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Arithmetic Progressions of b-Prodigious Numbers

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Abstract

A positive integer n is called a *b-prodigious number* if n is divisible by the product of its non-zero base-b digits. In this article, we investigate the maximum length of an arithmetic progression of b-prodigious numbers and the maximal length of consecutive sequences of b-prodigious numbers.

1 Introduction

In the *On-Line Encyclopedia of Integer Sequences*, sequence A055471 [8] is the sequence of positive integers that are divisible by the product of their nonzero digits. On the French website Diophante [4], this sequence is called *nombres prodigieux*, which translates to *prodigious numbers*. In recognition of this translation, for an integer $b \geq 2$, we say that a positive integer n is a *b-prodigious number* if it is divisible by $p_b(n)$, which is defined as the product of the nonzero digits of n in its base-b representation. Clearly, every positive integer is a 2-prodigious number. For $b = 10$, De Koninck and Luca [3] showed that for large enough values of x, the number of b-prodigious numbers less than x, denoted by $N_0(x)$, satisfies $x^{0.495} < N_0(x) < x^{0.901}$. This result was later improved and generalized to arbitrary base by Sanna [6].

In a similar manner, a positive integer n is called a *Niven number* if it is divisible by the sum of its digits. More generally, for an integer $b \geq 2$, a *b-Niven number* is a positive integer n that is divisible by $s_b(n)$, which is defined as the sum of the digits of n in its base-b representation. In 1992, Cooper and Kennedy [2] showed that there exist sequences of 20 consecutive integers that are all Niven numbers and further proved that no sequences of 21 consecutive Niven numbers exist. Their result was partially generalized in 1994 by Grundman [5], who showed that every sequence of consecutive integers of length at least $2b+1$ must contain a term that is not b-Niven. Grundman's bound on the maximum length of consecutive Niven numbers was shown to be tight for $b \in \{2, 3\}$ by Cai [1] and when $b \geq 4$ by Wilson [9]. In this paper, we establish analogous results for b-prodigious numbers.

Let $b \geq 3$ be a positive integer and let ℓ be the smallest positive integer that does not divide b. In Section 2 of this paper, we show that there exist sequences of $b + \ell$ consecutive integers that are all b-prodigious and we further show that no sequence of $b+\ell+1$ consecutive b-prodigious numbers exist. In fact, we obtain this result by considering consecutive terms in an arithmetic progression. In Section 3, we investigate maximal length of consecutive integers whose terms are all b-prodigious (here, *maximal* stands for sequences that cannot be extended on either ends).

2 The maximum length of arithmetic progressions of b-prodigious numbers

We begin this section with the following lemma without proof.

Lemma 1. *Let $b \geq 2$. Then for all integers $c > 0$, $1 \leq j \leq b - 1$, and $0 \leq \beta < \alpha$, we have*

$$p_b(cb^\alpha + jb^\beta) = p_b(c) \cdot j.$$

Furthermore, if $cb^\alpha + jb^\beta$ is a b-prodigious number, then

$$j \mid (cb^\alpha + jb^\beta) \quad \textit{and} \quad p_b(c) \mid (cb^\alpha + jb^\beta).$$

In view of Lemma 1, the relation $p_b(c) \mid (cb^\alpha + jb^\beta)$ is a necessary condition for $cb^\alpha + jb^\beta$ to be a b-prodigious number. For this reason, it will be useful to force $p_b(c) = 1$ while also guaranteeing that c satisfies certain congruence constraints.

