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Nesting Nonpartitions

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Abstract

Antichains of root posets associated with simple complex Lie algebras are a well-studied combinatorial object, famously counted by the Coxeter-Catalan numbers. In this paper, we study chains in root posets using standard enumerative techniques. The factorizations of the zeta polynomials of these root posets gives rise to sequences that have similar numerological properties as the exponents of the Weyl group.

1 Introduction

Call an interval $[a, b]$ *n-integral* if a and b are integers $1 \leq a < b \leq n$. We say that two n-integral intervals are *nesting* if one is contained in the other. In this language, *nonnesting partitions* may be defined as sets of distinct n-integral intervals with *no* pair of intervals nesting. Nonnesting partitions are a classical Catalan object—they are counted by the ubiquitous Catalan numbers, which appear as sequence A000108 in Sloane's *On-Line Encyclopedia of Integer Sequences* [9]:

$$\mathrm{Cat}(n) = \prod_{i=1}^{n-1} \frac{n+i+1}{i+1}.$$

We may ask instead about *nesting nonpartitions* on n points: sets of distinct n-integral intervals such that *every* pair of intervals nests. For example, there are $1, 2, 6, 20, 68,$ and 232 nesting nonpartitions on $1, 2, 3, 4, 5,$ and 6 points. The 20 nesting nonpartitions on four points are illustrated in Figure 1.

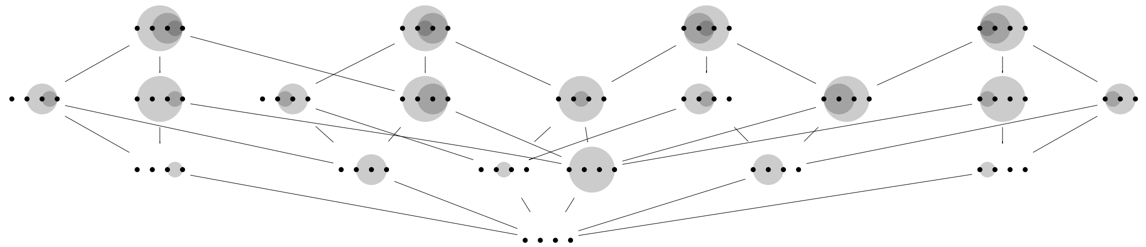


Figure 1: The 20 nesting nonpartitions on $n = 4$ points, ordered by inclusion.

It is not difficult to show that the number of nesting nonpartitions on $n+1$ points is given by

$$\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{n}{2i}\binom{n-i}{k-i} = \frac{\left(2+\sqrt{2}\right)^n + \left(2-\sqrt{2}\right)^n}{2}. \qquad \text{(A006012)}$$

Define a *k-multinesting nonpartition* to be a set of k n-integral intervals such that every pair of intervals nests (intervals may now appear with multiplicity). As we show in Section 4.1, the number of k-multinesting nonpartitions is given by the compact expression

$$\zeta_{A_n}(k) = \prod_{i=1}^{n-1} \frac{2k+i}{i}. \qquad (1)$$

Nesting nonpartitions on $n+1$ points are naturally interpreted as chains in the root poset of type A_n—we will review root systems and root posets in Section 2. This phrasing allows us to define the *k-multinesting nonpartitions* for a general irreducible root system Φ as the multichains of length k in the restriction of the root poset to the short roots, and state a root-theoretic generalization of Equation 1.

Theorem 1. *Let Φ be an irreducible crystallographic root system. Then the number of k-multinesting nonpartitions is given by*

$$\zeta_{\Phi_s^+}(k) = \prod_{i=1}^{g-2} \frac{2k+\delta_i}{\delta_i},$$

for the sequence of positive integers $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_{g-2} = h-2$ given in Table 1, where h is the Coxeter number of Φ, and g is the dual Coxeter number of the dual root system $\Phi^\vee$.

Our proofs are case-by-case, and—although we show in Proposition 10 that the sequences $(\delta_i)_{i=1}^{g-2}$ have connections to the numerology of the root system—we have been unable to find a representation-theoretic interpretation of these numbers.

