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An Exact Enumeration of the Unlabeled Disconnected Posets

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Abstract

We give an exact enumeration of the unlabeled disconnected posets according to the number of connected components of the posets. This result establishes that the enumeration of unlabeled posets belonging to a class that is closed under the direct sum depends mainly on the enumeration of unlabeled connected posets contained in the class. We also give an algorithm to determine the parameters involved in the enumeration formula, and finally, find the number of unlabeled disconnected posets with a certain number of elements. We show that the enumeration algorithm runs in polynomial time.

1 Introduction

We give an exact enumeration of the unlabeled disconnected posets belonging to a class of posets that is closed under the direct sum of posets. Let $\mathcal{P}_n$, $n \geq 1$ be the set of all n-element unlabeled posets. Also let $\mathcal{Q}_n$, $n \geq 1$ and $\mathcal{R}_n$, $n \geq 2$ be the sets of all n-element unlabeled connected and disconnected posets, respectively. Since the singleton poset is connected, we have $\mathcal{P}_1 = \mathcal{Q}_1$ and $|\mathcal{P}_1| = |\mathcal{Q}_1| = 1$. In general, for all $n \geq 2$, we have $\mathcal{P}_n = \mathcal{Q}_n \cup \mathcal{R}_n$ and hence $|\mathcal{P}_n| = |\mathcal{Q}_n| + |\mathcal{R}_n|$. We observe that every member of $\mathcal{R}_n$ can be expressed as the direct sum of two or more members from $\mathcal{Q}_r$, $1 \leq r \leq n-1$. In our enumeration method, for finite $n \geq 2$, we express $|\mathcal{R}_n|$ as a finite series consisting of the numbers $|\mathcal{Q}_r|$, $r \leq n-1$ that gives the enumeration of the posets belonging to $\mathcal{R}_n$ according to the number of connected direct terms (components) of the posets. Here, we establish in general the criterion for the pairwise nonisomorphic direct sum of unlabeled posets obtained by Mohammad [12] and used particularly for the enumeration of the class of P-series, a subclass of the class of series-parallel posets.

For common enumeration methods, we refer the readers to [3, 4, 7, 10] for the enumeration of finite posets, [1, 2] for graphs, and [6, 11] for topologies. In the most of these cases, the enumeration of a class of structures was done by generating and counting all the pairwise nonisomorphic structures belonging to the class. The running time of these algorithms increases rapidly even though the structures under consideration are significantly small in size. Mainly, the running time for generating pairwise nonisomorphic structures make these algorithms highly time-complex. We observe that the steps for generating pairwise nonisomorphic disconnected posets in an enumeration process can be skipped. Therefore, the proposed exact enumeration method for the unlabeled disconnected posets must reduce the time-complexities of the algorithms for enumeration of unlabeled posets. Further, this enumeration method is applicable for the enumeration and generation of any unlabeled mathematical structures (posets, graphs, networks, topologies, and so on) belonging to a class that is closed under the direct sum of the structures.

We also give an algorithm to determine the parameters involved in the enumeration formula and to compute the numbers $|\mathcal{R}_n|$ for $n \geq 2$. We show that the enumeration algorithm runs in polynomial time with complexity $\mathcal{O}(n^5)$. We implement the enumeration algorithm into the computer and gather some numerical results. Brinkmann and McKay [3] obtained the number of unlabeled posets up to 16 elements, the sequence A000112 in OEIS [13]. By using the number of unlabeled connected posets up to 16 elements, we determine the number of unlabeled disconnected posets up to 17 elements according to the number of connected direct terms of the posets, the sequences A349401 and A263864 in OEIS [13]. Khamis [9] obtained the number of unlabeled N-free posets up to 14 elements according to the height of the posets, the sequence A202182 in OEIS [13]. By using the number of unlabeled connected N-free posets up to 14 elements, we determine the number of unlabeled disconnected N-free posets up to 15 elements according to the number of connected direct terms of the posets, the sequences A349367 and A350783 in OEIS [13].

In Section 2, we recall some basic terminologies related to the posets and their direct sum. In Section 3, we give the criterion for pairwise nonisomorphic direct sums of connected posets. In Section 4, we establish the formulae giving the enumeration of disconnected posets. In Section 5, we give the enumeration algorithm and prove its time-complexity. In Section 6, we include the numerical results obtained by implementing the enumeration algorithm into the computer.

2 Preliminaries

A *poset* (*partially ordered set*) is a structure $\mathbf{A} = \langle A, \leqslant \rangle$ consisting of the nonempty set A with the order relation $\leqslant$ on A. A poset $\mathbf{A}$ is called *finite* if the underlying set A is finite. Here, we assume that every poset is finite. Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be any posets. A bijective map $\phi : A \to B$ is called an *order isomorphism* if for all $x, y \in A$, $x \leqslant_A y$ if and only if $\phi(x) \leqslant_B \phi(y)$. We write $\mathbf{A} \cong \mathbf{B}$ whenever $\mathbf{A}$ and $\mathbf{B}$ are order isomorphic. By saying that a collection of posets is isomorphic (analogously, nonisomorphic), we mean that the posets in the collection are *pairwise* isomorphic (nonisomorphic). For further details on posets, we refer the readers to the classical book by Davey and Priestley [5].

