Since $0 \leq \ell < p$, we know that $p \nmid a\ell!$. This implies that $D_p^{\ell+1}(x) = 0$ and thus $D_p^i(x) = 0$ for all $i \geq \ell + 1$. Therefore the D_p sequence of x is eventually periodic of period 1. $\square$

Suppose $\text{ord}_p(D_p^i(x)) \neq +\infty$ for all $i \geq 1$. To understand how each term in ord_p sequence changes, we study the sequence of increments (of consecutive terms) of the ord_p sequence

$$\text{ord}_p(D_p(x)) - \text{ord}_p(x), \ \text{ord}_p(D_p^2(x)) - \text{ord}_p(D_p(x)), \ \text{ord}_p(D_p^3(x)) - \text{ord}_p(D_p^2(x)), \ \ldots$$

and we call this the inc_p sequence of x. Recall that for integers $x \in \mathbb{Z}$, we let $x \bmod p$ denote the unique integer between 0 and $p - 1$ that is congruent to x modulo p. We denote $\lfloor x \rfloor_p := x - (x \bmod p)$. In other words, $\lfloor x \rfloor_p$ is the largest multiple of p that is less than or equal to x.

We are ready to handle the case $\text{ord}_p(x) \geq p$.

Lemma 5. *Let $x = ap^{bp^k}$ be a nonzero integer with $p \nmid ab$ and $k \geq 1$. Define $k' := (k - 1 \bmod p) + 1$. The first k' terms of the inc_p sequence of x are*

$$(k - 1, \underbrace{-1, -1, \ldots, -1}_{(k-1 \bmod p) \ copies}).$$

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Proof. Since $D_p(x) = abp^{bp^k+k-1}$ and $p \nmid ab$, we know $\mathrm{ord}_p(D_p(x)) = bp^k + k - 1$. If $k' \geq 2$, then $D_p^2(x) = ab(bp^k + k - 1)p^{bp^k+k-2}$. Since $p \nmid ab(bp^k + k - 1)$, we know $\mathrm{ord}_p(D_p^2(x)) = bp^k + k - 2$. Following this pattern, we know that when $1 \leq i \leq k'$, the $(i+1)$-th term of the D_p sequence is

$$ab \prod_{j=1}^{i-1}(bp^k + k - j)p^{bp^k+k-i}$$

because $p \nmid ab \prod_{j=1}^{i-1}(bp^k + k - j)$ for $1 \leq i \leq k'$. The first $k' + 1$ terms of the ord_p sequence are

$$(bp^k, bp^k + k - 1, bp^k + k - 2, \ldots, bp^k + k - k').$$

Therefore the first k' terms of the inc_p sequence of x are

$$(k - 1, \underbrace{-1, -1, \ldots, -1}_{(k-1 \bmod p) \text{ copies}}). \qquad \square$$

Corollary 6. *Let $x = ap^{bp^k}$ be a nonzero integer with $p \nmid ab$ and $1 \leq k \leq p$. The ord_p sequence and the inc_p sequence of x is periodic of period k.*

Proof. If $1 \leq k \leq p$, then $k' = (k - 1 \bmod p) + 1 = k - 1 + 1 = k$. The first $k + 1$ terms of the ord_p sequence are

$$(bp^k, bp^k + k - 1, bp^k + k - 2, \ldots, bp^k).$$

It is now clear that the ord_p sequence and the inc_p sequence of x is periodic of period k. $\quad \square$

Definition 7. *For integers $k \geq 1$, we call the following sequence*

$$\mathcal{S}_k := (k - 1, \underbrace{-1, -1, \ldots, -1}_{(k-1 \bmod p) \text{ copies}})$$

the k-segment.

For integers $\ell \geq p$, we define $\ell_1 := \mathrm{ord}_p(\lfloor \ell \rfloor_p)$ and for $i \geq 2$

$$\ell_i := \begin{cases} \mathrm{ord}_p(\lfloor \ell_{i-1} - 1 \rfloor_p), & \text{if } \ell_{i-1} < +\infty; \\ +\infty, & \text{if } \ell_{i-1} = +\infty. \end{cases} \tag{1}$$

It is clear that if $\ell_i < +\infty$, then $\ell_{i+1} < \log_p(\ell_i)$. For every integer $\ell \geq p$, there exists a unique positive integer $N = N(\ell)$ such that $1 \leq \ell_N \leq p$, and $\ell_i = +\infty$ for all $i > N$.

