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Partial Franel Sums

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Abstract

We derive analytical expressions for the position of irreducible fractions in the Farey sequence F_N of order N for a particular choice of N, obtaining an asymptotic behavior with a lower error bound than in previous results when these fractions are in the vicinity of $0/1$, $1/2$, or $1/1$.

Franel's famous formulation of Riemann's hypothesis uses the summation of distances between irreducible fractions and evenly spaced points in $[0, 1]$. We define "partial Franel sum" as a summation of these distances over a subset of fractions in F_N and we demonstrate that the partial Franel sum in the range $[0, i/N]$, with $N = \operatorname{lcm}(1, 2, \ldots, i)$, grows strictly slower than $O(\log N)$.

1 Introduction and statement of the main results

The Farey sequence F_N of order N is an ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed N [1]. Riemann's hypothesis implies that the irreducible fractions tend to be regularly distributed in $[0, 1]$. A formulation of this statement follows [2, 3],

$$\sum_{n=1}^{|F_N|} \left| F_N(n) - \frac{n}{|F_N|} \right| = O(N^{\frac{1}{2}+\epsilon}) ,$$

where $F_N(n)$ is the n^{th} irreducible fraction in F_N. Here we define the partial Franel sum in the range $[a_1/b_1, a_2/b_2]$ as

$$P\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) = \sum_{n=I_N(a_1/b_1)}^{I_N(a_2/b_2)} \left| F_N(n) - \frac{n}{|F_N|} \right| ,$$

where $I_N(a/b)$ is the position that a/b occupies in F_N. Dress [4] established the upper bound of the distance $|F_N(n) - n/|F_N||$ to be $1/N$ and to be located at $F_N(2) = 1/N$. This motivates the study of partial Franel sums in ranges including $1/N$. Furthermore, another equivalent formulation of the Riemann's hypothesis involving sums over irreducible fractions in the range $[0, 1/4]$ follows [5],

$$\sum_{n=1}^{I_N(1/4)} \left(F_N(n) - \frac{I_N(1/4)}{2|F_N|} \right) = O(N^{\frac{1}{2}+\epsilon}) ,$$

showing again the relevance of the vicinity of $1/N$.

Guthery [6, Chapter 6] attempted to find a closed expression for the i^{th} fraction in F_N ending in an "analytical hole". This paper achieves this goal for fractions in the range $[0, i/N]$, with $N = \operatorname{lcm}(1, 2, \ldots, i)$ as explained in the following. Note that $N = \operatorname{lcm}(1, 2, \ldots, i) = e^{\psi(i)}$, where $\psi(i)$ is the second Chebyshev function that fulfills the property $\psi(i) = (1+o(1))i$, and hence $i = (1 + o(1)) \log N$.

Let the subsequence $F_N^{a_1/b_1,\, a_2/b_2}$ of F_N, contain all the fractions of F_N in $[a_1/b_1,\ a_2/b_2]$. The cardinality of $F_N^{a_1/b_1,\, a_2/b_2}$ is well known to be [7]

$$\left| F_N^{a_1/b_1,\, a_2/b_2} \right| = \frac{3}{\pi^2} \left(\frac{a_2}{b_2} - \frac{a_1}{b_1} \right) N^2 + O(N \log N) .$$

As $I_N(a_2/b_2)$ is the position that a_2/b_2 occupies in F_N, it follows that

$$I_N\left(\frac{a_2}{b_2}\right) = \left| F_N^{0/1,\, a_2/b_2} \right| = \frac{3}{\pi^2} \frac{a_2}{b_2} N^2 + O(N \log N) . \quad (1)$$