2 Background

2.1 Root systems

Let E be a finite-dimensional vector space with inner product $\langle \cdot\,,\cdot\rangle$ and let $\alpha, \beta \in E$. The reflection σ_α through the hyperplane perpendicular to α sends any vector parallel to α to its negative while leaving any vector perpendicular to α unchanged. By linearity, we see that

$$\sigma_\alpha(\beta) = -\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha + \beta - \frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha = \beta - 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

W	$\epsilon_1, \epsilon_2, \ldots, \epsilon_{g-1}$
A_n	$1 \; 2 \; \ldots \; n$
B_n	$1 \; 3 \; \ldots \; 2n{-}1$
C_n	$1 \; 3 \; 5 \; \ldots \; 2n{-}3 \; 2n{-}1$ $3 \; 5 \; \ldots \; 2n{-}3$
D_n	$1 \; 3 \; 5 \; \ldots \; 2n{-}5 \; 2n{-}3$ $n{-}1$ $3 \; 5 \; \ldots \; 2n{-}5$
E_6	1 4 5 7 8 11 3 5 7 9 6
E_7	1 5 7 9 11 13 17 3 5 7 9 11 13 15 7 9 11
E_8	1 7 11 13 17 19 23 29 3 5 7 9 11 13 15 17 19 21 23 25 27 9 11 13 15 17 19 21 15
F_4	1 5 7 11 3 5 7 9
G_2	1 5 3

Table 1: The sequences $(\epsilon_i)_{i=1}^{g-1}$ for the irreducible crystallographic root systems, arranged to illustrate their symmetry, where $\epsilon_{i+1} = \delta_i + 1$ for $1 \le i \le g-2$ and $\epsilon_1 = 1$. The top line for each root system consists of its exponents.

Definition 2. A (crystallographic) *root system* is a finite set $\Phi \subseteq E$ of nonzero vectors spanning E such that for $\alpha, \beta \in \Phi$ we have

- $\mathrm{span}(\alpha) \cap \Phi = \{\alpha, -\alpha\}$,
- $\sigma_\alpha(\Phi) = \Phi$, and
- $2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}$ is an integer.

The *Weyl group* W associated with a root system Φ is the subgroup of $O(E)$ generated by the reflections σ_α for $\alpha \in \Phi$.

The condition that $2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}$ is an integer, is the crystallographic condition—it may be restated as "$\sigma_\alpha(\beta)$ may be obtained from β by adding an integer multiple of α." A root system is *reducible* if it is the union of two disjoint root systems, each of which spans one of a pair of orthogonal subspaces of E. Thus, a reducible root system can be viewed as the union of these two independent root systems. A root system which has no such decomposition is called *irreducible*.

By the well-known classification of irreducible root systems [5], there can be at most two different root lengths. If there are two different lengths, we call the corresponding sets of roots the *short roots* and *long roots*. A root system for which there is only one length of roots is called *simply laced*. The root systems A_n, D_n, E_6, E_7, and E_8 are simply laced, while B_n, C_n, F_4, and G_2 are not.

Example 3. Let E be the n-dimensional space of vectors in $\mathbb{R}^{n+1}$ whose entries sum to zero, and let e_i be the standard basis vectors of $\mathbb{R}^{n+1}$. Define $\alpha_i = e_i - e_{i+1}$, so that $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for E. This set generates (by reflecting in these roots) the full root system of type A_n:

$$\Phi_{A_n} = \{e_i - e_j : 1 \leq i, j \leq n+1\}.$$

Reflections in the hyperplane orthogonal to $e_i - e_j$ exchange the ith and jth coordinates, so that the Weyl group associated with the A_n root system is isomorphic to the symmetric group S_{n+1} (acting on E by permuting coordinates).

The root system A_n is often the first example of a root system, and combinatorics done in this context is often referred to as "Type A," in contrast to combinatorics done in the context of general root systems. More precisely, many classical combinatorial objects can be interpreted as arising from the symmetric group or Φ_{A_n} in some way, and—once phrased in that language—can often be generalized by passing to other root systems. In this spirit, this paper interprets nesting nonpartitions in the context of type A root systems and extends their definition to other root systems.