We use the notation $\mathbf{1}$ for the singleton poset, $\mathbf{C}_n$ ($n \geq 1$) for the n-element chain poset, $\mathbf{I}_n$ ($n \geq 1$) for the n-element antichain poset, $\mathbf{B}_{m,n}$ ($m \geq 1, n \geq 1$) for the complete bipartite poset with m minimal elements and n maximal elements. We write $\mathbf{A} + \mathbf{B}$ to denote the direct sum of $\mathbf{A}$ and $\mathbf{B}$. Here, $\mathbf{A}$ and $\mathbf{B}$ are called the *direct terms* (*components*) of the poset $\mathbf{A} + \mathbf{B}$. We write briefly $\sum_{i=1}^{r} \mathbf{A}_i$ for the direct sum $\mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_r$ and $r\mathbf{A}$ for the direct sum $\mathbf{A}+\mathbf{A}+\cdots+\mathbf{A}$ of r posets $\mathbf{A}$. For example, $\mathbf{I_n} \cong n\mathbf{1}$. A poset having two or more direct terms is called *disconnected*, otherwise, it is called *connected*. Note that, for all posets $\mathbf{A}_i, \mathbf{B}_i$, $1 \leq i \leq r$, since the direct sum of posets is commutative, we have $\sum_{i=1}^{r} \mathbf{A}_i \cong \sum_{i=1}^{r} \mathbf{B}_i$ if and only if $\mathbf{A}_i \cong \mathbf{B}_i$ for every $1 \leq i \leq r$.

3 Nonisomorphic direct sum criterion

For unlabeled connected posets, in particular, we have $\mathcal{Q}_1 = \{\mathbf{1}\}$, $\mathcal{Q}_2 = \{\mathbf{C}_2\}$, and $\mathcal{Q}_3 = \{\mathbf{B}_{1,2},\ \mathbf{B}_{2,1},\ \mathbf{C}_3\}$. For unlabeled disconnected posets, we have $\mathcal{R}_2 = \{2\mathbf{1}\}$, $\mathcal{R}_3 = \{1 + \mathbf{C}_2,\ 3\mathbf{1}\}$, and $\mathcal{R}_4 = \{1 + \mathbf{C}_3,\ 1 + \mathbf{B}_{1,2},\ 1 + \mathbf{B}_{2,1},\ \mathbf{C}_2 + \mathbf{C}_2,\ 2\mathbf{1} + \mathbf{C}_2,\ 4\mathbf{1}\}$. We observe that, for every $2 \leq n \leq 4$, every member of $\mathcal{R}_n$ can be expressed as the direct sum of some members of $\mathcal{Q}_r$, $1 \leq r \leq 3$. In general, for $\mathbf{R}_n \in \mathcal{R}_n$, $n \geq 2$, there exist $\mathbf{Q}_{n_i} \in \mathcal{Q}_{n_i}$, $1 \leq i \leq m$ such that

$$\mathbf{R}_n \cong \mathbf{Q}_{n_1} + \mathbf{Q}_{n_2} + \cdots + \mathbf{Q}_{n_m} = \sum_{i=1}^{m} \mathbf{Q}_{n_i}, \tag{1}$$

where $2 \leq m \leq n$ and $n = \sum_{i=1}^{m} n_i$. Here, m is the number of connected direct terms of $\mathbf{R}_n$. Since the direct sum of posets is commutative, we observe the following.

1. For $n = 2$, we have $\mathbf{R}_2 \cong \mathbf{Q}_1 + \mathbf{Q}_1$. Thus an $\mathbf{R}_2$ can be obtained only in one way with 2 connected direct terms.

2. For $n = 3$, we have $\mathbf{R}_3 \cong \mathbf{Q}_1 + \mathbf{Q}_2 \cong \mathbf{Q}_2 + \mathbf{Q}_1$ and $\mathbf{R}_3 \cong \mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1$. Thus, an $\mathbf{R}_3$ can be obtained in one way with 2 connected direct terms and in one way with 3 direct terms.

3. For $n = 4$, all the ways in which an $\mathbf{R}_4$ can be obtained are given in Table 1. Here, we see that an $\mathbf{R}_4$ can be obtained in two ways with 2 connected direct terms, in one way with 3 connected direct terms, and in one way with 4 connected direct terms.

Number of connected direct terms	Ways in which an $\mathbf{R}_4$ can be obtained
2	$\mathbf{Q}_2 + \mathbf{Q}_2$ and $\mathbf{Q}_1 + \mathbf{Q}_3 \cong \mathbf{Q}_3 + \mathbf{Q}_1$
3	$\mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_2 \cong \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_1 \cong \mathbf{Q}_2 + \mathbf{Q}_1 + \mathbf{Q}_1$
4	$\mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1$

Table 1: All the ways in which an $\mathbf{R}_4$ can be obtained as a direct sum of $\mathbf{Q}_r$, $r \leq 3$.