A first result of this paper is the derivation of an analytical expression for $I_N(1/q)$ where $N = \operatorname{lcm}(1, 2, \ldots, i)$ and $N/i \leq q \leq N$ in Theorem 3 as

$$I_N\left(\frac{1}{q}\right) = 2 + N \sum_{j=1}^{i} \frac{\varphi(j)}{j} - q\Phi(i) ,$$

where $\varphi(i)$ is the totient function and $\Phi(i)$ is the summatory totient function defined as

$$\Phi(i) = \sum_{j=1}^{i} \varphi(j) .$$

Theorem 3 also includes a more general expression giving the location of other fractions in F_N. To reach this relation a series of bijections are established in Theorem 1 between $F_{i'}$, with $i' \leq i$, and subsequences of F_N covering all elements in $F_N^{0/1,\, 1/q}$. Thanks to these bijections the cardinality of $F_N^{0/1,\, 1/q}$ can be expressed as function of all $|F_{i'}|$ as shown in Corollary 2. These bijections are illustrated in Table 1 for $N = \operatorname{lcm}(1, 2, \ldots, 5) = 60$. This result is used to derive the equivalent asymptotic estimate of (1) with a smaller residual error in Corollary 4 as follows:

$$I_N\left(\frac{1}{q}\right) \;=\; \frac{3}{\pi^2}q\left(\frac{N^2}{q^2}-\left\{\frac{N}{q}\right\}^2\right)+O\left(N\delta_A\left(\left\lfloor\frac{N}{q}\right\rfloor\right)\right) \;,$$

where $\{x\}=x-\lfloor x\rfloor$ and $\delta_A(x)$ is a decreasing function defined as

$$\delta_A(x) \;=\; \exp\left(-A\frac{\log^{0.6}x}{(\log\log x)^{0.2}}\right) \;, \tag{2}$$

where $A>0$.

As the final result of this work Theorem 5 establishes that the partial Franel sum in the range $[0,\,1/(N/i)]$ is given by

$$P\left(\frac{0}{1},\frac{1}{N/i}\right)=O(\log(N)\delta_B(\log N)) \;,$$

with $0<B<A$ and again $N=\mathrm{lcm}(1,2,\ldots,i)$. Therefore, this partial Franel sum grows strictly slower than $O(\log N)$. If we would assume the Riemann hypothesis and a uniform distribution density of Farey elements in $[0,1]$, we would expect this partial Franel sum to decrease as $O(\log(N)/N^{1/2-\epsilon})$. Theorem 5 includes equivalent results for partial Franel sums in ranges including 1/2 or 1/1. The generalization to compute partial Franel sums in the vicinity of any irreducible fraction is explored. Earlier results of this work were applied to resonance diagrams [8, 9].

The following definitions are used in the rest of the paper. We say that two elements of a Farey sequence, a_1/b_1 and a_2/b_2, form a Farey pair if $|a_1b_2-a_2b_1|=1$. In this report we exceptionally allow 0/1 and 1/0 to form a Farey pair even if 1/0 is not a proper fraction. The mediant of a Farey pair, a_1/b_1 and a_2/b_2, is given by

$$\frac{a_1+a_2}{b_1+b_2}$$

which is an irreducible fraction existing between a_1/b_1 and a_2/b_2 and forms two Farey pairs with a_1/b_1 and a_2/b_2.

2 Results

Theorem 1. *Let a_1/b_1 and a_2/b_2 be a Farey pair with $b_1>b_2$. Let N be multiple of $b_1i(i+1)$ with i being a natural number such $0<i<N$. Let q be an integer fulfilling*

$$\frac{N}{b_1(i+1)}<q\leq\frac{N}{b_1 i} \quad \textit{and} \quad b_1q+b_2\leq N \;.$$

Let F_i' be a subsequence of F_i defined as

$$F_i' = \left\{ \frac{h}{k} : \frac{h}{k} \in F_i \,,\ k(b_1 q + b_2) - b_1 h \leq N \right\} \ ,$$

noting that for $a_1/b_1 = 0/1$ and $a_2/b_2 = 1/0$, $F_i' = F_i$.
There is a bijective map M between F_i' and $F_N^{\frac{a_1 q + a_2}{b_1 q + b_2}, \frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}}$, given by

$$M: \ F_i' \to F_N^{\frac{a_1 q + a_2}{b_1 q + b_2}, \frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}} \ , \qquad \frac{h}{k} \mapsto \frac{k(a_1 q + a_2) - a_1 h}{k(b_1 q + b_2) - b_1 h} \ .$$

$$M^{-1}: \ F_N^{\frac{a_1 q + a_2}{b_1 q + b_2}, \frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}} \to F_i' \ , \qquad \frac{u}{l} \mapsto \frac{q(b_1 u - l a_1) + b_2 u - l a_2}{b_1 u - l a_1} \ .$$

The bijective map is order-preserving when $a_2/b_2 > a_1/b_1$ and order-inverting when $a_2/b_2 < a_1/b_1$.