2.2 Root posets

By choosing a generic hyperplane in E that does not contain any root, we can divide a root system Φ into positive and negative roots. The roots closest to the hyperplane form a basis for E, which allows us to partially order the positive roots.

Definition 4. A *simple system* is a subset $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq \Phi$ such that

- the elements of Δ are linearly independent, and
- every element of Φ can be written as a linear combination of elements of Δ, either with all nonnegative or all nonpositive coefficients.

Definition 5. A set of *positive roots* is a choice of roots $\Phi^+ \subseteq \Phi$ such that

- for $\alpha \in \Phi$, exactly one of α and $-\alpha$ is in Φ^+, and
- if a root is the sum of two roots in Φ^+, it is also in Φ^+.

We call the roots in the simple system the *simple roots*. The notions of positive roots and simple systems are closely related. It is easy to see that any choice of a simple system determines a set of positive roots; the roots which can be obtained from a nonnegative linear combination of simple roots may be designated as positive roots, and the rest negative. Conversely, any choice of positive roots uniquely determines a simple system [5, Section 1.3]. In particular, this means that a simple system exists, since we may choose a total order on E that is compatible with the vector space operations (for example, the lexicographic order), and this order determines a positive set and therefore a simple system. Moreover, all simple systems are essentially the same, differing only by the action of the Weyl group [5, Section 1.4]. More precisely, for any two simple systems Δ and Δ', there is some $w \in W$, the Weyl group of the root system, so that $\Delta' = w\Delta$. Since $W \subseteq O(E)$, the simple systems are orthogonal transformations of each other and have the same geometry. Thus, this choice is immaterial and we will make convenient choices of simple systems.

The choice of positive roots, along with vector space operations, allows us to define a partial order on the positive roots.

Definition 6. The *root poset* of an irreducible crystallographic root system Φ is the poset on the positive roots Φ^+ defined by $\alpha \leq \beta$ if $\beta - \alpha$ is a nonnegative sum of positive roots. The *short root poset* Φ_s^+ is the restriction of Φ^+ to the short roots when there are two different root lengths, and the root poset Φ^+ otherwise.

Example 7. Continuing Example 3, the set $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a convenient choice of simple roots (and will be assumed to be the set of simple roots for A_n for the rest of the paper). Writing $\alpha_{i,j} = e_i - e_j$, the corresponding positive roots are

$$\Phi^+_{A_n} = \{\alpha_{i,j} \mid 1 \leq i < j \leq n+1\}.$$

The difference between two positive roots, $\alpha_{i,j} - \alpha_{k,l}$, is itself a nonnegative sum of simple roots when $i \leq k < l \leq j$. The A_5 root poset is illustrated in Figure 2.

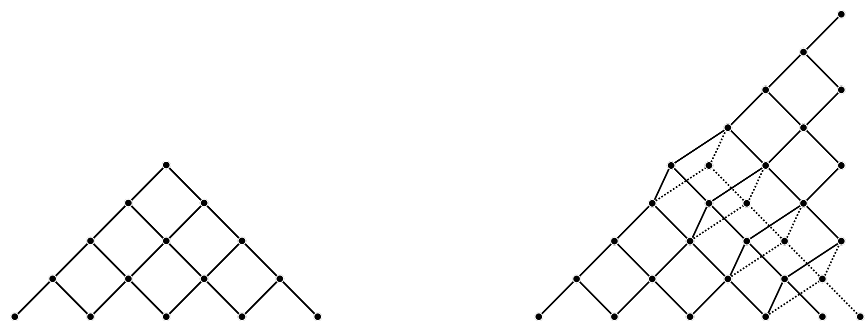


Figure 2: *Left*: the A_5 positive root poset. *Right*: the D_6 positive root poset.