Theorem 8. *Let $x = ap^\ell$ be a nonzero integer with $\ell \geq p$ and $p \nmid a$. The inc_p sequence of x is of the form*

$$(\underbrace{-1, -1, \ldots, -1}_{(\ell \bmod p) \text{ copies}}, \mathcal{S}_{\ell_1}, \mathcal{S}_{\ell_2}, \mathcal{S}_{\ell_3}, \ldots, \mathcal{S}_{\ell_N}, \mathcal{S}_{\ell_N}, \mathcal{S}_{\ell_N}, \ldots).$$

As a result, the ord_p sequence and the inc_p sequence of x are eventually periodic of period ℓ_N.

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Proof. Let $\ell_0' = \ell \bmod p$. Then the first $\ell_0' + 1$ terms of the D_p sequence of x are

$$(ap^\ell, a\ell p^{\ell-1}, a\ell(\ell-1)p^{\ell-2}, \ldots, a\prod_{i=1}^{\ell_0'}(\ell-i)p^{\ell-\ell_0'}).$$

The first $\ell_0' + 1$ terms of the ord_p sequence of x are

$$(\ell, \ell-1, \ell-2, \ldots, \ell-\ell_0').$$

Hence the first ℓ_0' terms of the inc_p sequence of x are

$$\underbrace{(-1, -1, \ldots, -1)}_{\ell_0' \text{ copies}}.$$

Note that $\ell - \ell_0' = \lfloor \ell \rfloor_p$ and $\ell_1 = \mathrm{ord}_p(\lfloor \ell \rfloor_p)$, we can write $D_p^{\ell_0'}(x) = a_0 p^{b_0 p^{\ell_1}}$ with $p \nmid a_0 b_0$. We can compute $D_p^{\ell_0'+1}(x) = a_0 b_0 p^{b_0 p^{\ell_1}+\ell_1-1}$. Note that $b_0 p^{\ell_1} + \ell_1 - 1 \bmod p = \ell_1 - 1 \bmod p$.

Let $\ell_1' := (\ell_1 - 1) \bmod p$. The $(\ell_0' + 1)$-th term to the $(\ell_0' + \ell_1' + 2)$-th term of the D_p sequence of x are

$$(a_0 p^{b_0 p^{\ell_1}}, a_0 b_0 p^{b_0 p^{\ell_1}+\ell_1-1}, \ldots, a_0 b_0 \prod_{i=1}^{\ell_1'}(b_0 p^{\ell_1}+\ell_1-i)p^{b_0 p^{\ell_1}+\ell_1-1-\ell_1'}).$$

The $(\ell_0' + 1)$-th term to the $(\ell_0' + \ell_1' + 2)$-th term of the ord_p sequence of x are

$$(b_0 p^{\ell_1}, b_0 p^{\ell_1} + \ell_1 - 1, \ldots, b_0 p^{\ell_1} + \ell_1 - 1 - \ell_1').$$

The next $\ell_1' + 1$ terms of the inc_p sequence of x is the ℓ_1-segment

$$(\ell_1 - 1, \underbrace{-1, -1, \ldots, -1}_{\ell_1' \text{ copies}}) = \mathcal{S}_{\ell_1}.$$

As $\ell_1 - 1 - \ell_1' = \lfloor \ell_1 - 1 \rfloor_p$ and $\ell_2 = \mathrm{ord}_p(\lfloor \ell_1 - 1 \rfloor_p)$, we can write $D_p^{\ell_0'+\ell_1'+1}(x) = a_1 p^{b_1 p^{\ell_2}}$ with $p \nmid a_1 b_1$. There exists a positive integer N such that $1 \leq \ell_N \leq p$. Using induction, we know that the initial terms of the inc_p sequence of x is of the form

$$(\underbrace{-1, -1, \ldots, -1}_{(\ell \bmod p) \text{ copies}}, \mathcal{S}_{\ell_1}, \mathcal{S}_{\ell_2}, \mathcal{S}_{\ell_3}, \ldots, \mathcal{S}_{\ell_N}).$$

Corollary 6 implies that if $a_{N-1} p^{b_{N-1} p^{\ell_N}}$, where $p \nmid a_{N-1} b_{N-1}$, is a term in the D_p sequence of x, then $\mathcal{S}_{\ell_N}$ will appear repeatedly in the inc_p sequence of x. This completes the proof of the theorem. $\qquad \square$

We summarize our results of Lemma 4 and Theorem 8 into the following theorem.

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Theorem 9. *The ord_p sequence of any integer x is eventually periodic of period at most p.*

As an application of Theorem 8, we can prove the following theorem that has a similar structure as Conjecture 1.