Proof. We first demonstrate that M is injective. The fractions $\frac{a_1 q + a_2}{b_1 q + b_2}$ and $\frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}$ form a Farey pair since a_1/b_1 and a_2/b_2 form a Farey pair:

$$|(a_1 q + a_2)(b_1(q-1) + b_2) - (b_1 q + b_2)(a_1(q-1) + a_2)| = |b_2 a_1 - a_2 b_1| = 1 \ .$$

Let u/l be the image of h/k under M,

$$\frac{u}{l} = \frac{k(a_1 q + a_2) - a_1 h}{k(b_1 q + b_2) - b_1 h} \ .$$

By virtue of this expression u/l is obtained by applying the mediant operation successively between $\frac{a_1 q + a_2}{b_1 q + b_2}$ and $\frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}$ in the same fashion as h/k is obtained by applying the mediant between $0/1$ and $1/1$, meaning

$$\begin{aligned} \frac{h}{k} &= \frac{(k-h)\cdot 0 + h \cdot 1}{(k-h)\cdot 1 + h \cdot 1} \ , \\ \frac{u}{l} &= \frac{(k-h)\cdot(a_1 q + a_2) + h \cdot (a_1(q-1) + a_2)}{(k-h)\cdot(b_1 q + b_2) + h \cdot (b_1(q-1) + b_2)} \ . \end{aligned}$$

Therefore u/l is a Farey fraction in the interval of interest:

$$\left[\frac{a_1 q + a_2}{b_1 q + b_2}, \frac{a_1(q-1) + a_2}{b_1(q-1) + b_2} \right] \ .$$

The fraction u/l belongs to F_N by definition of the domain F_i', meaning that h/k belonging to F_i' needs $l \leq N$. Therefore M is injective.

Now we demonstrate that M^{-1} is also injective. Let u/l belong to $F_N^{\frac{a_1q+a_2}{b_1q+b_2},\frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}}$ and assume $a_2/b_2 > a_1/b_1$, so that

$$\frac{a_1q+a_2}{b_1q+b_2} \leq \frac{u}{l} \leq \frac{a_1(q-1)+a_2}{b_1(q-1)+b_2} \quad . \tag{3}$$

Let h/k be the image of u/l under M^{-1},

$$\frac{h}{k} = \frac{q(b_1u-la_1)+ub_2-la_2}{b_1u-la_1} \quad . \tag{4}$$

This equality implies gcd(h,k)= gcd(ub_2-la_2, b_1u-la_1). Using

$$\gcd(u,l) = \gcd(a_1,b_1) = \gcd(a_2,b_2) = 1 \ ,$$

$a_2b_1 - a_1b_2 = 1$, and known equalities [10] implies $\gcd(h,k) = 1$. Hence, h/k is an irreducible fraction. Furthermore, operating with the inequalities in (3):

$$q(b_1u-la_1) \geq -(ub_2-la_2) \geq (b_1u-la_1)(q-1),$$

and therefore $0 \leq h \leq k$. This shows that h/k belongs to F_k. In the following we demonstrate that $k \leq i$ so that h/k belongs to F_i too.

From relations (3) and (4)

$$k = b_1u - la_1 \leq b_1 l \frac{a_1(q-1)+a_2}{b_1(q-1)+b_2} - la_1 = \frac{l}{b_1(q-1)+b_2}$$

and using that $l \leq N$ and $b_1(q-1) \geq \frac{N}{i+1}$, which derives from $q > \frac{N}{b_1(i+1)}$,

$$k \leq \frac{N}{\frac{N}{i+1}+b_2} = \frac{i+1}{1+\frac{i+1}{N}b_2} < i+1 \ .$$

If $b_2 > 0$ this implies $k \leq i$ and gathering the above results $0 \leq h \leq k \leq i$ and gcd(h,k)=1, hence $h/k \in F_i$. To demonstrate that h/k belongs to F'_i it is easy to verify that $k(b_1q+b_2) - b_1h \leq N$.