We see that if the posets $\mathbf{R}_n \in \mathcal{R}_n$ are obtained as above, some of the posets in $\mathcal{R}_n$ can be isomorphic even though the collection of direct terms $\mathbf{Q}_{n_i}$, $1 \leq i \leq m$ is nonisomorphic. In this section, we establish the criterion for the direct sum so that all the posets in $\mathcal{R}_n$ obtained as the direct sum of the posets $\mathbf{Q}_{n_i}$, $1 \leq i \leq m$ are nonisomorphic. Here, for every $1 \leq r \leq n-1$, we must assume that the collection $\mathcal{Q}_r$ is nonisomorphic. To make our intuition more precise, we observe the connected direct terms of the posets $\mathbf{R}_6 \in \mathcal{R}_6$. We see that an $\mathbf{R}_6$ can be obtained in all the ways given in Table 2.

Number of connected direct terms	Ways in which an $\mathbf{R}_6$ can be obtained
2	$\mathbf{Q}_1 + \mathbf{Q}_5$, $\mathbf{Q}_2 + \mathbf{Q}_4$, and $\mathbf{Q}_3 + \mathbf{Q}_3$
3	$\mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_4$, $\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3$, and $\mathbf{Q}_2 + \mathbf{Q}_2 + \mathbf{Q}_2$
4	$\mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_3$ and $\mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_2$
5	$\mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_2$
6	$\mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1 + \mathbf{Q}_1$

Table 2: All the ways in which an $\mathbf{R}_6$ can be obtained as a direct sum of $\mathbf{Q}_r$, $r \leq 5$.

Here, in Table 2, we see that all the other direct sums in which an $\mathbf{R}_6$ can be obtained are isomorphic to one of the direct sums given in the table. This observation shows that all the direct sums in which a poset $\mathbf{R}_n$ can be obtained will be nonisomorphic if the sequence $\langle n_1, n_2, \ldots, n_m \rangle$, as in the equation (1), is nondecreasing, that is, $n_1 \leq n_2 \leq \cdots \leq n_m$. We prove this conjecture in the following. Recall the assumption that, for every $1 \leq r \leq n-1$, the collection $\mathcal{Q}_r$ is nonisomorphic.

Theorem 1. *For all* $\mathbf{R}_n \in \mathcal{R}_n$*, let* $\mathbf{R}_n = \sum_{i=1}^{m} \mathbf{Q}_{n_i}$*, where* $n = \sum_{i=1}^{m} n_i$ *for some* $2 \leq m \leq n$*, such that the sequences* $\langle n_1, n_2, \ldots, n_m \rangle$ *are all nondecreasing and distinct. Then for every pair of posets* $\mathbf{R}_n, \mathbf{R'}_n \in \mathcal{R}_n$*, we have* $\mathbf{R}_n \ncong \mathbf{R'}_n$*.*

Proof. For $\mathbf{R}_n, \mathbf{R'}_n \in \mathcal{R}_n$, let $\mathbf{R}_n \cong \sum_{i=1}^{m} \mathbf{Q}_{n_i}$ and $\mathbf{R'}_n \cong \sum_{i=1}^{m'} \mathbf{Q}_{r_i}$, as in the hypothesis, such that $L = \langle n_1, n_2, \ldots, n_m \rangle \neq \langle r_1, r_2, \ldots, r_{m'} \rangle = L'$. If $m \neq m'$ then $\mathbf{R}_n$ and $\mathbf{R'}_n$ have different numbers of connected direct terms and, clearly, $\mathbf{R}_n \ncong \mathbf{R'}_n$. Otherwise, let $m = m'$. In this case, since both L and L' contain nondecreasing lengths, there exist $1 \leq s, t \leq m$, such that $n_i \neq r_i$ when $s \leq i \leq t$ and $n_i = r_i$ otherwise (in the simplest case, for example, consider the sequences $\langle 1, 2, 3, 4, 5 \rangle$ and $\langle 1, 2, 2, 5, 5 \rangle$ where $s = 3$ and $t = 4$). Also, $r_i < n_s$ or $n_i < r_s$ for all $1 \leq i \leq s-1$ (when $s > 1$); and $n_t < r_i$ or $r_t < n_i$ for all $t+1 \leq i \leq m$ (when $t < m-1$). Thus, there exist either $s \leq u \leq t$ such that $\mathbf{Q}_{n_u} \ncong \mathbf{Q}_{r_i}$ for all $1 \leq i \leq m$, or $s \leq v \leq t$ such that $\mathbf{Q}_{r_v} \ncong \mathbf{Q}_{n_i}$ for all $1 \leq i \leq m$. This shows that $\mathbf{R}_n \ncong \mathbf{R'}_n$. □