2.3 Numerology

Associated with a root system Φ are two important invariants, the sequence of *degrees* $(d_i)_{i=1}^n$ and the sequence of *exponents* $(e_i)_{i=1}^n$. It turns out that we can easily define both using the positive root poset (although this is not how they were defined historically). A positive root α can be expressed as a nonnegative sum of simple roots: $\alpha = \sum_{i=1}^n a_i\alpha_i$. Define the *height* of a positive root α to be $\sum_{i=1}^n a_i$—the number of simple roots that must be added to get α (this is the rank of α in the positive root poset).

If k_i is the number of positive roots of height i, we always have

$$k_1 \geq k_2 \geq \cdots \geq k_{h-1} = 1.$$

This defines a partition of $|\Phi|/2$, the number of positive roots. From this partition, we define the sequence of exponents (in descending order) by the *dual* partition $e_n \geq e_{n-1} \geq \cdots \geq e_1$ of $|\Phi^+|$ with $k_1 = n$ parts. We define the *Coxeter number* by $h = e_n + 1 = |\Phi|/n$, the degrees by $d_i = e_i + 1$, and the *dual Coxeter number of the dual root system* g as one plus the height of the highest short root. In fact, it turns out that these numbers satisfy

$$\sum_{i=1}^n e_i = nh/2 \text{ and } e_i + e_{n+1-i} = h. \tag{2}$$

As explained in [5, Chapter 3], there are two other algebraic settings in which the exponents and degrees appear.

The first comes from the invariant theory of the Weyl group. Let Φ be an irreducible crystallographic root system spanning an n-dimensional vector space E, and let $W \subseteq O(E)$ be its Weyl group, generated by the reflections of Φ. We associate E with $\mathbb{R}^n$, and W acts on the polynomial ring $S = \mathbb{R}[x_1, x_2, \ldots, x_n]$ in a natural way by transforming the vectors $(x_1, x_2, \ldots, x_n)$. Let R be the ring of polynomials invariant under the action of W and $f_1, f_2, \ldots, f_n$ be a set of homogeneous, algebraically independent polynomials that generate R. Then the sequence $d_1 \leq d_2 \leq \cdots \leq d_n$, where d_i is the degree of f_i, is independent of the choice of generating polynomials (up to reordering). These degrees are therefore an important invariant of the root system.

The second comes from the eigenvalues of certain elements in the Weyl group. Since any $w \in W$ is an orthogonal transformation of finite order, its eigenvalues must be roots of unity. Fix a choice of simple roots $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ with corresponding reflections $\sigma_1, \sigma_2, \dots, \sigma_n$. Any element that is the product of all simple reflections (in any order) is called a *Coxeter element*, and all such elements are conjugate, with the same order h. The eigenvalues of a Coxeter element are powers of ζ, where ζ is a primitive hth root of unity. If $(\zeta^{e_i})_{i=1}^n$ are the eigenvalues of a Coxeter element, the sequence $e_1 \leq e_2 \leq \cdots \leq e_n$ recovers the exponents of W.

2.4 Posets

Recall that a *chain* in a poset P is a sequence of elements $p_1 < p_2 < \cdots < p_k$. A *multichain* is a chain with repetitions allowed. For a finite poset $(P, \leq)$ of height n, write $\mathrm{ch}_k(P) = \{p_1 \leq p_2 \leq \cdots \leq p_k\}$ for the set of multichains in P of length k, and $\mathrm{ch}(P)$ for the set of maximal strict chains in P. The *zeta polynomial* $\zeta_P(q)$ of a finite poset P is characterized as the unique polynomial satisfying

$$\zeta_P(q) = |\mathrm{ch}_{q-1}(P)|,$$

and it has leading coefficient $\frac{|\mathrm{ch}(P)|}{n!}$. Write $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ for the number of ways to choose k elements from a set of size n, with repetitions allowed. If a_k is the number of strict chains in P with k elements, then

$$\zeta_P(q) = \sum_{k=0}^{\ell} a_k \binom{q-2}{k-1}, \tag{3}$$

since there are

$$\left(\!\binom{k}{q-1-k}\!\right) = \binom{q-2}{k-1}$$

ways to choose $q-1$ elements from a set of size k with repetitions allowed, subject to the requirement that every element must be selected at least once.

