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Equal Sums of Two Distinct Like Powers

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Abstract

We study the equation $a^j + b^j = c^k + d^k$ for positive integers a, b, c, d, and $2 < j < k$. Two heuristic arguments correctly predict the cases in which the equation has primitive solutions.

1 Introduction

Let $(S_n^{(k)})_{n\geq 1}$ denote the sequence of sums of two positive k-th powers of integers. We consider mostly the sequences $(S_n^{(k)})_{n\geq 1}$ with $3 \leq k \leq 6$: A004999, A003336, A003347, and A003358 in the OEIS [12]. For integers $2 < j < k$, we investigate the numbers common to the two sequences $(S_n^{(j)})_{n\geq 1}$ and $(S_n^{(k)})_{n\geq 1}$. The numbers $(a^k)^j + (b^k)^j = (a^j)^k + (b^j)^k$, for positive integers a and b, appear trivially in both sequences. We are more interested in nontrivial common elements.

Equivalently, we study the equation $a^j + b^j = c^k + d^k$ for positive integers a, b, c, d. We are interested mostly in the case $2 < j < k$, but we begin by reviewing the case $j = k$, which has been studied extensively.

Many solutions in integers are known for the equation $a^j + b^j = c^j + d^j = N$ when $j = 2$, 3, and 4. We exclude trivial solutions with $\{a, b\} = \{c, d\}$. Only primitive solutions, those with $\gcd(a, b, c, d) = 1$, are considered because all other solutions may be derived from the primitive ones. For example, the solution $399^4 + 402^4 = 177^4 + 474^4$ would be ignored because it is just 3^4 times the solution $133^4 + 134^4 = 59^4 + 158^4$.

Theorem 412 in Hardy and Wright [10] asserts that some integers have arbitrarily many representations as the sum of two cubes. Parametric solutions are known for $j = 2$, 3, and 4. Equation (13.7.11) in Hardy and Wright [10] gives a parametric solution for the case $j = 4$ and Choudhry [3] offers another one. Wroblewski [14] lists all primitive solutions to $a^4 + b^4 = c^4 + d^4$ with $a, b, c, d \leq 10^{14}$. No primitive solution to $a^j + b^j = N = c^j + d^j$ is known with $j > 4$. According to Guy [8, Sec. D1], people have searched at least up to $N < 10^{25}$ for solutions in the case $j = 5$. Fermat's Last Theorem says there are no solutions with $d = 0$. Browning [2] used algebraic geometry to show that primitive solutions with $j > 4$ are rare if they exist at all.

We have searched for solutions to the equation $a^j + b^j = c^k + d^k$ in integers a, b, c, d with $2 < j < k < 11$. There are many solutions with $j = 2$ and all $k > 2$. We found primitive solutions in the three cases $j = 3$, $k = 4$, 5, and 6, but none for other values of $2 < j < k < 11$.

When one studies the equation $a^j + b^j = c^k + d^k$ with $j < k$, one ignores trivial identities such as $(a^k)^j + (b^k)^j = (a^j)^k + (b^j)^k$, even when $\gcd(a, b) = 1$.

In case $j < k$ and $j \mid k$ some solutions with $\gcd(a, b, c, d) > 1$ might be considered if one wants to see all solutions, because sometimes a common factor cannot be canceled. For example, one solution to $a^3 + b^3 = c^6 + d^6$ is

$$102^3 + 330^3 = 12^6 + 18^6 \tag{1}$$

and the common factor 6 of the four numbers cannot be canceled since it appears to different powers on the two sides of the equation. However, this solution comes from a solution to $w^3 + x^3 = y^3 + z^3$. If $k = \gcd(a, b, c, d) > 1$ for a solution to $a^3 + b^3 = c^6 + d^6$, then this equation may be rewritten as $(kw)^3 + (kx)^3 = (ke)^6 + (kf)^6$ or $k^3w^3 + k^3x^3 = (k^2e^2)^3 + (k^2f^2)^3$ from which k^3 may be canceled to give $w^3 + x^3 = (ke^2)^3 + (kf^2)^3$, which is primitive.