4 Enumeration of unlabeled disconnected posets

To determine $|\mathcal{R}_n|$, $n \geq 2$, the observations in the previous section suggest that, for certain $2 \leq m \leq n$ (the number of connected direct terms of the $\mathbf{R}_n \in \mathcal{R}_n$), we must consider only the distinct nondecreasing sequences $\langle n_1, n_2, \ldots, n_m \rangle$, as given in the equation (1). In particular, we see that $|\mathcal{R}_6|$ can be computed by using the direct sums given in Table 2 and the numbers $|\mathcal{Q}_1| = |\mathcal{Q}_2| = 1$, $|\mathcal{Q}_3| = 3$, $|\mathcal{Q}_4| = 10$, and $|\mathcal{Q}_5| = 44$, see [3, 4, 7]. Here, we use the notation $\mathcal{R}_n^m$ to denote the set of all posets $\mathbf{R}_n \in \mathcal{R}_n$ with m connected direct terms. Then we have $\mathcal{R}_n = \bigcup_{m=2}^{n} \mathcal{R}_n^m$. Since the collections $\mathcal{R}_n^m$, $2 \leq m \leq n$ of unlabeled posets are pairwise disjoint, we have $|\mathcal{R}_n| = \sum_{m=2}^{n} |\mathcal{R}_n^m|$. Firstly, we compute $|\mathcal{R}_6^2|$. For $\mathbf{R}_6 \in \mathcal{R}_6^2$, we have the following cases.

1. $\mathbf{R}_6 \cong \mathbf{Q}_1 + \mathbf{Q}_5$.
   Since the direct terms are the posets with unequal numbers of elements, in this case, we have $|\mathcal{Q}_1| \times |\mathcal{Q}_5| = 1 \times 44 = 44$ disconnected posets.

2. $\mathbf{R}_6 \cong \mathbf{Q}_2 + \mathbf{Q}_4$.
   Due to the reason same to the previous case, here, we have $|\mathcal{Q}_2| \times |\mathcal{Q}_4| = 1 \times 10 = 10$ disconnected posets.

3. $\mathbf{R}_6 \cong \mathbf{Q}_3 + \mathbf{Q}_3$.
   Since both the direct terms are the posets with the same number of elements (that is, a direct term in the expression is repeated), in this case, we have $\binom{|\mathcal{Q}_3|+2-1}{2} = \binom{3+1}{2} = 6$ disconnected posets.

These give

$$|\mathcal{R}_6^2| = 44 + 10 + 6 = 60.$$

Similarly, we have

$$\begin{aligned}
|\mathcal{R}_6^3| &= 10+3+1 = 14,\\
|\mathcal{R}_6^4| &= 3+1 = 4,\\
|\mathcal{R}_6^5| &= 1, \text{ and}\\
|\mathcal{R}_6^6| &= 1.
\end{aligned}$$

Finally, we have

$$|\mathcal{R}_6| = \sum_{m=2}^{6} |\mathcal{R}_6^m| = 60+14+4+1+1 = 80.$$

In the following, we establish, in general, the above observations consecutively.

Lemma 2. *For given $2 \le t \le n$, let $\tilde{\mathcal{R}}_n^t \subseteq \mathcal{R}_n$ be the collection of posets such that for $\mathbf{R}_n \in \tilde{\mathcal{R}}_n^t$, we have $\mathbf{R}_n \cong \sum_{i=1}^t \mathbf{Q}_{n_i}$, where $n = \sum_{i=1}^t n_i$ and the sequence $\langle n_1, n_2, \ldots, n_t\rangle$ is strictly increasing. Then $|\tilde{\mathcal{R}}_n^t| = \prod_{i=1}^t |\mathcal{Q}_{n_i}|$.*

Proof. Let $\mathbf{R}_n \in \tilde{\mathcal{R}}_n^t$. As the sequence $\langle n_1, n_2, \ldots, n_t\rangle$ is strictly increasing, the direct terms $\mathbf{Q}_{n_i}$, $1 \le i \le t$ of $\mathbf{R}_n$ have different cardinalities. Thus, all the t direct terms of a poset $\mathbf{R}_n$ can be chosen consecutively from one of the disjoint collections $\mathcal{Q}_{n_1}$, $\mathcal{Q}_{n_2}$, ..., and $\mathcal{Q}_{n_t}$ each of which consists of all nonisomorphic connected posets. Therefore, $|\tilde{\mathcal{R}}_n^t|$ equals the number of the collections consisting of t distinct items each of which is chosen consecutively from one of the collections consisting of $|\mathcal{Q}_{n_1}|$, $|\mathcal{Q}_{n_2}|$, ..., and $|\mathcal{Q}_{n_t}|$ distinct items, respectively. Therefore, we have $|\tilde{\mathcal{R}}_n^t|$ as follows:

$$|\tilde{\mathcal{R}}_n^t| = |\mathcal{Q}_{n_1}| \times |\mathcal{Q}_{n_2}| \times \cdots \times |\mathcal{Q}_{n_t}| = \prod_{i=1}^{t} |\mathcal{Q}_{n_i}|. \tag{2}$$

□

Lemma 3. *For given $2 \le t \le n$, let $\bar{\mathcal{R}}_n^t \subseteq \mathcal{R}_n$ be the collection of posets such that for $\mathbf{R}_n \in \bar{\mathcal{R}}_n^t$, we have $\mathbf{R}_n \cong \sum_{i=1}^t \mathbf{Q}_{n_i}$, where $n = \sum_{i=1}^t n_i$ and the sequence $\langle n_1, n_2, \ldots, n_t\rangle$ is constant. Then $|\bar{\mathcal{R}}_n^t| = \binom{|\mathcal{Q}_r|+t-1}{t}$, where $r = n_i$, $1 \le i \le t$.*