If $b_2 = 0$ we are in the exceptional case included in this report of $a_1/b_1 = 0/1$ and $a_2/b_2 = 1/0$, that implies $h/k = (qu-l)/u$. Note that $k = u$. We only need to show that $k \leq i$ also in this case. From the inequalities in (3) and $\frac{N}{i} \geq q > \frac{N}{i+1}$,

$$\frac{i}{N} \leq \frac{1}{q} \leq \frac{u}{l} \leq \frac{1}{q-1} \leq \frac{i+1}{N} \ .$$

The fraction $(i+1)/N$ is not irreducible, as N is taken as a multiple of $i(i+1)$, and therefore it does not belong to F_N. Similarly for i/N when $i > 1$. In the range $[i/N, (i+1)/N]$ there cannot be fractions with denominator N other than $1/N$ when $i = 1$. Therefore if $i = 1$ we directly have $k = u \leq i$ and for $i > 1$ we have $l \leq N-1$ and hence

$$k = u \leq l\frac{i+1}{N} \leq i \ .$$

□

Corollary 2. *The cardinalities of F_i, F_i' and $F_N^{\frac{a_1q+a_2}{b_1q+b_2},\frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}}$ are related as follows:*

- *If $q = N/(b_1 i)$, then*

$$|F_i| \geq |F_i'| = \left| F_N^{\frac{a_1q+a_2}{b_1q+b_2},\frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}} \right| > |F_i| - i \ .$$

- *If $q < N/(b_1 i)$ or $b_2 = 0$, then*

$$|F_i| = |F_i'| = \left| F_N^{\frac{a_1q+a_2}{b_1q+b_2},\frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}} \right| \ .$$

Proof. The first inequality is evident from the definition of F_i'. The first equality derives from the bijective map in Theorem 1.

If $q = N/(b_1 i)$, let u/l be the image of h/k via the map M in Theorem 1, then $l = k(N/i + b_2) - b_1 h$. To prove that $|F_i'| > |F_i| - i$ we should count how many $h/k \in F_i$ fulfill $k(N/i + b_2) - b_1 h > N$. Dividing both sides of the later inequality by k and operating we obtain

$$b_2 - b_1\frac{h}{k} > \frac{N}{k} - \frac{N}{i} = N\frac{i-k}{ki} \ ,$$
$$b_2 \geq b_2 - b_1\frac{h}{k} > N\frac{i-k}{ki} \geq 0 \ .$$

To fulfill these inequalities it is required that $k = i$. Otherwise for any $k < i$ and recalling that N is a multiple of $b_1 i(i+1)$:

$$b_2 > N\frac{i-k}{ki} \geq b_1\frac{i+1}{k}(i-k) > b_1 \ ,$$

which is inconsistent with the assumption $b_2 < b_1$ in Theorem 1. Then, $k = i$ implies $h/i < b_2/b_1 < 1$ and in F_i there are fewer than i irreducible fractions of the form h/i below b_2/b_1, hence $|F_i'| > |F_i| - i$.

If $q < N/(b_1 i)$ we define $g > 0$ such that $q = N/(b_1 i) - g$; therefore, $l = k(N/i - gb_1 + b_2) - b_1 h$. Now, we need to count how many h/k in F_i have $l > N$,

$$b_2 - b_1\frac{h}{k} - b_1 g > N\frac{i-k}{ki} \ ,$$

and there are no h/k which can fulfill this equation as $b_2 - b_1 g < 0$. Hence, $|F_i| = |F_i'|$ when $q < N/(b_1 i)$.