A solution $(k^2a)^3 + (k^2b)^3 = (kc)^6 + (kd)^6$ for some integer $k > 1$ is easily derived from the solution $a^3 + b^3 = c^6 + d^6$ by multiplying by k^6, so it is ignored. For example, $918^3 + 2970^3 = 36^6 + 54^6$ is just the solution $102^3 + 330^3 = 12^6 + 18^6$ multiplied by 3^6. Likewise, we ignore solutions to $a^3 + b^3 = c^6 + d^6$ with $a = c^2$ and $b = d^2$.

Similar reductions apply also whenever $j < k$ and $j \mid k$. We tested the 11089 solutions to $w^4 + x^4 = y^4 + z^4$ found by Wroblewski [14] to see whether one would give a solution to $a^4 + b^4 = c^8 + d^8$. None of them worked (because none had square values for both w and x).

Lander [11] gave (1) as a solution when $j = 3$ and $k = 6$. He found parametric solutions to $a^j + b^j = c^k + d^k$ (and to similar equations with any number of terms on each side) for every $2 \leq j \leq k$, but none of his solutions are primitive. His solution to $a^j + b^j = c^k + d^k$ is

$$(pw^B)^j + (qw^B)^j = (w^A)^k + (w^A)^k,$$

where A, B and w are positive integers that depend on j, k, and the parameters p and q. For $j = 5$, $k = 8$, he gave a typical numerical solution $(3 \cdot 122^3)^5 + (122^3)^5 = (122^2)^8 + (122^2)^8$. For $j = 3$, $k = 5$, another numerical solution would be $(4 \cdot 36^3)^3 + (2 \cdot 36^3)^3 = (36^2)^5 + (36^2)^5$.

The tables in the next section list only primitive solutions—those with $\gcd(a, b, c, d) = 1$ (other than $a = b = c = d = 1$).

2 The results

We used the methods described in Bernstein [1] in the search for solutions. One heap (data structure) held the sums $a^j + b^j$ and another one held the sums $c^k + d^k$. The program compared their least elements as they were removed. We used Algorithm 6 of Eisermann [4] to reduce the memory requirement of the program. The computation was done on the Brown cluster at Purdue's RCAC.

We used local constraints to accelerate the algorithm. For instance, only five of the eleven residue classes modulo 11 are sums $c^5 + d^5$, namely, -2, -1, 0, 1, 2 modulo 11. The map $f(x) = x^3 \bmod 11$ is a permutation of the eleven classes. Thus for each $a \pmod{11}$, if a^3+b^3 equals a sum of two fifth powers, then there are only five possible values of $b \pmod{11}$. For example, when $a \equiv 3 \pmod{11}$, we have $a^3 \equiv 5 \pmod{11}$. Thus $b^3 \bmod 11$ must be one of 4, 5, 6, 7, 8 (mod 11), so $b \bmod 11$ must be one of 5, 3, 8, 6, 2 (mod 11). To search these cases we ran five jobs, one for each possible value of $b \bmod 11$ (and all possible values for a, c, and d). We used the moduli 16, 11, 27, 29, 32, 19, 25, for $k = 4, 5, 6, 7, 8, 9, 10$, respectively, and most values of $j < k$.

If $c \le d \le M$, then $a^j + b^j = c^k + d^k \le 2M^k$, so $a \le b \le L = (2M^k)^{1/j}$. The number of pairs a, b is about $L^2/2$. The search for solutions examines more a, b pairs than c, d pairs since $j < k$, and the naive running time would be $O(L^2)$ or $O(M^{2k/j})$. However, we rewrote the equation as $a^j - c^k = d^k - b^j$ and computed these differences instead of the sums. This reduced the work to $O(M^{1+k/j})$.