Theorem 10. *For any $x \in \mathbb{Z}$, exactly one of the following three cases will happen:*

(i) $D_p^i(x) = 0$ *for sufficiently large i;*

(ii) $D_p^i(x) = ap^p$ *where $\gcd(a, p) = 1$ for sufficiently large i;*

(iii) $\lim\limits_{i \to +\infty} D_p^i(x) = \pm\infty$.

Proof. Suppose $x = 0$ or $0 \le \mathrm{ord}_p(x) < p$. Lemma 4 implies that $D_p^i(x) = 0$ for sufficiently large i.

Suppose $\mathrm{ord}_p(x) \ge p$. Theorem 8 implies that the ord_p sequence of x is eventually periodic of period ℓ_N. Let $x_1 = ap^{bp^k}$ be the first term of the periodic cycle where $1 \le k \le p$ and $p \nmid ab$. If $b = k = 1$, then $D_p(x_1) = D_p(ap^p) = ap^p = x_1$. We know that $D_p^i(x) = ap^p$ for sufficiently large i. Suppose $b > 1$ or $k > 1$. If $x_1 > 0$, then

$$x_1 < D_p(x_1) < D_p^2(x_1) < \cdots < D_p^{\ell_{N_0}-1}(x_1) < D_p^{\ell_{N_0}}(x_1) = \alpha x_1,$$

for some $\alpha > 1$. Therefore

$$\lim_{i \to +\infty} D_p^i(x) = \lim_{i \to +\infty} \alpha^i x_1 = +\infty.$$

Similarly, if $x_1 < 0$, then

$$x_1 > D_p(x_1) > D_p^2(x_1) > \cdots > D_p^{\ell_{N_0}-1}(x_1) > D_p^{\ell_{N_0}}(x_1) = \alpha x_1,$$

for some $\alpha > 1$. Therefore

$$\lim_{i \to +\infty} D_p^i(x) = \lim_{i \to +\infty} \alpha^i x_1 = -\infty. \qquad \square$$

Theorem 8 gives the form of the inc_p sequence of any integer x such that $\mathrm{ord}_p(x) \ge p$. It is a sequence of nonnegative integers that rapidly decreases until it is less than p, after which it is constant, where the first term determines all subsequent terms. This sequence can be interrupted by strings of -1's, where the length of the string is equal to the previous nonnegative integer modulo p.

The question remains about whether all sequences of the form

$$(\underbrace{-1, -1, \ldots, -1}_{(\ell \bmod p) \text{ copies}}, \mathcal{S}_{\ell_1}, \mathcal{S}_{\ell_2}, \mathcal{S}_{\ell_3}, \ldots, \mathcal{S}_{\ell_N}, \mathcal{S}_{\ell_N}, \mathcal{S}_{\ell_N}, \ldots)$$

has an x such that this is its inc_p sequence. Since the initial ℓ_1 determines all the subsequent ℓ_i's, the answer is no. But we do have some control over the length of the interrupting -1's and the number of jumps N.

Theorem 11. *Let $N > 0$ be an integer. For $N + 1$ integers $0 \leq i_0, i_1, i_2, \ldots, i_N \leq p - 1$, there exists a positive integer ℓ such that the inc_p sequence of $x = p^\ell$ is of the form*

$$(\underbrace{-1, -1, \ldots, -1}_{i_0 \text{ copies}}, \mathcal{S}_{\ell_1}, \mathcal{S}_{\ell_2}, \mathcal{S}_{\ell_3}, \ldots, \mathcal{S}_{\ell_N}, \mathcal{S}_{\ell_N}, \mathcal{S}_{\ell_N}, \ldots),$$

such that for all $1 \leq j \leq N$, $\mathcal{S}_{\ell_j}$ is the ℓ_j-segment with i_j consecutive copies of -1.

Proof. Given a sequence of $N + 1$ nonnegative integers $0 \leq i_0, i_1, i_2, \ldots, i_N \leq p - 1$, we construct a sequence $k_N, k_{N-1}, \ldots, k_1, k_0$ as follows. Let $k_N = i_N + 1$ and $k_j = p^{k_{j+1}} + i_j + 1$ for $0 \leq j \leq N - 1$. Let $\ell := k_0 - 1$ and $x = p^{k_0 - 1}$. We first show that $\ell_j = k_j$ for all $1 \leq j \leq N$. By definition of ℓ_1, we can compute

$$\ell_1 = \mathrm{ord}_p(\lfloor \ell \rfloor_p) = \mathrm{ord}_p(\lfloor k_0 - 1 \rfloor_p) = \mathrm{ord}_p(\lfloor p^{k_1} + i_0 \rfloor_p) = \mathrm{ord}_p(p^{k_1}) = k_1.$$