3 Nesting nonpartitions

3.1 Nonnesting partitions

Recall from the introduction that an interval $[a, b]$ is *n-integral* if a and b are integers $1 \leq a < b \leq n$, and two n-integral intervals are called nesting if one is contained in the other. In this language, the *nonnesting partitions* are sets of n-integral intervals with no pair of intervals nesting.

Associate a positive root $\alpha_{i,j}$ in the A_n root system with the interval $[i, j]$, so that the order structure on positive roots is equivalent to ordering intervals by inclusion. Then the nonnesting partitions may be rephrased as antichains in the A_{n-1} root poset, giving a natural generalization to other root systems Φ.

Definition 8. The *nonnesting partitions* associated with an irreducible crystallographic root system Φ are the antichains in the positive root poset Φ^+.

The nonnesting partitions turn out to be counted by a root-theoretic generalization of the Catalan numbers:

$$\mathrm{Cat}(\Phi) = \prod_{i=1}^{n-1} \frac{h + d_i}{d_i},$$

where the $(d_i)_{i=1}^n$ are the *degrees* of Φ, and $h = d_n$ is the Coxeter number [7, 2]. In type A_n, we obtain the sequence A000108, in type B_n and C_n we get the central binomial coefficients A000984, in type D_n the sequence A051924, while in the exceptional types we have the numbers $\mathrm{Cat}(\Phi_{E_6}) = 833, \mathrm{Cat}(\Phi_{E_7}) = 4160, \mathrm{Cat}(\Phi_{E_8}) = 25080, \mathrm{Cat}(\Phi_{F_4}) = 105$, and $\mathrm{Cat}(\Phi_{G_2}) = 6$.

Nonnesting partitions have been widely studied, in part due to their connection to noncrossing partitions and cluster combinatorics. Armstrong's monograph gives a comprehensive survey [1].

3.2 Nesting nonpartitions

Recall that the *nesting nonpartitions* are collections of n-integral intervals such that *every* pair of intervals nest, as illustrated in Figure 1. We can again interpret this definition in the context of the A_{n-1} root system—the interval $[a, b]$ corresponds again to the positive root $\alpha_{a,b}$, and two intervals nest if and only if the corresponding positive roots are comparable. Thus, it is natural to study *chains* of positive roots—to our knowledge, these have not been systematically enumerated before.

Definition 9. The *nesting nonpartitions* associated with an irreducible crystallographic root system Φ are the chains in the short positive root poset Φ_s^+. The *k-multinesting nonpartitions* are multichains with k roots in Φ_s^+.

We emphasize that the nesting nonpartitions are counted by the zeta polynomial of the positive root poset, in contrast to the *order polynomial*, which counts order ideals in $P \times [q]$. When $P = \Phi^+$ is a positive root poset, this order polynomial counts plane partitions in Φ^+; it recovers $\mathrm{Cat}(\Phi)$ for $q = 1$, but has a simple product form only for roots systems of "coincidental" type [4].

The condition requiring chains in the *short* positive root poset may seem a bit strange, but we motivate it as follows. By Equation 3, it suffices to study the zeta polynomial to count the objects of Definition 9. As we saw in Equation 1, the zeta polynomials of the type A root poset have a simple factorization; we will show the same for the other classical types

B_n, C_n, and D_n. But if we compute the zeta polynomial for the full F_4 positive root poset, we find it has a factor

$$q^4 + \frac{310}{39}q^3 + \frac{270}{13}q^2 + \frac{935}{39}q + \frac{142}{13},$$

which is irreducible over $\mathbb{Q}$. By restricting to the short roots, we obtain a simple factorization over $\mathbb{Q}$ with desirable numerical properties, similar to the exponents. We note that a similar phenomenon appears when considering the order polynomial of the positive root poset of type F_4: while this order polynomial doesn't factor into linear factors over $\mathbb{Q}$, it does after restricting to the short roots.