For $j = 3$, we used $M = 10000, 5000, 1400, 700, 350, 200, 100$ for $k = 4, 5, 6, 7, 8, 9, 10$, respectively. For $j = 4$, we used $M = 18100, 3500, 1100, 460, 230, 135$ for $k = 5, 6, 7, 8, 9, 10$, respectively. We made similar effort in the other cases. These limits kept the sums in range of 64-bit integer arithmetic in the three cases where we found solutions and in range of 128-bit integer arithmetic in the other cases. They also kept the search times reasonable.

Tables 1 and 2 show the primitive solutions to $0 \le a^3 + b^3 = c^4 + d^4 = N$ with $0 \le c \le d \le 10000$ in order by the size of N.

Note the repeated values of c or d in these solutions.

$$
\begin{aligned}
45^3 + 133^3 &= 39^4 + 19^4 \\
161^3 + 176^3 &= 39^4 + 52^4 \\
887^3 + 6457^3 &= 456^4 + 690^4 \\
2941^3 + 14771^3 &= 456^4 + 1338^4 \\
6377^3 + 7977^3 &= 227^4 + 935^4 \\
5528^3 + 8529^3 &= 398^4 + 935^4
\end{aligned}
$$

a	b	c	d	a	b	c	d
17	24	8	11	3658	29849	825	2262
45	133	19	39	7202	29977	225	2286
161	176	39	52	26224	27217	167	2486
641	960	133	170	18226	33641	396	2577
993	1400	32	247	16101	35149	1225	2595
840	1681	73	270	4401	37096	1644	2573
1417	1634	78	291	11986	39881	288	2841
41	3801	193	481	23097	42217	195	3059
4272	4337	35	632	14521	46850	1959	3090
887	6457	456	690	31249	44928	215	3318
6377	7977	227	935	39893	42669	2261	3275
5528	8529	398	935	32321	47936	981	3458
8669	9043	60	1086	23645	56133	401	3713
2941	14771	456	1338	35653	61979	1470	4086
9928	16849	1145	1418	17432	65969	1806	4097
2201	19721	1239	1519	39777	69704	2131	4418
6885	22253	571	1831	23281	74176	266	4529
5381	23901	913	1903	217	76679	2418	4518
20368	20449	371	2030	51072	79217	950	5009
10745	25113	1457	1883	69793	70440	3618	4771
17777	25104	1469	2024	12721	90136	4076	4627

Table 1: Primitive solutions to $a^3 + b^3 = c^4 + d^4$, Part 1.

a	b	c	d	a	b	c	d
15248	91169	3207	5060	111875	133933	3606	7764
77169	77912	3964	5117	120089	136087	5496	7602
77393	77904	4633	4670	74384	157697	6811	6834
15333	100205	1781	5623	77375	157633	6252	7308
7849	101257	257	5677	104528	154625	1791	8336
72753	103520	3809	5986	89673	161689	675	8387
12269	116357	3021	6217	127992	143009	1757	8414
70181	109325	3177	6275	85529	172402	2691	8688
1129	118857	4087	6117	10320	183793	6732	8029
45176	129521	1222	6897	145373	183379	570	9804
109009	110017	1861	7151	64989	210277	2379	9883
17717	138189	557	7171	18403	216605	4092	9972
47233	137824	3388	7135	162754	180713	7365	9234
25176	142801	197	7356	94201	215481	8459	8697
41361	145361	407	7487				

Table 2: Primitive solutions to $a^3 + b^3 = c^4 + d^4$, Part 2.

Table 3 shows the primitive solutions to $0 \le a^3+b^3 = c^5+d^5 = N$ with $0 \le c \le d \le 5000$ in order by the size of N. Note that $c = 187$ occurs in both the second and third solutions.

a	b	c	d	a	b	c	d
2467	3071	115	119	315515	410576	96	2515
7755	15102	187	326	324657	450095	1458	2600
14475	29190	187	488	111359	526113	1286	2700
9301	40290	80	581	74484	597517	293	2924
41144	144677	653	1244	480397	480926	1690	2909
39032	150265	1065	1158	50387	710804	123	3244
177898	451093	1765	2404	265936	1225217	3838	4001

Table 3: Primitive solutions to $a^3 + b^3 = c^5 + d^5$.