Lemma 2. *Let $\{x \equiv r_v \pmod{m_v} : 1 \le v \le w\}$ be a consistent system of congruences such that $r_v = 0$ if $\gcd(m_v, b) \ne 1$. Then there exists a positive integer c such that c satisfies this system of congruences and $p_b(c) = 1$.*

Proof. For each $1 \le v \le w$, let $m_v = m_{v,1}m_{v,2}$ such that $m_{v,1}$ divides a nonnegative power of b and $\gcd(m_{v,2}, b) = 1$. Let $M_1 = \operatorname{lcm}\{m_{v,1} : 1 \le v \le w\}$ and $M_2 = \operatorname{lcm}\{m_{v,2} : 1 \le v \le w\}$. By the conditions on r_v, the system of congruences is equivalent to

$$\{x \equiv 0 \pmod{M_1},\ x \equiv R \pmod{M_2}\}$$

for some $1 \le R \le M_2$. Then $c = \sum_{u=t}^{R+t-1} b^{u\varphi(M_2)}$ satisfies the desired properties, where φ is the Euler's totient function and t is a nonnegative integer such that $M_1 \mid b^{t\varphi(M_2)}$. □

For a positive integer d, we say that a sequence $\mathcal{S}$ of integers is a d-AP if all consecutive terms of $\mathcal{S}$ have a common difference d. When $d \mid b$ and $d < b/2$, the following theorem establishes the maximum length of a d-AP whose terms are all b-prodigious numbers.

Theorem 3. *Let $b > 2$ and d be positive integers such that $d \mid b$ and $d < b/2$. Define $D_b = \{0\} \cup \{1 \le j \le b-1 : j \mid b\}$, and for each $j \in D_b$, let $k_j = \min\{\kappa \in \mathbb{N} : (j + \kappa d) \nmid b\}$. Then the maximum length of a b-prodigious d-AP is $b/d + k$, where $k = \max\{k_j : j \in D_b\}$.*

Proof. Suppose $\mathcal{S}$ is a b-prodigious d-AP of length at least $b/d + k$. Let the first term of $\mathcal{S}$ be $ab + j$ for some nonnegative integers a and j with $0 \le j \le b-1$. Then $\mathcal{S}$ contains $ab + j, ab + (j + d), \ldots, (a+1)b + (j + (k-1)d)$.

If there exists $1 \le i \le k-1$ such that $j + (i-1)d < b \le j + id$, then $j + (i-1)d \ge b - d > b/2$. Thus, $(j + (i-1)d) \nmid b$. Since $j + (i-1)d$ is a factor of both $p_b(ab + (j + (i-1)d))$ and $p_b((a+1)b + (j + (i-1)d))$ by Lemma 1, we deduce that $ab + (j + (i-1)d)$ and $(a+1)b + (j + (i-1)d)$ cannot be both b-prodigious, a contradiction. Hence $j + id \le b - 1$ for all $0 \le i \le k-1$. Furthermore, since $ab + (j + id)$ and $(a+1)b + (j + id)$ are both b-prodigious, we have $j, j+d, \ldots, j + (k-1)d \in D_b$. This implies that $j + (k-1)d \le b/2$, so $j + kd < b$. Therefore, $ab + (j + kd)$ and $(a+1)b + (j + kd)$ cannot be both prodigious since $(j + kd) \nmid b$ by the definition of k. In other words, $\mathcal{S}$ cannot be of length greater than $b/d + k$.

Finally, note that there exists a positive integer c such that c satisfies the congruence $x \equiv 0 \pmod{(b-1)!}$ and $p_b(c) = 1$ by Lemma 2. As a result, if $j \in D_b$ satisfies $k_j = k$, then

$$cb^2 + j, cb^2 + (j + d), \ldots, cb^2 + b + (j + (k-1)d)$$

is a b-prodigious d-AP of length $b/d + k$. This is because $p_b(cb^2 + (j + id)) = j + id$ by Lemma 1 when $0 < j + id < b$; together with $(b-1)! \mid c$, we have $(j + id) \mid (cb^2 + (j + id))$. Likewise, $p_b(cb^2 + b + (j + id)) = j + id$ when $0 < j + id < j + kd$; together with $(j + id) \mid b$, we have $(j + id) \mid (cb^2 + b + (j + id))$. □