If $b_2 = 0$ we should show that there are no h/k in $|F_i|$ fulfilling $kb_1 q - b_1 h > N$. The largest possible value of q is $N/(b_1 i)$ and therefore $kb_1 q - b_1 h \leq kN/i - b_1 h < N$, for $i > 1$, so there is no h/k fulfilling the previous condition and $|F_i| = |F_i'|$. Note that $i = 1$ and $h/k = 0/1$ would not have given $kb_1 q - b_1 h > N$ as $b_1 q + b_2 \leq N$ from the assumptions in Theorem 1. □

Theorem 3. *Let* $N = b_1 \mathrm{lcm}(1, 2 \ldots, i_{\max})$*,* $\frac{N}{b_1(i+1)} < q \leq \frac{N}{b_1 i}$*, with* a_1/b_1 *and* a_2/b_2 *forming a Farey pair,* $b_1 > b_2$ *and* $i < i_{\max}$*. Then*

- *For* $b_1 > 1$*:*

$$I_N\left(\frac{a_1 q + a_2}{b_1 q + b_2}\right) = I_N\left(\frac{a_1}{b_1}\right) + s\left(\frac{N}{b_1}\sum_{j=1}^{i}\frac{\varphi(j)}{j} - q\Phi(i)\right) + O(i^2)\ , \tag{5}$$

  with $s = +1$ *when* $a_1/b_1 < a_2/b_2$ *and* $s = -1$ *otherwise.*

- *For* $a_1/b_1 = 0/1$ *and* $a_2/b_2 = 1/0$*:*

$$I_N\left(\frac{1}{q}\right) = 2 + N\sum_{j=1}^{i}\frac{\varphi(j)}{j} - q\Phi(i)\ .$$

Proof. To simplify equations we assume $s = +1$ in the following. We count the number of elements in $F_N^{\frac{a_1}{b_1},\frac{a_1 q + a_2}{b_1 q + b_2}}$ using the bijective maps described in Theorem 1 and adding up the cardinalities of the sets involved from Corollary 2. Thanks to the fact that N is a multiple of all natural numbers i' such that $i' \leq i$ we can establish bijections between F'_i and $F_N^{\frac{a_1 p + a_2}{b_1 p + b_2},\frac{a_1(p-1)+a_2}{b_1(p-1)+b_2}}$ where p can take all values fulfilling $\frac{N}{b_1(i'+1)} < p \leq \frac{N}{b_1 i'}$, covering all elements in $F_N^{\frac{a_1}{b_1},\frac{a_1 q + a_2}{b_1 q + b_2}}$ when scanning over all $i' \leq i$ and the corresponding p. For a given i' the number of values p takes is given by

$$\frac{N}{b_1 i'} - \frac{N}{b_1(i'+1)} = \frac{N}{b_1}\left(\frac{1}{i'} - \frac{1}{i'+1}\right)\ .$$

In a first step we compute the number of elements in $F_N^{\frac{a_1}{b_1},\frac{a_1 q' + a_2}{b_1 q' + b_2}}$ with $q' = N/(b_1 i)$,

$$\begin{aligned} I_N\left(\frac{a_1 q' + a_2}{b_1 q' + b_2}\right) - I_N\left(\frac{a_1}{b_1}\right) &= \frac{N}{b_1}\sum_{i'=1}^{i-1}\left(\frac{1}{i'} - \frac{1}{i'+1}\right)(|F'_{i'}| - 1) \\ &= \frac{N}{b_1}\sum_{i'=1}^{i-1}\left[\left(\frac{1}{i'} - \frac{1}{i'+1}\right)\Phi(i') + O(i')\right] \\ &= \frac{N}{b_1}\sum_{j=1}^{i-1}\frac{\varphi(j)}{j} - \frac{N}{b_1}\frac{\Phi(i-1)}{i} + O(i^2)\ . \end{aligned}$$

In particular, when $b_2 = 0$ the term $O(i^2)$ does not appear according to Corollary 2. In a second step we compute the number of elements in $F_N^{\frac{a_1q'+a_2}{b_1q'+b_2},\ \frac{a_1q+a_2}{b_1q+b_2}}$, that is, $\Phi(i)(q'-q)+O(i)$. Adding both contributions gives

$$\begin{aligned} I_N\left(\frac{a_1q+a_2}{b_1q+b_2}\right) - I_N\left(\frac{a_1}{b_1}\right) &= \frac{N}{b_1}\sum_{j=1}^{i-1}\frac{\varphi(j)}{j} - \frac{N}{b_1}\frac{\Phi(i-1)}{i} \\ &\quad + \Phi(i)\left(\frac{N}{b_1 i} - q\right) + O(i^2) \\ &= \frac{N}{b_1}\sum_{j=1}^{i}\frac{\varphi(j)}{j} - q\Phi(i) + O(i^2)\ , \end{aligned}$$

which demonstrates the theorem for $s = 1$. For $s = -1$, following the same steps leads to the desired result. □