Table 4 shows the primitive solutions to $0 \le a^3+b^3 = c^6+d^6 = N$ with $0 \le c \le d \le 1400$ in order by the size of N.

a	b	c	d	a	b	c	d
3441	7708	57	88	265008	500137	636	653
28105	28596	40	189	85656	620785	534	775
50145	350428	286	591	296305	1233132	238	1113
225681	458812	233	690				

Table 4: Primitive solutions to $a^3 + b^3 = c^6 + d^6$.

3 The first heuristic argument

We estimate the probability that $a^j + b^j = c^k + d^k$ has a nontrivial solution using a form of the Birthday Paradox. This argument is old and well known. Weaver [13], page 135, told an amusing anecdote about the Birthday Paradox during World War II.

Our first heuristic assumption is that the four terms a^j, b^j, c^k, d^k, are random integers of about the same size B, say, between $B/2$ and $3B/2$. Then a and b will be near $B^{1/j}$ and there will be about that many possible values for each of them, so that there are about $B^{2/j}$ pairs (a, b). Likewise, there are about $B^{2/k}$ pairs (c, d) with c and d near $B^{1/k}$.

The sums $a^j + b^j$, $c^k + d^k$ will be random integers in the interval $[B, 2B]$ of length B, and we assume they are independent. Then the probability that one particular sum $a^j + b^j$ differs from every one of the $B^{2/k}$ sums $c^k + d^k$ is $(1 - 1/B)^{B^{2/k}}$. By independence, the probability that every sum $a^j + b^j$ differs from every sum $c^k + d^k$ is

$$((1 - 1/B)^{B^{2/k}})^{B^{2/j}} = (1 - 1/B)^{B^{2/j+2/k}},$$

and the probability that the equation $a^j + b^j = c^k + d^k$ has at least one solution with numbers of this size is 1 minus this probability.

Since B is large we may approximate $1 - 1/B$ by $\exp(-B^{-1})$. The probability of at least one solution becomes $1 - \exp(-B^{-1+2/j+2/k})$. When $2/j + 2/k < 1$ the probability is near 0, so we expect no solution. When $2/j + 2/k > 1$ the probability is near 1, so we expect many solutions. When $2/j + 2/k = 1$ we have a borderline case where there may be a few solutions or none at all.

This heuristic argument explains why there are many solutions to $a^j + b^j = c^k + d^k$ when $j = 2$: $2/j + 2/k > 1$ for every integer $k > 1$.

When $j = 3$ the argument predicts many solutions for $k = 4$ and 5, just a few solutions when $k = 6$, and no solution when $k > 6$. It also predicts no solution for $3 < j < k$. This is just what we found.

The case $j = k = 4$ is also a borderline case and it has infinitely many solutions perhaps because there is a parametric solution. Wroblewski [14] found 11089 solutions with a, b, c, $d \le 10^{14}$.

Note that the comparison of number of solutions is a bit unfair because of different search limits M. Tables 1, 2, 3, and 4 give all solutions with roughly the same limit on the sum N.

4 The second heuristic argument

The argument in this section is modeled on that of Erdős and Ulam [6]. We will show that for most sequences having the growth rates of $(S_n^{(j)})_{n\ge1}$ and $(S_n^{(k)})_{n\ge1}$ there is a nontrivial intersection if and only if $2/j + 2/k \le 1$. This theorem proves nothing about particular sequences like $(S_n^{(j)})_{n\ge1}$ or $(S_n^{(k)})_{n\ge1}$, but it suggests that the same statement is likely to hold for them.