Proof. Let $\mathbf{R}_n \in \bar{\mathcal{R}}_n^t$. As the sequence $\langle n_1, n_2, \ldots, n_t\rangle$ is constant, we assume $r = n_i$, $1 \le i \le t$. Thus, every poset $\mathbf{Q}_{n_i}$, $1 \le i \le t$ consists of r elements. This shows that all the t direct terms of a poset $\mathbf{R}_n$ can be chosen from the same collection $\mathcal{Q}_r$ consisting of $|\mathcal{Q}_r|$ nonisomorphic connected posets. Therefore, $|\bar{\mathcal{R}}_n^t|$ equals the number of t-element combinations of $|\mathcal{Q}_r|$ objects, with repetition. This gives $|\bar{\mathcal{R}}_n^t|$ as follows:

$$|\bar{\mathcal{R}}_n^t| = \binom{|\mathcal{Q}_r|+t-1}{t}. \tag{3}$$

□

Theorem 4. *For given $2 \leq m \leq n$, let $\mathcal{R}_n^m \subseteq \mathcal{R}_n$ be the collection of posets such that for $\mathbf{R}_n \in \mathcal{R}_n^m$, we have $\mathbf{R}_n \cong \sum_{i=1}^m \mathbf{Q}_{n_i}$, where $n = \sum_{i=1}^m n_i$ and the sequence $\langle n_1, n_2, \ldots, n_m \rangle$ is nondecreasing. Then, there exist $q \leq m$ and r_k, t_k, $1 \leq k \leq q$ such that $|\mathcal{R}_n^m| = \prod_{k=1}^q \binom{|\mathcal{Q}_{r_k}|+t_k-1}{t_k}$.*

Proof. Let $\mathbf{R}_n \in \mathcal{R}_n^m$. Since the sequence $\langle n_1, n_2, \ldots, n_m \rangle$, as in the hypothesis, is nondecreasing, there exist q and r_k, t_k, $1 \leq k \leq q$ such that

$$\begin{aligned}
r_1 &= n_1 = n_2 = \cdots = n_{t_1}, \\
r_2 &= n_{t_1+1} = n_{t_1+2} = \cdots = n_{t_1+t_2}, \\
r_3 &= n_{t_1+t_2+1} = n_{t_1+t_2+2} = \cdots = n_{t_1+t_2+t_3}, \\
&\vdots \\
r_q &= n_{t_1+\cdots+t_{q-1}+1} = n_{t_1+\cdots+t_{q-1}+2} = \cdots = n_m,
\end{aligned}$$

where $r_1 < r_2 < \cdots < r_q$ and $m = t_1 + \cdots + t_q$. Let $\mathbf{R}_{r_k t_k} \in \bar{\mathcal{R}}_{r_k t_k}^{t_k}$ be the k-th subcollection of t_k consecutive connected direct terms of the poset $\mathbf{R}_n$. Then, for every $1 \leq k \leq q$, the poset $\mathbf{R}_{r_k t_k}$ can be expressed as follows:

$$\mathbf{R}_{r_k t_k} \cong \underbrace{\mathbf{Q}_{r_k} + \mathbf{Q}_{r_k} + \cdots + \mathbf{Q}_{r_k}}_{t_k \text{ terms}}. \tag{4}$$

Therefore, the poset $\mathbf{R}_n$ can be expressed as follows:

$$\mathbf{R}_n \cong \mathbf{R}_{r_1 t_1} + \mathbf{R}_{r_2 t_2} + \cdots + \mathbf{R}_{r_q t_q}. \tag{5}$$

Since the sequence $\langle r_k,\ r_k,\ \cdots,\ r_k \rangle$ is constant, all of the t_k direct terms $\mathbf{Q}_{r_k}$ in (4) are connected posets with r_k elements. Then, by Lemma 3, we have $|\bar{\mathcal{R}}_{r_k t_k}^{t_k}|$ as follows:

$$|\bar{\mathcal{R}}_{r_k t_k}^{t_k}| = \binom{|\mathcal{Q}_{r_k}| + t_k - 1}{t_k}. \tag{6}$$

Since the sequence $\langle r_1,\ r_2,\ \cdots,\ r_q \rangle$ is strictly increasing, every direct term $\mathbf{R}_{r_k t_k}$ in (5) is itself a direct sum that consists of t_k connected direct terms each consisting of r_k elements. Then, by Lemma 2, we have $|\mathcal{R}_n^m|$ as follows:

$$|\mathcal{R}_n^m| = \prod_{k=1}^q |\bar{\mathcal{R}}_{r_k t_k}^{t_k}|.$$

Then, by using the equation (6), we have $|\mathcal{R}_n^m|$ as follows:

$$|\mathcal{R}_n^m| = \prod_{k=1}^q \binom{|\mathcal{Q}_{r_k}| + t_k - 1}{t_k}. \tag{7}$$

□

Note that the value of the parameter q, as in the equation (7), equals the number of blocks with the same connected direct terms in the direct sum with m terms as considered above. The following example illustrates Theorem 4.