As an example of how to use Theorem 3, consider the case $b = 12$ and $d = 4$. Then $D_b = \{0, 1, 2, 3, 4, 6\}$, $k_1 = k_3 = k_4 = k_6 = 1$, and $k_0 = k_2 = 2$. Therefore, $k = 2$ and the maximum length of a 12-prodigious 4-AP is $12/4 + 2 = 5$. Following the definitions in the proof of Lemma 2, we have $m_1 = 11!$, $M_1 = 2^8 \cdot 3^4$, and $M_2 = 5^2 \cdot 7 \cdot 11$. Then $R = M_2$, $\varphi(M_2) = 1200$, and $t = 1$. Hence we have $c = \sum_{u=1}^{1925} 12^{1200u}$, and

$$144c, 144c + 4, 144c + 8, 144c + 12, 144c + 16$$

forms a 12-prodigious 4-AP of length 5. Note that although our theorem provides a method to construct a b-prodigious d-AP of maximum length, the sequence constructed using this method may not be the simplest. For example, the sequence $2, 6, 10, 14, 18$ forms a 12-prodigious 4-AP of length 5.

When $d = 1$, the statement of Theorem 3 simplifies to the following corollary.

Corollary 4. *Let $b > 2$. The maximum length of a sequence of consecutive b-prodigious numbers is $b + \ell$, where ℓ is the smallest positive integer that does not divide b.*

Proof. From Theorem 3 we have $k_0 = \ell \leq k$. On the other hand, for all $j \in D_b$, there always exists a multiple of ℓ in $\{j + \kappa : 1 \leq \kappa \leq \ell\}$; thus $k_j \leq \ell$. Combining both directions, our result follows from Theorem 3, since $k = \ell$. □

Since Theorem 3 only applies when $d \mid b$ and $d < b/2$, we end this section by addressing the maximum length of a b-prodigious $(b/2)$-AP.

Theorem 5. *Let $b > 2$ be even. Then the maximum length of a b-prodigious $(b/2)$-AP is 6.*

Proof. Suppose $\mathcal{S}$ is a b-prodigious $(b/2)$-AP of length at least 6. Then $\mathcal{S}$ contains $ab + (j + b/2)$ and $(a + 1)b + (j + b/2)$ for some nonnegative integers a and j with $0 \leq j < b/2$. By Lemma 1, $j + b/2$ divides both $ab + (j + b/2)$ and $(a + 1)b + (j + b/2)$. Hence $(j + b/2) \mid b$, which implies that $j = 0$.

If $\mathcal{S}$ contains both $cb^2 + ib$ and $cb^2 + ib + b/2$ for some nonnegative integers n and $1 \leq i \leq b - 1$, then $i \mid (cb^2 + ib)$ and $(i \cdot b/2) \mid (cb^2 + ib + b/2)$ by Lemma 1. Hence i divides $(cb^2 + ib + b/2) - (cb^2 + ib) = b/2$. Moreover, $cb^2 + ib + b/2 = mi \cdot b/2$ for some integer m. Dividing both sides of the last equation by $b/2$, we have $2cb+2i+1 = mi$, so $i \mid (2cb+2i+1)$.

Combining the established observations that $i \mid (b/2)$ and $i \mid (2cb+2i+1)$, we have $i = 1$. Therefore, $\mathcal{S}$ is of the form

$$(c - 1)b^2 + (b - 1)b + b/2, cb^2, cb^2 + b/2, cb^2 + b, cb^2 + b + b/2, cb^2 + 2b$$

for some positive integer c. Note that this is a b-prodigious $(b/2)$-AP of length 6 if $c = \sum_{u=0}^{b/2-1} b^u$. □

3 Maximal lengths of consecutive b-prodigious numbers

We say that the sequence $n, n+1, n+2, \ldots, n+t-1$ of consecutive b-prodigious numbers is *maximal of length* t if $n-1$ and $n+t$ are not b-prodigious numbers. Recall from Corollary 4 that $b+\ell$ is the maximum length of consecutive b-prodigious numbers, where ℓ is the smallest positive integer that does not divide b. In this section, we determine the existence of maximal sequences of consecutive b-prodigious numbers of lengths $b \le t \le b+\ell-1$.

We begin this section with the following three lemmas.