Suppose for $2 \leq j \leq m - 1 < N$, we have $\ell_j = k_j$. Then we have

$$\ell_m = \mathrm{ord}_p(\lfloor \ell_{m-1} - 1 \rfloor_p) = \mathrm{ord}_p(\lfloor k_{m-1} - 1 \rfloor_p) = \mathrm{ord}_p(\lfloor p^{k_m} + i_{m-1} \rfloor_p) = \mathrm{ord}_p(p^{k_m}) = k_m.$$

It remains to show that the inc_p sequence of x is of the desired form. As $k_0 - 1 = p^{k_1} + i_0$, it is clear that the first i_0 terms of the inc_p sequence of x are -1 and $\mathrm{ord}_p(D_p^{i_0}(x)) = p^{k_1}$. Suppose $D_p^{i_0}(x) = \alpha p^{p^{k_1}}$. Set $k_1' := (k_1 - 1 \bmod p) + 1 = (i_1 \bmod p) + 1 = i_1 + 1$. By Lemma 5, we know that the next k_1' terms of the inc_p sequence is a k_1-segment $\mathcal{S}_{k_1} = \mathcal{S}_{\ell_1}$. The number of consecutive -1 in this k_1-segment $\mathcal{S}_{k_1}$ is equal to $k_1' - 1 = i_1$ as desired. The proof is complete by induction. $\qquad\square$

For a nonzero integer $x = ap^\ell$ with $p \nmid a$ and $\ell \geq 1$, Theorem 8 implies that the ord_p sequence increases (not necessarily consecutively) exactly $N - 1$ times before it reaches the periodic cycle. Therefore it is impossible for the ord_p-sequence to continue to increase indefinitely. However, we see that the number of jumps can be made arbitrarily large. And as the following corollary tells us, we can even prescribe these jumps to be consecutive.

Corollary 12. *For every integer $N \geq 1$, there exists $x \in \mathbb{Z}$ such that*

$$\mathrm{ord}_p(x) < \mathrm{ord}_p(D_p(x)) < \cdots < \mathrm{ord}_p(D_p^N(x)).$$

Proof. Use Theorem 11 by choosing $i_0 = i_1 = \cdots = i_N = 0$. $\qquad\square$

3 Anti-partial derivatives

Not every integer has an anti-partial derivative with respect to p. We would like to have a criterion to determine when an integer does have an anti-partial derivative (Question 1). If we know an integer has an anti-partial derivative, a natural extension of Question 1 is how many anti-partial derivatives can it have (Question 3)? In order to answer that, it

would be useful to know when two integers have the same partial derivatives (Question 2). Furthermore, if we know that there exists at least one integer that has exactly n anti-partial derivatives, then we can also ask how many integers there are that have exactly n anti-partial derivatives (Question 4). In this section, we will attempt to answer these four questions:

1. When does an integer have an anti-partial derivative?

2. When do two integers have the same partial derivatives?

3. How many anti-partial derivatives can an integer have?

4. For a positive integer n, how many integers have exactly n anti-partial derivatives?

For a nonzero integer x, if $\mathrm{ord}_p(x) > 0$, then there exist unique integers $a \neq 0$, $b > 0$, and $k \geq 0$ with $p \nmid ab$ such that $x = ap^{bp^k}$. We call this the *standard form* of x.

3.1 When does an integer have an anti-partial derivative?

We start this subsection with an example of a family of integers that does not have an anti-partial derivative.

Example 13. Suppose $y = a_0 p^{p-1}$ for some integer a_0 where $p \nmid a_0$. If $D_p(x) = y$, then it is clear that $x \neq 0$ and $\mathrm{ord}_p(x) > 0$. Let $x = ap^{bp^k}$ be in standard form such that $D_p(x) = a_0 p^{p-1}$. Then $abp^{bp^k+k-1} = a_0 p^{p-1}$. This implies that $bp^k + k = p$. As $b \geq 1$, we know that $k = 0$. Therefore $b = p$ and this contradicts to $p \nmid ab$. Therefore $a_0 p^{p-1}$ does not have an (integral) anti-partial derivative (with respect to p).

The main goal of this subsection is to determine when an integer y has an anti-partial derivative. We first consider two simple cases.

- If $y = 0$, then $D_p(a) = 0$ for all $a \in \mathbb{Z}$ with $p \nmid a$.