4 Proof of Theorem 1.1

In this section we prove Theorem 1, that the number of k-multinesting nonpartitions is given by

$$\zeta_{\Phi_s^+}(k) = \prod_{i=1}^{g-2} \frac{2k + \delta_i}{\delta_i},$$

for the sequence of positive integers $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_{g-2} = h - 2$ given in Table 1, where h is the Coxeter number of Φ, and g is the dual Coxeter number of the dual root system $\Phi^\vee$. Just as the Coxeter number h satisfies that $h-1$ is the height of the highest long root, the appearance of the dual Coxeter number g is due to the fact that the height of the highest short root is $g-1$.

In analogy to the relationship between exponents and degrees, let $\epsilon_{i+1} = \delta_i + 1$ for $1 \leq i \leq g-2$ and $\epsilon_1 = 1$. These positive integers $(\epsilon_i)_{i=1}^{g-1}$ are given in Figure 1; these sequences contain repeated entries, and these are shown as stacked vertically in the table. The sequence $(\epsilon_i)_{i=1}^{g-1}$ contains the usual exponents as a subsequence; this is shown as the top line of numbers in each entry. For types A and B, $(\epsilon_i)_{i=1}^{g-1}$ are exactly the exponents, while the other root systems contain more entries. We have the following analogue of Equation 2:

Proposition 10. *The exponents $(e_i)_{i=1}^{n}$ are a subsequence of $(\epsilon_i)_{i=1}^{g-1}$. Moreover,*

$$\sum_{i=1}^{g-1} \epsilon_i = (g-1)h/2 \textit{ and } \epsilon_i + \epsilon_{g-i} = h.$$

Proposition 10 is easily verified by examining Figure 1, and gives further evidence that the short roots are the right thing to consider. For example, in the case of G_2, where $h = 6$ and $g = 4$, if we do not restrict ourselves to the short roots, the sequence $(\epsilon_i)_{i=1}^{h-1}$ becomes $(1, 3, 5, 5, 7)$, which satisfies no such nice properties and loses the symmetry. Similarly, for types B_n and C_n (which have isomorphic positive root posets when not restricted to short roots), the sequence $(\epsilon_i)_{i=1}^{h-1}$ is $(1, 3, 3, 5, 5, \ldots, 2n-1, 2n-1)$. Only types B_n, C_n, F_4, and G_2 are not simply laced, so these are the only root systems for this modification is needed.

4.1 Proof in type A

The Hasse diagram for the A_n positive root poset is a triangle, as illustrated in Figure 2 for A_5. The sequence $(\delta_i)_{i=1}^{g-2}$ for type A is given by $\delta_i = i$ and $g = n+1$. The count given in Theorem 1 becomes

$$\prod_{i=1}^{n-1} \frac{2k+i}{i} = \binom{2k+n-1}{n-1} = \left(\!\!\binom{n}{2k}\!\!\right).$$

The multichains in type A are in correspondence with multisets of size $2k$ whose elements are simple roots. The A_n root poset is a join semilattice, with n simple roots. An arbitrary choice of two (possibly identical) simple roots can be identified with their join, and for any element of the poset, there is a unique pair of (not necessarily distinct) simple roots whose join is that element. This gives a way for k not necessarily distinct elements of the poset to be projected onto a multiset of size $2k$, whose elements are all simple roots. In particular, a multichain of length k can be projected in this way, starting with the smallest element and projecting one by one until the largest element has been projected. There are $\left(\!\binom{n}{2k}\!\right)$ ways to multichoose $2k$ of the n simple roots.

This map is bijective, which establishes the count. To go from a multiset M of $2k$ simple roots to a chain, pair up the leftmost and rightmost element of M and take their join to get the largest element of the chain, repeatedly pairing off the leftmost and rightmost remaining elements of M to get the rest of the elements of the chain. This recovers the chain that generates the multiset, since it undoes the projection operation that generated the multiset. Furthermore, this process can be applied to any multiset M of size $2k$ and must give a multichain back, showing that there is a chain that generates every multiset. Thus, there are $\left(\!\binom{n}{2k}\!\right)$ multichains of length k in the A_n root poset.

4.2 Proof in types B and C

The poset of short roots in B_n is just a chain of length n, so we see that the number of multichains of length k is $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{n-1}$. This agrees with the factorization given in Theorem 1 with $\delta_i = 2i$ and $g = n+1$.