Lemma 6. *Let $b \ge 2$. If $ab+j$ and $ab+(j+1)$ are both b-prodigious numbers, where a is a positive integer and $0 \le j \le b-2$, then $p_b(a) = 1$.*

Proof. If $ab+j$ and $ab+(j+1)$ are b-prodigious, then $p_b(a) \mid (ab+j)$ and $p_b(a) \mid (ab+(j+1))$ by Lemma 1. It follows that $p_b(a) \mid 1$. Thus $p_b(a) = 1$. $\square$

Lemma 7. *Let $b > 2$, let ℓ be the smallest positive integer that does not divide b, and let $\mathcal{S}$ be a maximal sequence of consecutive b-prodigious numbers of length at least b. If the first term of $\mathcal{S}$ is of the form cb^2 for some positive integer c, then $\mathcal{S}$ is of length $b+\ell$. If the first term of $\mathcal{S}$ is of the form cb^2+1 for some nonnegative integer c, then $c = 0$ and $\mathcal{S}$ is of length at $b+\ell-1$.*

Proof. If the first term of $\mathcal{S}$ is of the form cb^2 for some positive integer c, then since $\mathcal{S}$ has at least b terms, $p_b(c) = 1$ by Lemma 6. Thus, by Lemma 1, we have $p_b(cb^2+b+j) = j$ for $1 \le j \le b-1$. Hence, if ℓ is the smallest positive integer that does not divide b, then $cb^2, cb^2+1, \ldots, cb^2+b+(\ell-1)$ are each b-prodigious.

If the first term of $\mathcal{S}$ is of the form cb^2+1 for some nonnegative integer c, then $c = 0$ since cb^2 is not b-prodigious. Noting that $b+\ell$ is not b-prodigious since $p_b(b+\ell) = \ell$ by Lemma 1 and $\ell \nmid (b+\ell)$, our result follows. $\square$

Lemma 8. *Let $b > 2$ and let $\mathcal{S}$ be a maximal sequence of consecutive b-prodigious numbers of length at least $b+1$. Then the first term of $\mathcal{S}$ is either 1, of the form cb^2, or of the form cb^2+b for some positive integer c.*

Proof. Let $ab+j$ be the first term of $\mathcal{S}$ for some nonnegative integers a and j with $j \le b-1$. Then the last term of $\mathcal{S}$ is at least $(a+1)b+j$. This implies that $j = 0$ or $j \mid b$; thus $j \le b/2$. If $1 \le j \le b/2$, then Lemma 6 implies that $p_b(a) = p_b(a+1) = 1$, and if $j = 0$, then $p_b(a) = 1$. Therefore, the first term of $\mathcal{S}$ is of the forms cb^2+j or cb^2+b for some nonnegative integers c and j with $0 \le j \le b/2$.

If $c = 0$, then since every integer $1 \le j \le b-1$ is b-prodigious, the first term of the maximal sequence $\mathcal{S}$ of consecutive b-prodigious numbers is 1. Hence it remains to consider $c > 0$, and we have $p_b(c) = 1$. If $2 \le j \le b/2$, then since $cb^2+2(j-1) \in \mathcal{S}$ is b-prodigious, we know that $j-1$ divides $cb^2+2(j-1)-(j-1) = cb^2+(j-1)$. As a result, $cb^2+(j-1)$ is also b-prodigious, contradicting that cb^2+j is the first term of the maximal sequence $\mathcal{S}$. Finally, $j \ne 1$ by Lemma 7. $\square$

Theorem 9. *Let $b > 2$ and let $\mathcal{S}$ be a maximal sequence of consecutive b-prodigious numbers of length m for some $b+1 \leq m \leq b+\ell-1$. Then*

(*a*) *m cannot be strictly between $b+1$ and $b+\ell-1$;*

(*b*) *$m = b+\ell-1$ if and only if the first term of $\mathcal{S}$ is 1; and*

(*c*) *$m = b+1$ if and only if either b is odd and the first term of $\mathcal{S}$ is 1 or b is a power of 2.*

Proof. By Lemmas 7 and 8, the first term of $\mathcal{S}$ is either 1 or of the form cb^2+b for some positive integer c. If the first term of $\mathcal{S}$ is 1, then $m = b+\ell-1$ by Lemma 7. In the remainder of this proof, assume that the first term of $\mathcal{S}$ is cb^2+b.