- If $\mathrm{ord}_p(y) = 0$, then $D_p(x) = y$ if and only if $x = yp$.

Now we assume that $\mathrm{ord}_p(y) > 0$. If $D_p(x) = y$, then $\mathrm{ord}_p(x) > 0$.

Lemma 14. *Let $y = a_0 p^{\ell_0}$ where $p \nmid a_0$ and $\ell_0 > 0$. For $x = ap^{bp^k} \in \mathbb{Z}$ in standard form, we have $D_p(x) = y$ if and only if*

$$ab = a_0, \tag{2}$$
$$bp^k + k - 1 = \ell_0. \tag{3}$$

Proof. Since $D_p(x) = abp^{bp^k+k-1}$, it is clear that $abp^{bp^k+k-1} = a_0 p^{\ell_0}$ if and only if (2) and (3) hold. $\square$

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Using Lemma 14, we can show that every nonzero integer has finitely many anti-partial derivatives. This is already known as a result of [1, Theorem 1].

Corollary 15. *For each nonzero $y \in \mathbb{Z}$, the set $\{x \in \mathbb{Z} : D_p(x) = y\}$ is finite (possibly empty).*

Proof. If $\operatorname{ord}_p(y) = 0$, then $D_p(x) = y$ if and only if $x = yp$. If $\operatorname{ord}_p(y) > 0$, then Eq. (3) implies that $k \leq \log_p(\ell_0)$. Since b is determined by k by (3) and a is determined by b by (2), there are only finitely many a, b, k such that (2) and (3) hold. $\square$

Corollary 16. *For two nonzero integers x_1 and x_2 with $\operatorname{ord}_p(x_1), \operatorname{ord}_p(x_2) > 0$, let $x_1 = a_1 p^{b_1 p^{k_1}}$ and $x_2 = a_2 p^{b_2 p^{k_2}}$ be in standard form such that $D_p(x_1) = D_p(x_2)$. The following three statements are equivalent.*

$$\text{(i) } k_1 < k_2 \qquad \text{(ii) } b_1 p^{k_1} > b_2 p^{k_2} \qquad \text{(iii) } b_1 > b_2$$

If $D_p(x_1) = D_p(x_2) > 0$, then the above three statements are further equivalent to the following two statements.

$$\text{(iv) } a_1 < a_2 \qquad \text{(v) } x_1 < x_2$$

Proof. If $D_p(x_1) = D_p(x_2)$, then we have

$$a_1 b_1 = a_2 b_2 \tag{4}$$
$$b_1 p^{k_1} + k_1 - 1 = b_2 p^{k_2} + k_2 - 1, \tag{5}$$

Eq. (5) implies that (i) and (ii) are equivalent. If (i) and (ii) hold, then (iii) holds. If (iii) holds, then Eq. (5) implies that (i) holds.

Now suppose $D_p(x_1) = D_p(x_2) > 0$. Hence $a_1, a_2, x_1, x_2 > 0$. Eq. (4) implies that (iii) and (iv) are equivalent. It remains to show that (v) is equivalent to (ii). Note that

$$\frac{x_1}{x_2} = \frac{a_1}{a_2} \cdot \frac{p^{b_1 p^{k_1}}}{p^{b_2 p^{k_2}}} = \frac{b_2}{b_1} p^{b_1 p^{k_1} - b_2 p^{k_2}} = \frac{b_2}{b_1} p^{k_2 - k_1} = \frac{b_2 p^{k_2}}{b_1 p^{k_1}}.$$

Therefore (v) is equivalent to (ii). $\square$

If $D_p(x_1) = D_p(x_2) < 0$, then (iv) and (v) of Corollary 16 need to be replaced by $a_1 > a_2$ and $x_1 > x_2$ respectively.

3.2 When do two integers have the same partial derivatives?

Anti-partial derivatives do not exist uniquely. For example

$$D_p(p^{p^{p+1}+p}) = D_p((p^p + 1)p^{p^{p+1}}).$$

The main goal of this subsection is to determine when two integers have the same partial derivatives. If $D_p(x) = 0$, then $x = 0$ or $\mathrm{ord}_p(x) = 0$. This means that the set of anti-partial derivatives of 0 is

$$\{x \in \mathbb{Z} : p \nmid x\} \cup \{0\}.$$

Now we suppose that $\mathrm{ord}_p(x) > 0$ and let $y = D_p(x)$. By Corollary 16, there is an element x_0 in the set of all anti-partial derivative of y with the smallest value k. We call x_0 the *primitive* anti-partial derivative of y and denote $x_0 = a_0 p^{b_0 p^{k_0}}$ in the standard form.