We define a probability measure on the space of sequences of positive integers. See Erdős and Rényi [5] or Halberstam and Roth [9] for more about this measure. Let $\gamma > 1$ be a real number and n be a positive integer. Let the measure of the set of all sequences containing n be $c_1 n^{-1+1/\gamma}$ and the measure of the complement be $1 - c_1 n^{-1+1/\gamma}$. Here c_1 and other c_i used later are appropriate positive constants. In this case, c_1 is chosen so that the measure of the set of all sequences is 1. Call this measure the γ measure. The phrase "almost all sequences $\mathcal{A}$" will mean "for all $\mathcal{A}$ except for a set of sequences of γ measure 0." Let $P_\gamma(n) = c_1 n^{-1+1/\gamma}$ be the probability that $n \in \mathcal{A}$.

If $\mathcal{A}$ is a sequence of positive integers and x is a real number, let $A(x)$ be the number of $a \in \mathcal{A}$ with $a \le x$. It is easy to see that $A(x) = (1 + o(1))c_1\gamma x^{1/\gamma}$ for almost all $\mathcal{A}$. Hence the n-th term of $\mathcal{A}$ is $(1 + o(1))(n/c_1\gamma)^\gamma$ for almost all $\mathcal{A}$.

Let $2\mathcal{A}$ denote the sequence of all sums $a + a'$ with $a \in \mathcal{A}$ and $a' \in \mathcal{A}$. Note that if $\gamma = j \ge 2$, then, for almost all $\mathcal{A}$, $\mathcal{A}$ and $2\mathcal{A}$ have growth rates similar to those of $(n^j)_{n\ge1}$ and $(S_n^{(j)})_{n\ge1}$, respectively.

Now let $1 < \alpha \le \beta$. Write $P_\alpha(n) = c_1 n^{-1+1/\alpha}$ and $P_\beta(n) = c_2 n^{-1+1/\beta}$.

Theorem 1. *Let $1 < \alpha \leq \beta$. If $2/\alpha + 2/\beta < 1$, then for almost all sequences $\mathcal{A}$ in α measure and almost all sequences $\mathcal{B}$ in β measure the intersection $2\mathcal{A} \cap 2\mathcal{B}$ is finite. But if $2/\alpha + 2/\beta \geq 1$, then for almost all sequences $\mathcal{A}$ in α measure and almost all sequences $\mathcal{B}$ in β measure the intersection $2\mathcal{A} \cap 2\mathcal{B}$ is infinite.*

The proof is based on that in Erdős and Ulam [6].

Proof. We will prove the first statement by showing that the expected number E of integers n in the intersection is finite. We have

$$\begin{aligned}
E &= \sum_{n=1}^{\infty} \left(\sum_{u+v=n} P_\alpha(u) P_\alpha(v) \right) \left(\sum_{u+v=n} P_\beta(u) P_\beta(v) \right) \\
&= \sum_{n=1}^{\infty} c_1^2 c_2^2 \left(\sum_{u+v=n} (uv)^{-1+1/\alpha} \right) \left(\sum_{u+v=n} (uv)^{-1+1/\beta} \right) \\
&< c_3 \sum_{n=1}^{\infty} n^{-1+2/\alpha} n^{-1+2/\beta} \\
&= c_3 \sum_{n=1}^{\infty} n^{-2+2/\alpha+2/\beta},
\end{aligned}$$

where the inner sums were estimated by integrals. For example, the first inner sum is estimated by $\int_0^n (u(n-u))^{-1+1/\alpha} du = n^{-2+2/\alpha} n c_4 = c_4 n^{-1+2/\alpha}$. Since $2/\alpha + 2/\beta < 1$, $E < \infty$ and the intersection is finite by the Borel-Cantelli lemma. See Feller [7] for the Borel-Cantelli lemma.