If $m > b+1$, then $\mathcal{S}$ contains both cb^2+2b and cb^2+2b+1, contradicting Lemma 6. This implies that $m = b+1$ and establishes part (a). Hence $m = b+\ell-1$ if and only if $\ell = 2$, which is equivalent to b being odd. However, b cannot be odd since $2 \mid (cb^2+b+2)$ and $2 \mid (cb^2+2b)$. Therefore, if $m = b+\ell-1$, the first term of $\mathcal{S}$ cannot be cb^2+b. This establishes part (b).

If $b = 2^\alpha m$ for some integer $\alpha \geq 1$ and odd integer $m > 1$, then note that $2^{\alpha+1} \leq b-1$ and $2^{\alpha+1} \nmid (cb^2+b+2^{\alpha+1})$ since $2^{\alpha+1} \mid b^2$ but $2^{\alpha+1} \nmid b$. This contradicts that $\mathcal{S}$ contains $cb^2+b+2^{\alpha+1}$. Hence it remains to show that such a maximal sequence $\mathcal{S}$ exists when $b = 2^\alpha$ for some integer $\alpha \geq 2$. This can be achieved by finding a positive integer c such that $j \mid (cb^2+b+j)$ for each $1 \leq j \leq b-1$ and $p_b(c) = 1$.

For each $1 \leq j \leq b-1$, let $j = 2^{\beta_j} j'$ for some nonnegative integer β_j and odd integer j'. Note that $\beta_j < \alpha$, so $j \mid (cb^2+b+j)$ is equivalent to $cb^2+b \equiv 0 \pmod{j'}$. Since $\gcd(b, j') = 1$, we have $c \equiv -b^{-1} \pmod{j'}$. Such an integer c exists by Lemma 2, which completes the proof of part (c). $\square$

Lemma 10. *Let $b > 2$ and let*

$$ab+(b-1), (a+1)b, (a+1)b+1, \ldots, (a+1)b+(b-2)$$

be a b-prodigious sequence. Then $b-1$ is a prime.

Proof. Suppose by way of contradiction that $b-1$ is not a prime. Then there exists $2 \leq j \leq b-2$ such that $j \mid b-1$. We deduce that j divides $ab+(b-1)$ since $(b-1) \mid ab+(b-1))$ by Lemma 1. Since j also divides $(a+1)b+j$ by Lemma 1, it follows that j divides $(a+1)b+j-(ab+(b-1)) = j+1$, a contradiction. $\square$

Theorem 11. *Let $b > 2$. Then there exists a maximal sequence $\mathcal{S}$ of consecutive b-prodigious numbers of length b if and only if either $b-1$ is a prime or b is a power of an odd prime.*

Proof. Let $\mathcal{S}$ be a maximal sequence of consecutive b-prodigious numbers of length b. By Lemma 6, $\mathcal{S}$ is a subsequence of

$$(c-1)b^2+(b-1)b+(b-1), cb^2, cb^2+1, \ldots, cb^2+2b$$

for some nonnegative integer c such that $c = 0$ or $p_b(c) = 1$. Suppose by way of contradiction that $b-1$ is not a prime and b is not a power of an odd prime. Then Lemma 10 implies that the first term of $\mathcal{S}$ cannot be $(c-1)b^2 + (b-1)b + (b-1)$ nor $cb^2 + (b-1)$. Also, the first term of $\mathcal{S}$ cannot be cb^2 nor $cb^2 + 1$ since $cb^2 + b$ and $cb^2 + b + 1$ are b-prodigious, and the first term of $\mathcal{S}$ cannot be $cb^2 + b + 1$ since $cb^2 + b$ is b-prodigious.