Theorem 17. *Let $x_0 = a_0 p^{b_0 p^{k_0}} \in \mathbb{Z}$ be the standard form of the primitive anti-partial derivative of $y = D_p(x_0)$. Define*

$$C = C(x_0) := \left\{ c \in [0, b_0) \cap \mathbb{Z} : p^{p^{k_0} c} \,||\, (b_0 - c), \; \frac{b_0 - c}{p^{p^{k_0} c}} \,\Big|\, a_0 b_0 \right\}.$$

There is a one-to-one correspondence between C and the set of all anti-partial derivatives of y. Furthermore, if $x = a p^{b p^k}$ (in its standard form) is an anti-partial derivative of y, then there exists a unique $c \in C$ such that

$$k = p^{k_0} c + k_0, \quad b = \frac{b_0 - c}{p^{k-k_0}} = \frac{b_0 - c}{p^{p^{k_0} c}} \in \mathbb{Z}, \quad a = \frac{a_0 b_0}{b} \in \mathbb{Z}.$$

Proof. We show that a anti-partial derivative $x = a p^{b p^k}$ of y is associated with a unique $c \in C$. If $x = x_0$, then we associated x with $c = 0$. If $x \neq x_0$, since x_0 is the primitive anti-partial derivative, we know that $k > k_0$. Then $p^{k_0}(b_0 - b p^{k-k_0}) = k - k_0$. Since $p \nmid b b_0$, we know that $p \nmid (b_0 - b p^{k-k_0})$. Therefore $p^{k_0} \,||\, (k - k_0)$. Let $k - k_0 = p^{k_0} c$ where $c > 0$ and $p \nmid c$. By plugging $k - k_0 = p^{k_0} c$ into $p^{k_0}(b_0 - b p^{k-k_0}) = k - k_0$, we get $b_0 - b p^{k-k_0} = c$. Since $b p^{k-k_0} > 0$, we know that $c < b_0$. Since $p \nmid b$, we know that $p^{k-k_0} \,||\, (b_0 - c)$. Since $a \in \mathbb{Z}$, we also have $b \mid a_0 b_0$, that is, $\frac{b_0 - c}{p^{p^{k_0} c}} \mid a_0 b_0$.

Then we show that for each $c \in C$, we can define a unique $x = a p^{b p^k}$ such that $D_p(x) = y$. Since $c < b_0$ and $p^{p^{k_0} c} \,||\, (b_0 - c)$, we know that $b > 0$ is an integer and $\mathrm{ord}_p(b) = 0$. Since $b \mid a_0 b_0$, we know that $a \in \mathbb{Z}$ and $\mathrm{ord}_p(a) = 0$ as well. We can compute

$$b p^k + k - 1 = \frac{b_0 - c}{p^{k-k_0}} p^k + k - 1 = (b_0 - c) p^{k_0} + p^{k_0} c + k_0 - 1 = b_0 p^{k_0} + k_0 - 1.$$

Therefore

$$D_p(x) = a b p^{b p^k + k - 1} = a_0 b_0 p^{b_0 p^{k_0} + k_0 - 1} = D_p(x_0) = y. \qquad \square$$

3.3 How many anti-partial derivatives can an integer have?

By Corollary 15, we know that every nonzero integer can only have finitely many anti-partial derivatives. This partially answers Question 3, but we can further ask for a given positive integer n, is there an integer x with exactly n anti-partial derivatives? The answer is yes. In

order to prove that, we will construct k_0 such that x_0 is the primitive anti-partial derivative of $D_p(x_0)$ and then construct a_0 and b_0 so that $D_p(x_0)$ has exactly n anti-partial derivatives. We first show that for certain values of k_0, the quantity x_0 is always the primitive anti-partial derivative of $D_p(x_0)$ no matter how a_0 and b_0 are defined as long as $p \nmid a_0 b_0$.