Now suppose $2/\alpha + 2/\beta \geq 1$. We will give the proof for the case $2/\alpha + 2/\beta = 1$. The case $2/\alpha + 2/\beta > 1$ is similar. Let $E(x)$ denote the expected number of integers $n \leq x$ in the intersection. We have

$$\begin{aligned}
E(x) &= \sum_{n=1}^{x} \left(\sum_{u+v=n} P_\alpha(u) P_\alpha(v) \right) \left(\sum_{u+v=n} P_\beta(u) P_\beta(v) \right) \\
&= \sum_{n=1}^{x} c_1^2 c_2^2 \left(\sum_{u+v=n} (uv)^{-1+1/\alpha} \right) \left(\sum_{u+v=n} (uv)^{-1+1/\beta} \right) \\
&= (1 + o(1)) c_5 \sum_{n=1}^{x} n^{-2+2/\alpha+2/\beta} \\
&= (1 + o(1)) c_5 \sum_{n=1}^{x} n^{-1} = (1 + o(1)) c_5 \log x.
\end{aligned}$$

Now we use a second moment argument to show that for almost all $\mathcal{A}$ and almost all $\mathcal{B}$ the size $f(\mathcal{A}, \mathcal{B}, x)$ of $2\mathcal{A} \cap 2\mathcal{B} \cap [1, x]$ satisfies $f(\mathcal{A}, \mathcal{B}, x) = (1 + o(1)) c_5 \log x$ so that

$\lim_{x\to\infty} f(\mathcal{A},\mathcal{B},x)/E(x) = 1$. The expected value of $f(\mathcal{A},\mathcal{B},x)$ is $E(x)$, which we just computed. Let $E^2(x)$ be the expected value of $f(\mathcal{A},\mathcal{B},x)^2$. Then $E^2(x) =$

$$\sum_{n_1=1}^{x}\sum_{n_2=1}^{x}\sum_{u_1+v_1=n_1}\sum_{u_2+v_2=n_1}\sum_{u_3+v_3=n_2}\sum_{u_4+v_4=n_2} P(u_1,u_2,u_3,u_4,v_1,v_2,v_3,v_4),$$

where $P(u_1,u_2,u_3,u_4,v_1,v_2,v_3,v_4)$ is the probability that u_1, v_1, u_2, and v_2 are in $\mathcal{A}$ and that u_3, v_3, u_4, and v_4 are in $\mathcal{B}$. If these eight integers were distinct, we would have $P(u_1,u_2,u_3,u_4,v_1,v_2,v_3,v_4) =$

$$P_\alpha(u_1)P_\alpha(v_1)P_\alpha(u_2)P_\alpha(v_2)P_\beta(u_3)P_\beta(v_3)P_\beta(u_4)P_\beta(v_4),$$

and the sum would be $(E(x))^2$, but if some integers are repeated, the probability is larger. Hence, $E^2(x) > (E(x))^2$. To get the opposite inequality we add terms to $E^2(x)$ to account for possible repeated values.

If the eight integers were distinct, we could pair the sums

$$\left(\sum_{u_1+v_1=n_1} P_\alpha(u_1)P_\alpha(v_1)\right)\left(\sum_{u_3+v_3=n} P_\beta(u_3)P_\beta(v_3)\right)$$

and obtain $(1+o(1))c_6\log x$ as in the calculation for $E(x)$. But if say, $u_3 = u_1$, then $v_3 = v_1$ and the two sums become

$$\begin{aligned}\sum_{u_1+v_1=n_1} P_\alpha(u_1)P_\alpha(v_1)P_\beta(u_1)P_\beta(v_1) &= \sum_{u_1+v_1=n_1}(u_1v_1)^{-1+1/\alpha}(u_1v_1)^{-1+1/\beta}\\ &= \sum_{u_1+v_1=n_1}(u_1v_1)^{-2+1/\alpha+1/\beta}\\ &= \sum_{u_1+v_1=n_1}(u_1v_1)^{-3/2}.\end{aligned}$$

When we approximate this sum by the integral $\int_0^{n_1}(u_1(n_1-u_1))^{-3/2}du$ we get a constant times n_1^{-2}, and the sum on n_1 is finite. The sum on n_2 pairs the other two sums and gives $(1+o(1))c_7\log x$.