It is convenient to prove a more general statement when considering the short roots in type C_n. For $a \geq b \geq 0$, an $a \times b$ *trapezoid* is the poset $T_{a,b}$ given by the integer points (x, y) in $\mathbb{Z}^2$ such that $0 \leq y \leq b$ and $b - y \leq x \leq a + y$, ordered by the usual product order. As an example, the Hasse diagram for the 5×2 trapezoid is shown in Figure 3.

The C_n short root poset is an $(n-1)\times(n-2)$ trapezoid, and the number of multichains in a general $a \times b$ trapezoid is useful for the type D case, so we count the number of multichains in an $a \times b$ trapezoid. There is a terse sketch of a proof of the count given by Proctor [6]; this count was also obtained by Stembridge [10]. We reproduce this proof here with additional details.

The Hasse diagram of $T_{a,b}$ has $b+1$ long diagonals, as drawn in Figure 3, which we may label 0 through b. Diagonal d is a chain of $a + b - 2d + 1$ elements. Let $E_d(m)$ denote the number of length m multichains whose minimal element is on diagonal d. To count these,

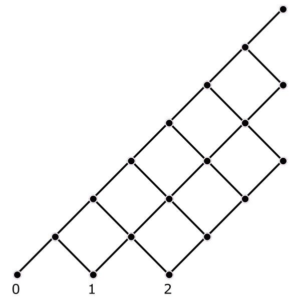
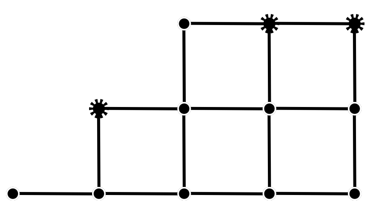


Figure 3: The 5×2 trapezoid.

we need to know how many multichains there are in the $u \times v$ rectangle (points in $\mathbb{Z}^2$ with $0 \le x \le u$ and $0 \le y \le v$) with the elements strictly above the line $y = x$ deleted (see Figure 4). Call this poset $R_{u,v}$, with $u \ge v$, and let $F_{u,v}(m)$ be the number of length m multichains in it.

Multichains in $R_{u,v}$ are in bijection with certain pairs of non-intersecting lattice paths. Let $a_1 = (0, 1)$ and $b_1 = (v, m + 1)$ be points in $\mathbb{Z}^2$. Upward steps in paths between these points correspond to the x-coordinate of a point in $R_{u,v}$. Let $a_2 = (1, 0)$ and $b_2 = (u+1, m)$, so steps here correspond to y-coordinates. Specifically, if P_1 and P_2 are a pair of non-intersecting paths (no shared vertices) from a_1 to b_1 and a_2 to b_2 respectively, we may construct a multichain in $R_{u,v}$ by considering the the first "up" step in each path as specifying the x and y coordinates of the first element of the multichain, with the x or y coordinates given by how many total "right" steps there had been before that "up" step in the respective paths (see Figure 4). We see that these coordinates are legal so long as P_1 stays above P_2, as this ensures that the x-coordinate is always larger than the y-coordinate. We may replace a_1 by the point $(0, 0)$ and get the same count, since the non-intersecting property requires that the first step be north, and we recover the previous situation. This modification simplifies a sum later.



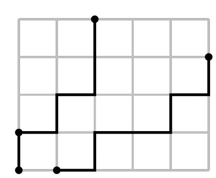


Figure 4: *Left*: A chain in $R_{4,2}$, where northeast is 'up'. *Right*: The corresponding lattice paths.