Lemma 18. *Fix an integer $m \geq 2$ and let $k_0 := p + p^2 + p^3 + \cdots + p^m$. For integers a_0, b_0 with $b_0 > 0$ such that $p \nmid a_0 b_0$, let $x_0 := a_0 p^{b_0 p^{k_0}}$. Then x_0 is the primitive anti-partial derivative of $D_p(x_0)$.*

Proof. Suppose x_0 is not the primitive anti-partial derivative of $D_p(x_0)$. Let $x = a p^{b p^k}$ be an anti-partial derivative of $D_p(x_0)$ such that $0 \leq k < k_0$. Since $D_p(x) = D_p(x_0)$, we get $b p^k + k = b_0 p^{k_0} + k_0$. This means that $p^k \mid\mid k_0 - k$. It is clear that $k \neq 0$ because $p \mid\mid k_0 - 0$. It is also clear that $k \neq 1$ because $p \nmid k_0 - 1$. Suppose $k \geq 2$. If $p^k \mid\mid k_0 - \ell$ for some $\ell \in \mathbb{Z}$, then $\ell \geq p + p^2 + \cdots + p^{k-1} > k$. Therefore there does not exist an anti-partial derivative $x = a p^{b p^k}$ with $k < k_0$. This means that x_0 is the primitive anti-partial derivative of $D_p(x_0)$. $\square$

We now construct b_0 such that there are exactly $n - 1$ different possible values of $c \in (0, b_0) \cap \mathbb{Z}$ such that $p^{p^{k_0} c} \mid\mid (b_0 - c)$ for $k_0 \geq 0$. This means that the set $C(x_0)$ (as defined in Theorem 17) has at most n elements (including 0).

Lemma 19. *Fix integers $n > 0$ and $k_0 \geq 0$. Let $c_1 = 0$. For $2 \leq i \leq n + 1$, let $c_i := p^{p^{k_0} c_{i-1}} + c_{i-1}$ and $b_0 := c_{n+1}$. We have*

$$\{c \in (0, b_0) \cap \mathbb{Z} \ : \ p^{p^{k_0} c} \mid\mid b_0 - c\} = \{c_2, \ldots, c_n\}.$$

Proof. First, we prove $\supseteq$. By the definition of c_j, we know that $c_{j+1} > c_j$ and $p^{p^{k_0} c_j} \mid\mid c_{j+1} - c_j$ for all $1 \leq j \leq n$. For $1 \leq i \leq n$, we have

$$b_0 - c_i = \sum_{j=i}^{n} (c_{j+1} - c_j) = \sum_{j=i}^{n} p^{p^{k_0} c_j}.$$

Therefore $p^{p^{k_0} c_i} \mid\mid b_0 - c_i$.

Second, we prove $\subseteq$. Let $c \in (0, b_0)$ be an integer such that $p^{p^{k_0} c} \mid\mid b_0 - c$. If $c \notin \{c_2, \ldots, c_n\}$, then there exists a unique $3 \leq i \leq n + 1$ such that $c_{i-1} < c < c_i$. As

$$b_0 - c = \sum_{j=i}^{n} (c_{j+1} - c_j) + (c_i - c) = \sum_{j=i}^{n} p^{p^{k_0} c_j} + (c_i - c).$$

Since $c < c_j$ for all $i \leq j \leq n$, we have $p^{p^{k_0} c} \mid p^{p^{k_0} c_j}$. Since $p^{p^{k_0} c} \mid\mid b_0 - c$, we know that $p^{p^{k_0} c} \mid\mid c_i - c$. Let $\ell > 0$ with $p \nmid \ell$ such that $c_i = \ell p^{p^{k_0} c} + c$. By definition of c_i, we know that $c_i = p^{p^{k_0} c_{i-1}} + c_{i-1}$. This means that

$$\ell p^{p^{k_0} c} + c = p^{p^{k_0} c_{i-1}} + c_{i-1}.$$

Since $c_{i-1} < c$, we know that $\ell p^{p^{k_0} c} < p^{p^{k_0} c_{i-1}}$. But this is not possible as $\ell \geq 1$ and $c > c_{i-1}$. Hence there does not exist an integer $c \in (0, b_0)$ such that $p^{p^{k_0} c} \mid\mid b_0 - c$ and $c \notin \{c_2, \ldots, c_n\}$. This completes the proof of $\subseteq$. $\square$

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In order for $D_p(x_0)$ to have exactly n anti-partial derivatives, we want to construct a_0 so that

$$\frac{b_0 - c}{p^{p^{k_0}c}} \mid a_0 b_0$$

for every $0 < c < b_0$ satisfying $p^{p^{k_0}c} \| b_0 - c$. This can be achieved by the next lemma.