Example 5. Let $\mathcal{S} = \{\mathbf{R}_{23} \in \mathcal{R}_{23}^{6} : \mathbf{R}_{23} \cong \mathbf{Q}_3 + \mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_4 + \mathbf{Q}_4 + \mathbf{Q}_5\}$. Here, we describe the computation of $|\mathcal{S}|$ by using the equation (7). We have

$$\begin{aligned}\mathbf{R}_{23} &\cong \mathbf{Q}_3 + \mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_4 + \mathbf{Q}_4 + \mathbf{Q}_5\\ &\cong (\mathbf{Q}_3 + \mathbf{Q}_3) + (\mathbf{Q}_4 + \mathbf{Q}_4 + \mathbf{Q}_4) + (\mathbf{Q}_5)\\ &\cong \mathbf{R}_6 + \mathbf{R}_{12} + \mathbf{Q}_5 \text{ (say)},\end{aligned}$$

where $\mathbf{R}_6 \cong \mathbf{Q}_3 + \mathbf{Q}_3$ and $\mathbf{R}_{12} \cong \mathbf{Q}_4 + \mathbf{Q}_4 + \mathbf{Q}_4$. Note that, in the case of the direct sum considered above, the value of the parameter q (the number of distinct blocks with repeated connected direct terms), as in the equation (7), equals 3. Now we assume

$$\begin{aligned}&\mathcal{S}_1 = \{\mathbf{R}_6 \in \mathcal{R}_6^2 : \mathbf{R}_6 \cong \mathbf{Q}_3 + \mathbf{Q}_3\},\\ &\mathcal{S}_2 = \{\mathbf{R}_{12} \in \mathcal{R}_{12}^3 : \mathbf{R}_{12} \cong \mathbf{Q}_4 + \mathbf{Q}_4 + \mathbf{Q}_4\}, \text{ and}\\ &\mathcal{S}_3 = \mathcal{Q}_5.\end{aligned}$$

Then

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3,$$

where the collections $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ are pairwise disjoint. This implies

$$|\mathcal{S}| = |\mathcal{S}_1| \times |\mathcal{S}_2| \times |\mathcal{S}_3|.$$

Here, we have

$$\begin{aligned}&|\mathcal{S}_1| = \binom{|\mathcal{Q}_3| + 2 - 1}{2} = \binom{3+1}{2} = 6,\\ &|\mathcal{S}_2| = \binom{|\mathcal{Q}_4| + 3 - 1}{3} = \binom{10+2}{3} = 220, \text{ and}\\ &|\mathcal{S}_3| = |\mathcal{Q}_5| = 44.\end{aligned}$$

Therefore,

$$|\mathcal{S}| = 6 \times 220 \times 44 = 58,080.$$

Now, we give the enumeration formula for the unlabeled disconnected posets, in general, as follows:

Theorem 6. *Let $\mathcal{R}_n$ be the collection of posets such that for $\mathbf{R}_n \in \mathcal{R}_n$, we have $\mathbf{R}_n \cong \sum_{i=1}^{m} \mathbf{Q}_{n_{ij}}$, $1 \le j \le p_m$, $2 \le m \le n$ for some $p_m \le \binom{n-1}{m}$, where $n = \sum_{i=1}^{m} n_{ij}$, $1 \le j \le p_m$ and all the sequences $\langle n_{1j}, n_{2j}, \dots, n_{mj}\rangle$, $1 \le j \le p_m$ are nondecreasing. Then, there exist $q_{mj} \le m$ and r_{mjk}, t_{mjk}, $1 \le k \le q_{mj}$, $1 \le j \le p_m$, $2 \le m \le n$ such that $|\mathcal{R}_n| = \sum_{m=2}^{n} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{|\mathcal{Q}_{r_{mjk}}| + t_{mjk} - 1}{t_{mjk}}$.*

Proof. Let $\mathbf{R}_n \in \mathcal{R}_n$. Then $\mathbf{R}_n \in \mathcal{R}_n^{mj}$ for some $1 \leq j \leq p_m$ and $2 \leq m \leq n$. Since the sequences $\langle n_{1j}, n_{2j}, \ldots, n_{mj} \rangle$, $1 \leq j \leq p_m$, as in the hypothesis, are nondecreasing, there exist q_{mj} and r_{mjk}, t_{mjk}, for $1 \leq k \leq q_{mj}$, $1 \leq j \leq p_m$, $2 \leq m \leq n$ such that

$$\begin{aligned} r_{mj1} &= n_{ij}, 1 \leq i \leq t_{mj1}, \\ r_{mj2} &= n_{ij}, t_{mj1} + 1 \leq i \leq t_{mj2}, \\ &\vdots \\ r_{mjq_{mj}} &= n_{ij}, t_{mj(q-1)} + 1 \leq i \leq t_{mjq_{mj}}, \end{aligned}$$