This allows us to use the Lindström-Gessel-Viennot lemma [3], which states that the number of tuples of non-intersecting lattice paths from a starting set to an ending set is equal to the determinant of the matrix whose i, jth entry is the number of lattice paths from the ith starting point to the jth ending point. Thus,

$$F_{u,v}(m) = \binom{m+u}{u}\binom{m+v+1}{v} - \binom{m+u+1}{u+1}\binom{m+v}{v-1}.$$

To find $E_d(m)$, we observe that there are $F_{a+b-d,d}(m)$ m-multichains whose elements are on or above diagonal d, and $F_{a+b-d,d-1}(m)$ multichains whose elements are above but not on diagonal d, so there are $E_d(m) = F_{a+b-d,d}(m) - F_{a+b-d,d-1}(m)$ multichains whose minimal element is on diagonal d. This gives

$$E_d(m) = \binom{m+a+b-d}{a+b-d}\binom{m+d}{d} - \binom{m+a+b-d+1}{a+b-d+1}\binom{m+d-1}{d-1}.$$

Summing over d, we find that the sum telescopes, and the only nonzero term left is the number of length m multichains in an $a \times b$ trapezoid,

$$\binom{m+a}{a}\binom{m+b}{b}. \tag{4}$$

Since the short root poset for C_n is an $(n-1)\times(n-2)$ trapezoid, Theorem 1 follows by noting that the sequence $(\delta_i)_{i=1}^{g-2}$ for type C is given by $(2, 2, 4, 4, \ldots, 2n-4, 2n-4, 2n-2)$, and that

$$\prod_{i=1}^{n-1}\frac{2k+2i}{2i}\cdot\prod_{j=1}^{n-2}\frac{2k+2j}{2j} = \binom{k+n-1}{n-1}\binom{k+n-2}{n-2}.$$

Finally, we observe that the full B_n and C_n root posets are $(n-1)\times(n-1)$ trapezoids, and we can see that the factorization using the full root poset does not give sequences $(\epsilon_i)_{i=1}^{h-1}$ with the desired numerological properties.

4.3 Proof in type D

The D_n root poset is similar to an $(n-2)\times(n-2)$ trapezoid, but with a doubled southeast diagonal, as shown in Figure 2. We draw one of these diagonals solid, and the other dashed. We will refer to the dashed southeast diagonal as the "dashed diagonal" and the corresponding solid diagonal as the "solid diagonal." Any multichain cannot include elements from both the solid and dashed diagonal, and we note that if we delete one of these diagonals, the resulting poset is an $(n-2)\times(n-2)$ trapezoid. If we delete both diagonals, then we have an $(n-2)\times(n-3)$ trapezoid. The total number of multichains in the type D poset is the number of chains which do not pass through solid diagonal, plus the number which do not pass through the dashed diagonal, minus the number which do not pass through either diagonal. This gives the number of length m multichains in the D_n poset as

$$\begin{aligned}
2\binom{k+n-2}{n-2}^2 &- \binom{k+n-2}{n-2}\binom{k+n-3}{n-3} \\
&= \binom{k+n-2}{n-2}\binom{k+n-3}{n-3}\left(2\cdot\frac{k+n-2}{n-2}-1\right) \\
&= \binom{k+n-2}{n-2}\binom{k+n-3}{n-3}\left(\frac{2k+n-2}{n-2}\right).
\end{aligned}$$

This exactly matches the factorization listed in Figure 1, verifying the main theorem for type D and thus all of the infinite families of root systems.

4.4 Verification in the exceptional types

The factorizations of the zeta polynomials of the short root posets for the exceptional types E_6, E_7, E_8, F_4 and G_2 were all evaluated in SageMath [8]. This completes the proof of the main theorem and the numerological properties for all irreducible crystallographic root systems.

5 Future work

The sequences associated with the zeta polynomial factorizations for the short root posets show remarkable numerological properties and relationship to the exponents across all root systems, but we have been unable to find any representation-theoretic explanation for these properties, or for why the short root poset is a more natural object to study in the root systems which are not simply-laced. It would be interesting to find extensions of this to other root-theoretic posets, such as minuscule posets or crystals (the preceding analysis could be interpreted as multichains in the top half of the crystal associated with the highest short root), or to weighted enumerations as in [11].

It would also be interesting to compute the zeta polynomial of the poset of nesting nonpartitions, as in Figure 1. In type A, this zeta polynomial appears to be related to the Chebyshev polynomials, while for types B and C these zeta polynomials appear to be related to Legendre polynomials.

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