Lemma 20. *Fix integers $n > 0$ and $k_0 \geq 0$. Let $c_1, c_2, \ldots, c_{n+1} = b_0$ be defined as in Lemma 19. For each $2 \leq i \leq n$, let $a_i := (b_0 - c_i)/p^{p^{k_0}c_i}$ and let $a_0 := \prod_{j=2}^{n} a_j$. Then $p \nmid a_0$ and the set $C(x_0)$ (as defined in Theorem 17) has exactly n elements.*

Proof. Lemma 19 shows that $C(x_0) \subseteq \{c_1, c_2, \ldots, c_n\}$. For each $2 \leq i \leq n$, we know that $(b_0 - c_i)/p^{p^{k_0}c_i} = a_i$ divides a_0 by the definition of a_0. Hence it divides $a_0 b_0$. Since $p \nmid a_i$ for $2 \leq i \leq n$, we know that $p \nmid a_0$. Therefore $C(x_0) = \{c_1, c_2, \ldots, c_n\}$. $\qquad\square$

Now we are ready to give a fuller answer to Question 3 as well as an answer to Question 4 at the same time.

Theorem 21. *For each positive integer n, there are infinitely many integers x_0 such that $D_p(x_0)$ has exactly n anti-partial derivatives.*

Proof. Lemma 18 implies that there are infinitely many k_0 such that x_0 is the primitive anti-partial derivative of $D_p(x_0)$. Lemmas 19 and 20 imply that for a given k_0, we can construct a_0 and b_0 so that $x_0 = a_0 p^{b_0 p^{k_0}}$ is the primitive anti-partial derivative of $D_p(x_0)$ and $D_p(x_0)$ has exactly n anti-partial derivatives. Therefore, for each positive integer n, there exists infinitely many integers $x_0 = a_0 p^{b_0 p^{k_0}}$ such that $D_p(x_0)$ has exactly n anti-partial derivatives. $\qquad\square$

Remark 22. For an nteger x_0, let $\mathcal{D}_{\mathbb{Z},p}(x_0)$ (resp., $\mathcal{D}_{\mathbb{Q},p}(x_0)$) be the set of all integral (resp., rational) anti-partial derivatives of $D_p(x_0)$. It is clear that $\mathcal{D}_{\mathbb{Z}}(x_0) \subset \mathcal{D}_{\mathbb{Q}}(x_0)$. Therefore Theorem 21 implies that [1, Conjecture 27] is false.

Theorem 21 remains true even if we allow rational anti-partial derivatives. We give a brief explanation as follows. Let $x_0 := a_0 p^{b_0 p^{k_0}}$ be defined as in Lemma 18, so x_0 is the primitive anti-partial derivative of $D_p(x_0)$ (even among all the rational anti-partial derivatives) with $b_0 > 0$. Let $x \neq x_0$ be a rational anti-partial derivative of $D_p(x_0)$. We can write $x = a p^{b p^k}$ where $0 \neq a \in \mathbb{Q}$, $0 \neq b \in \mathbb{Z}$, $k \in \mathbb{Z}$, $k \geq 0$, and $\mathrm{ord}_p(ab) \neq 0$. Since x_0 is the primitive anti-partial derivative of $D_p(x_0)$, we have $k > k_0$. As $D_p(x) = D_p(x_0)$, we have $b_0 p^{k_0} + k_0 - 1 = b p^k + k - 1$, or $p^{k_0}(b_0 - b p^{k-k_0}) = k - k_0$. If $b < 0$, then $k - k_0 < -b p^{k-k_0} < b_0 - b p^{k-k_0} < p^{k_0}(b_0 - b p^{k-k_0})$. This is a contradiction, so $b > 0$. This is saying that if x_0 is the primitive rational anti-partial derivative of $D_p(x_0)$ with $\mathrm{ord}_p(x_0) > 0$, then every other rational anti-partial derivative x of $D_p(x_0)$ also satisfies $\mathrm{ord}_p(x) > 0$. In this case, there is a one-to-one correspondence between the set of all rational anti-partial derivatives of $D_p(x_0)$ and the following set

$$C_{\mathbb{Q}}(x_0) := \left\{ c \in [0, b_0) \cap \mathbb{Z} : p^{p^{k_0}c} \| (b_0 - c) \right\}.$$

If we compare $C_{\mathbb{Q}}(x_0)$ with $C(x_0)$ defined in Theorem 17, we have dropped the extra condition $\frac{b_0-c}{p^{p^{k_0}c}} \mid a_0 b_0$. This is because we no longer need $a = a_0 b_0 / b$ to be an integer. Therefore Lemma 19 still applies, that is, we can find b_0 such that $C_{\mathbb{Q}}(x_0)$ has exactly n elements for every positive integer n. Hence, for every positive integer n, there are infinitely many integers x_0 such that $D_p(x_0)$ has exactly n *rational* anti-partial derivatives.

4   Acknowledgments

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