where $r_{mj1} < r_{mj2} < \cdots < r_{mjq_{mj}}$ and $m = t_{mj1} + \cdots + t_{mjq_{mj}}$. For certain m and j, let $\mathbf{R}_{r_{mjk}t_{mjk}} \in \bar{\mathcal{R}}^{t_{mjk}}_{r_{mjk}t_{mjk}}$ be the k-th subcollection of t_{mjk} consecutive direct terms of the poset $\mathbf{R}_n$. Then, for every $1 \leq k \leq q_{mj}$, the poset $\mathbf{R}_{r_{mjk}t_{mjk}}$ can be expressed as follows:

$$\mathbf{R}_{r_{mjk}t_{mjk}} \cong \underbrace{\mathbf{Q}_{r_{mjk}} + \mathbf{Q}_{r_{mjk}} + \cdots + \mathbf{Q}_{r_{mjk}}}_{t_{mjk} \text{ terms}}. \tag{8}$$

Therefore, the poset $\mathbf{R}_n$ can be expressed as follows:

$$\mathbf{R}_n \cong \mathbf{R}_{r_{mj1}t_{mj1}} + \mathbf{R}_{r_{mj2}t_{mj2}} + \cdots + \mathbf{R}_{r_{mjq_{mj}}t_{mjq_{mj}}}. \tag{9}$$

Since the sequence $\langle r_{mjk}, r_{mjk}, \cdots, r_{mjk} \rangle$ is constant, all of the t_{mjk} direct terms $\mathbf{Q}_{r_{mjk}}$ in (8) are connected posets with r_{mjk} elements. Again, since the sequence $\langle r_{mj1}, r_{mj2}, \cdots, r_{mjq_{mj}} \rangle$ is strictly increasing, each of the direct terms $\mathbf{R}_{r_{mjk}t_{mjk}}$ in (9) itself is a direct sum consisting of the connected direct terms having r_{mjk} elements. Then, by Theorem 4, we have $|\mathcal{R}_n^{mj}|$ as follows:

$$|\mathcal{R}_n^{mj}| = \prod_{k=1}^{q_{mj}} \binom{|\mathcal{Q}_{r_{mjk}}| + t_{mjk} - 1}{t_{mjk}}. \tag{10}$$

Since $|\mathcal{R}_n|$ equals the sum of $|\mathcal{R}_n^{mj}|$ for all possible values of m and j, where $2 \leq m \leq n$ and $1 \leq j \leq p_m$ for some $p_m \leq \binom{n-1}{m}$, we have $|\mathcal{R}_n|$ as follows:

$$|\mathcal{R}_n| = \sum_{m=2}^{n} \sum_{j=1}^{p_m} |\mathcal{R}_n^{mj}|, n \geq 2.$$

Finally, by using the equation (10), we have $|\mathcal{R}_n|$ as follows:

$$|\mathcal{R}_n| = \sum_{m=2}^{n} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{|\mathcal{Q}_{r_{mjk}}| + t_{mjk} - 1}{t_{mjk}}, n \geq 2. \tag{11}$$

□

Note that for every $2 \leq m \leq n$, the value of the parameter p_m, as in (11), equals the number of nondecreasing sequences $\langle n_1, n_2, \ldots, n_m \rangle$, as in (1), where $n_i \in \{1, 2, \ldots, n-1\}$ for all $1 \leq i \leq m$. Also, for every $2 \leq m \leq n$ and $1 \leq j \leq p_m$, the value of the parameter q_{mj}, as in (11), equals the number of blocks with same connected direct terms in the direct sum corresponding to the j-th nondecreasing sequence with m numbers as constructed above.

5 Enumeration algorithm

Recall that we do not determine explicitly the parameters p_m, q_{mj}, and r_{mjk}, where $1 \leq k \leq q_{mj}$, $1 \leq j \leq p_m$, and $2 \leq m \leq n$, as in the equation (11). Therefore, for given $n \geq 2$, the computation of $|\mathcal{R}_n|$ depends on determining these parameters by constructing mainly the nondecreasing sequences $\langle n_{1j}, n_{2j}, \ldots, n_{mj} \rangle$ for all $1 \leq j \leq p_m$ and $2 \leq m \leq n$. Note that, by inspection, we have $p_m \leq n^2$ for all $2 \leq m \leq n$. Also, we have $q_{mj} \leq m + 1$ for all $1 \leq j \leq p_m$ and $2 \leq m \leq n$. Here, by using Algorithm 7 given below, we construct the nondecreasing sequences $\langle n_{1j}, n_{2j}, \ldots, n_{mj} \rangle$ and determine the parameters p_m, q_{mj}, r_{mjk} for all $1 \leq k \leq q_{mj}$, $1 \leq j \leq p_m$, $2 \leq m \leq n$, and finally compute the numbers $|\mathcal{R}_n|$ for $n \geq 2$.

Algorithm 7. To compute $V = |\mathcal{R}_n|$, the number of n-element unlabeled disconnected posets, where $n \geq 2$ is fixed.

(1) Initialize $V = 0$.