If the first term of $\mathcal{S}$ is $cb^2 + b$, then b cannot be even; otherwise, $cb^2 + 2b$ is b-prodigious. Since b is not a power of an odd prime, there exist distinct primes $p < q$ such that $pq \mid b$. Let α be the greatest integer such that $p^\alpha \mid b$. Then $p^{\alpha+1} \leq b - 1$, but $cb^2 + b + p^{\alpha+1}$ is not b-prodigious, a contradiction. Hence $\mathcal{S}$ is given by

$$cb^2 + j, cb^2 + (j+1), \ldots, cb^2 + b + (j-1)$$

for some $2 \leq j \leq b - 2$. Since $j - 1$ divides both $cb^2 + b + (j-1)$ and $cb^2 + (j-1)$ by Lemma 1, this implies that $(j-1) \nmid b$ since $cb^2 + b + (j-1)$ is b-prodigious while $cb^2 + (j-1)$ is not.

Since $cb^2 + (b-1)$ is b-prodigious, we have by Lemma 1 that $(b-1) \mid c$. If there exists $2 \leq \tilde{j} \leq j$ such that $\tilde{j} \mid b-1$, then $\tilde{j} \nmid b$; thus $\tilde{j} \nmid (cb^2 + b + \tilde{j})$, a contradiction. Hence j is less than every prime factor of $b-1$, implying that $j < \sqrt{b-1}$ since $b-1$ is not a prime. Note that $b - 1 \geq 4$, so $j < \sqrt{b-1} \leq \frac{b-1}{2}$. As a result, $j \leq 2(j-1) < b-1$, so $cb^2 + 2(j-1)$ is b-prodigious. Together with $cb^2 + b + (j-1)$ being b-prodigious, we deduce from Lemma 1 that $(j-1) \mid b$, a contradiction.

Conversely, we assume that $b-1$ is a prime or b is a power of an odd prime. If $b-1$ is a prime, then let

$$B = \max\{\beta \in \mathbb{Z} : p^\beta \mid j \text{ for some prime } p \mid b \text{ and } 1 \leq j \leq b-2\}$$

and

$$N = \max\{n \in \mathbb{N} : n \mid \operatorname{lcm}\{j \in \mathbb{N} : 1 \leq j \leq b-2\} \text{ and } \gcd(n, b) = 1\}.$$

This construction ensures that $j \mid b^B N$ for all $1 \leq j \leq b-2$. By Lemma 2, there exists a positive integer $\overline{c}$ such that $\overline{c} \equiv b^{-2}(-b^2+1) - b^{B-2} + 1 \pmod{(b-1)^B}$, $\overline{c} \equiv -b^{B-2} \pmod{N}$, $\overline{c} \equiv 0 \pmod{b^{B-1}}$, and $p_b(\overline{c}) = 1$. Define $c = \overline{c} + b^{B-2}$; thus $p_b(c) = 1$. Note that

$$\begin{aligned}
p_b\big((c-1)b^2 + (b-1)b + (b-1)\big) &= p_b\big((\overline{c} + b^{B-2} - 1)b^2 + (b-1)b + (b-1)\big) \\
&= p_b\big(\overline{c}b^2 + b^B - 1\big) \\
&= p_b\left(\overline{c}b^2 + \sum_{t=0}^{B-1}(b-1)b^t\right) \\
&= (b-1)^B
\end{aligned}$$

and $p_b(cb^2+j)=j$ for all $1\le j\le b-2$ by Lemma 1. Furthermore,

$$\begin{aligned}(c-1)b^2+(b-1)b+(b-1)&=(\overline{c}+b^{B-2}-1)b^2+(b-1)b+(b-1)\\&=\overline{c}b^2+b^B-1\\&\equiv(b^{-2}(-b^2+1)-b^{B-2}+1)b^2+b^B-1 &\pmod{(b-1)^B}\\&\equiv(-b^2+1)-b^B+b^2+b^B-1 &\pmod{(b-1)^B}\\&\equiv 0 &\pmod{(b-1)^B}\end{aligned}$$