Likewise, the other added terms are all less than constants times $\log x$. Therefore, $(E(x))^2 < E^2(x) < (E(x))^2 + c_8\log x$. By the Tchebycheff inequality the α measure of the set of $\mathcal{A}$ and the β measure of the set of $\mathcal{B}$ for which

$$|f(\mathcal{A},\mathcal{B},x) - E(x)| > \epsilon\log x$$

are less than $c_9/\epsilon^2\log x$. Let $x_k = 2^{k(\log k)^2}$. By the Borel-Cantelli lemma we have

$$\lim_{k\to\infty} f(\mathcal{A},\mathcal{B},x_k)/E(x_k) = 1.$$

Therefore, since $f(\mathcal{A},\mathcal{B},x_k) \le f(\mathcal{A},\mathcal{B},x) \le f(\mathcal{A},\mathcal{B},x_{k+1})$ when $x_k < x < x_{k+1}$, for almost all $\mathcal{A}$ and almost all $\mathcal{B}$ we have $\lim_{x\to\infty} f(\mathcal{A},\mathcal{B},x)/E(x) = 1$. In the same way, $\lim_{x\to\infty} f(\mathcal{A},\mathcal{B},x)/E(x) = 1$ when $2/\alpha + 2/\beta > 1$. Since $E(x)$ is unbounded as $x\to\infty$, the intersection $2\mathcal{A}\cap 2\mathcal{B}$ is infinite for almost all $\mathcal{A}$ and almost all $\mathcal{B}$ when $2/\alpha+2/\beta \ge 1$. □

Now let $\alpha = j$ and $\beta = k$. If $(n^j)_{n\geq 1}$ and $(n^k)_{n\geq 1}$ were typical sequences, this theorem would predict infinitely many solutions to $a^j + b^j = c^k + d^k$ when $2/j + 2/k \geq 1$ and finitely many solutions when $2/j + 2/k < 1$. Of course, Lander [11] found infinitely many solutions to this equation for all j and k. Thus $(n^j)_{n\geq 1}$ and $(n^k)_{n\geq 1}$ are special sequences in the exceptional set of measure 0 when $2/j + 2/k < 1$. But since the theorem predicts that there should be no solutions, perhaps Lander found all solutions in this case.

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References

[1] D. J. Bernstein, Enumerating solutions to $p(a) + q(b) = r(c) + s(d)$. *Math. Comp.* **70** (2000), 389–394.

[2] T. D. Browning, Equal sums of two kth powers. *J. Number Theory* **96** (2002), 293–318.

[3] A. Choudhry, The Diophantine equation $A^4 + B^4 = C^4 + D^4$. *Indian J. Pure Appl. Math.* **22** (1991), 9–11.

[4] P. Eisermann, Bimonotone enumeration. *Math. Comp.* **78** (2009), 591–613.

[5] P. Erdős and A. Rényi, Additive properties of random sequences of positive integers. *Acta Arith.* **6** (1960), 83–110.

[6] P. Erdős and S. Ulam, Some probabilistic remarks on Fermat's last theorem. *Rocky Mountain J. Math.* **1** (1971), 613–616.

[7] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 2nd edition. Wiley, 1957.

[8] R. K. Guy, *Unsolved Problems in Number Theory*. Springer, 2nd edition, 1994.

[9] H. Halberstam and K. F. Roth, *Sequences*, Vol. 1. Oxford, 1966.

[10] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*. 4th edition, Oxford, 1960.

[11] L. J. Lander, Equal sums of unlike powers. *Fibonacci Quart.* **28** (1990), 141–150.

[12] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, 2022. Available at https://oeis.org.

[13] W. Weaver, *Lady Luck, the Theory of Probability*. Anchor Books, 1963.

[14] J. Wroblewski, 2006. Available at `http://www.math.uni.wroc.pl/~jwr/422/422.zip`.

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