(2) Repeat (A) for $m = 2$ to n.

  (A) Repeat (i) to (iv) for every nondecreasing sequence $L(m, j) = \langle n_{1j}, n_{2j}, \ldots, n_{mj} \rangle$, $1 \leq j \leq p_m$ as is constructed in (i). (Here, the total number of repetitions equals the value of the parameter p_m in the equation (11)).

    (i) Construct j-th nondecreasing sequence $L(m, j)$ consisting of m integers chosen from the integers 1, 2, ..., $n - 1$.

    (ii) Initialize $S(m, j)$ as $S(m, j) = 1$.

    (iii) Compute t_{mjk} and repeat (a) below for every distinct r_{mjk} in the sequence $L(m, j)$. (Here, the total number of distinct r_{mjk} equals the value of the parameter q_{mj} in the equation (11)).

      (a) Update $S(m, j)$ with $S(m, j) \times \binom{|\mathcal{Q}_{r_{mjk}}| + t_{mjk} - 1}{t_{mjk}}$.

    (iv) Increase V by $S(m, j)$.

(3) Return V.

Lemma 8. *Algorithm 7 runs in time $\mathcal{O}(n^5)$.*

Proof. The constructions of the sequences $L(m, j)$ in the step (i) have complexity equal to $m(n - 1)$. Since $1 \leq t_{mjk}, q_{mj} \leq m + 1$ and t_{mjk} is inversely proportional to q_{mj}, the computations of $S(m, j)$ in the step (iii) have complexity equal to $m + 1$. Then $m \leq n$ implies that the complexity $m(n - 1) \approx \mathcal{O}(n(n - 1)) \approx \mathcal{O}(n^2)$ and the complexity $m + 1 \approx \mathcal{O}(n + 1) \approx \mathcal{O}(n)$. Since $1 \leq p_m \leq n^2$, the repetitions in the step (A) increase the complexity to $n^2(\mathcal{O}(n^2) + \mathcal{O}(n)) \approx \mathcal{O}(n^4)$. Finally, the repetitions in the step (3) increase the complexity to $n(\mathcal{O}(n^4)) \approx \mathcal{O}(n^5)$. □

6 Data

We implemented the enumeration algorithm into the computer and determined the number of unlabeled disconnected posets up to 17 elements (Table 3 and Table 4) and the number of unlabeled disconnected N-free posets up to 15 elements (Table 5) according to the number of connected direct terms of the posets.

$m \diagdown n$	2	3	4	5	6	7	8	9	10	11	12	13
2	1	1	4	13	60	312	2075	17316	186173	2594568	47041877	1108710868
3		1	1	4	14	63	328	2159	17801	189406	2620368	47298156
4			1	1	4	14	64	331	2175	17885	189906	2623701
5				1	1	4	14	64	332	2178	17901	189990
6					1	1	4	14	64	332	2179	17904
7						1	1	4	14	64	332	2179
8							1	1	4	14	64	332
9								1	1	4	14	64
10									1	1	4	14
11										1	1	4
12											1	1
13												1
Total:	1	2	6	19	80	395	2487	19890	206565	2804453	49872647	1158843214

Table 3: The number of n-element unlabeled disconnected posets for $2 \leq n \leq 13$ according to the number of connected direct terms m, $2 \leq m \leq 13$.

$m \diagdown n$	14	15	16	17
2	33887448384	1339579736074	68314951618033	4484639396830962
3	1111998749	33943069332	1340826402638	68351780482060
4	47324726	1112260708	33946401908	1340882409188
5	2624201	47328080	1112287428	33946665077
6	190006	2624285	47328580	1112290782
7	17905	190009	2624301	47328664
8	2179	17905	190010	2624304
9	332	2179	17905	190010
10	64	332	2179	17905
11	14	64	332	2179
12	4	14	64	332
13	1	4	14	64
14	1	1	4	14
15		1	1	4
16			1	1
17				1
Total:	35049606566	1374685228988	69690886873398	4554367168841547

Table 4: The number of n-element unlabeled disconnected posets for $14 \leq n \leq 17$ according to the number of connected direct terms m, $2 \leq m \leq 17$.

$m \diagdown n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1	1	4	12	46	173	727	3195	15017	74404	388895	2134070	12266637	73642052
3		1	1	4	13	49	188	795	3502	16436	81146	421816	2300237	13134628
4			1	1	4	13	50	191	810	3570	16758	82655	429138	2336477
5				1	1	4	13	50	192	813	3585	16826	82977	430668
6					1	1	4	13	50	192	814	3588	16841	83045
7						1	1	4	13	50	192	814	3589	16844
8							1	1	4	13	50	192	814	3589
9								1	1	4	13	50	192	814
10									1	1	4	13	50	192
11										1	1	4	13	50
12											1	1	4	13
13												1	1	4
14													1	1
15														1
Total:	1	2	6	18	65	241	984	4250	19590	95484	491459	2660030	15100494	89648378

Table 5: The number of n-element unlabeled disconnected N-free posets for $2 \leq n \leq 15$ according to the number of connected direct terms m, $2 \leq m \leq 15$.

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