and $cb^2+j=(\overline{c}+b^{B-2})b^2+j=\overline{c}b^2+b^B+j$ is divisible by j since $\overline{c}b^2+b^B\equiv 0 \pmod{N}$ and $\overline{c}b^2+b^B\equiv 0 \pmod{b^B}$. Hence

$$(c-1)b^2+(b-1)b+(b-1), cb^2, cb^2+1, \ldots, cb^2+(b-2)$$

forms a sequence of consecutive b-prodigious numbers of length b. This sequence is maximal since $(c-1)b^2+(b-1)b+(b-2)$ and $(c-1)b^2+(b-1)b+(b-1)$ cannot both be b-prodigious by Lemma 6 and $cb^2+(b-1)=\big((c-1)b^2+(b-1)b+(b-1)\big)+b$ is not divisible by $b-1$.

Lastly, if $b=p^\alpha$ for some odd prime p, then for each $1\le j\le b-1$, let β_j be the maximum integer such that $p^{\beta_j}\mid j$ and $j=p^{\beta_j}j'$. By Lemma 2, there exists a positive integer c such that $c\equiv -p^{-\alpha} \pmod{j'}$ for all $1\le j\le b-1$ and $p_b(c)=1$. Then

$$cb^2+b, cb^2+b+1, \ldots, cb^2+b+(b-1)$$

forms a maximal sequence of consecutive b-prodigious numbers of length b, since $cb^2+b+(b-1)$ is even, and hence cb^2+2b is not b-prodigious. □

4 Concluding remarks

Although the main focus of our paper is on consecutive b-prodigious numbers, there are many other questions about b-prodigious numbers that could be investigated. One such question is to consider the intersection of b-prodigious numbers with other integer sequences. In this direction, we end our paper with a theorem that considers the b-prodigious-Niven numbers, i.e., the intersection of the set of b-prodigious numbers and the set of b-Niven numbers.

Lemma 12 ([7])**.** *Let $b\ge 2$. If x and y are two b-Niven numbers such that $s_b(x)=s_b(y)$, then $s_b(x)\mid(x-y)$.*

Theorem 13. *The maximum length of a sequence of consecutive b-prodigious-Niven numbers is exactly*

(a) 4 *if $b=2$;*

(b) b *if $b>2$ is even; and*

(*c*) $b+1$ *if* $b>2$ *is odd.*

Proof. Every positive integer is 2-prodigious, so part (a) follows from the result by Cai [1] on 2-Niven numbers as mentioned in the introduction. In the rest of the proof, we consider $b>2$.

To prove parts (b) and (c), we first note that $1, 2, \ldots, b$ and $1, 2, \ldots, b+1$ are maximal sequences of consecutive b-prodigious-Niven numbers when b is even and b is odd, respectively. We call these the "trivial" sequences. To search for nontrivial sequences, let a and j be positive integers with $1 \le j \le b-1$ such that the sequence $ab+j, ab+j+1, \ldots, (a+1)b+j-1$ is b-prodigious-Niven. Since $ab+(b-1)$ is b-prodigious, we have by Lemma 1 that $(b-1) \mid a$ and thus $(b-1) \mid s_b(a)$. If $j = b-1$, then $(a+1)b+b-1$ is not b-prodigious since $(b-1) \nmid (a+1)$. If $1 \le j \le b-2$, then by Lemma 6 we see that $p_b(a) = 1$, and we deduce that $s_b(ab+j) = s_b((a+1)b+(j-1)) = s_b(a)+j$. Thereforewe have $(s_b(a)+j) \mid (b-1)$ by Lemma 12, contradicting that $(b-1) \mid s_b(a)$. In other words, every nontrivial sequence of consecutive b-prodigious-Niven numbers with length at least b is a subsequence of $ab, ab+1, \ldots, ab+(b-1)$ or $ab+(b-1), (a+1)b, \ldots, (a+1)b+(b-2)$ with maximum length b. $\square$

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