Corollary 4. *Let $N = b_1 \mathrm{lcm}(1, 2, \ldots, i_{\max})$ and $\frac{N}{b_1(i+1)} < q \le \frac{N}{b_1 i}$, with $i < i_{\max}$. Then,*

$$I_N\left(\frac{a_1q+a_2}{b_1q+b_2}\right) = I_N\left(\frac{a_1}{b_1}\right) + s\frac{3}{\pi^2}q\left(\frac{N^2}{b_1^2q^2} - \left\{\frac{N}{b_1q}\right\}^2\right) + O(N\delta_A(i))\ ,$$

with $\delta_A(x)$ defined in (2). In particular, for $a_1/b_1 = 0/1$ and $a_2/b_2 = 1/0$,

$$I_N\left(\frac{1}{q}\right) = \frac{3}{\pi^2}q\left(\frac{N^2}{q^2} - \left\{\frac{N}{q}\right\}^2\right) + O(N\delta_A(i))\ ,$$

and for $a_1/b_1 = 1/2$ and $a_2/b_2 = 1/1$,

$$I_N\left(\frac{q+1}{2q+1}\right) = \frac{|F_N|}{2} + \frac{3}{\pi^2}q\left(\frac{N^2}{2^2q^2} - \left\{\frac{N}{2q}\right\}^2\right) + O(N\delta_A(i))\ .$$

Proof. The following known relations [11, 12] are needed:

$$\begin{aligned} \sum_{k=1}^{N}\varphi(k) &= \frac{3}{\pi^2}N^2 + E(N), \\ \sum_{k=1}^{N}\frac{\varphi(k)}{k} &= \frac{6}{\pi^2}N + H(N)\ , \\ E(x) &= O(x\log^{2/3}x(\log\log x)^{4/3})\ , \\ E(x) &= xH(x) + O(x\delta_A(x))\ , \end{aligned} \tag{6}$$

with $A > 0$ and $\delta_A(x)$ is a decreasing factor. From the definitions of i, q, and N it follows that

$$\begin{aligned} i &= \left\lfloor \frac{N}{qb_1} \right\rfloor = \frac{N}{qb_1} + O(1) \ , \\ i &< i_{\max} = (1 + o(1)) \log N / b_1 \ . \end{aligned}$$

Inserting the above equalities in expression (5) of Theorem 3,

$$\begin{aligned} I_N\left(\frac{a_1 q + a_2}{b_1 q + b_2}\right) &= I_N\left(\frac{a_1}{b_1}\right) + s\frac{N}{b_1}\frac{6}{\pi^2} i - sq\frac{3}{\pi^2} i^2 + s\frac{N}{b_1} H(i) - sqE(i) + O(i^2) \\ &= I_N\left(\frac{a_1}{b_1}\right) + s\frac{N}{b_1}\frac{6}{\pi^2} i - sq\frac{3}{\pi^2} i^2 + sq(iH(i) - E(i)) + O(i^2) \\ &= I_N\left(\frac{a_1}{b_1}\right) + s\frac{N}{b_1}\frac{6}{\pi^2} i - sq\frac{3}{\pi^2} i^2 + O(N\delta_A(i)) \\ &= I_N\left(\frac{a_1}{b_1}\right) + s\frac{3}{\pi^2}\frac{N^2}{b_1^2 q} - s\frac{3}{\pi^2} q \left\{\frac{N}{b_1 q}\right\}^2 + O(N\delta_A(i)) \ . \end{aligned}$$

□

Theorem 5. *Let N be $N = b_1 \mathrm{lcm}(1, 2, \ldots, i)$. Then the partial Franel sum over all Farey fractions in the range* $\left[\frac{a_1}{b_1}, \frac{a_1 \frac{N}{b_1 i} + a_2}{b_1 \frac{N}{b_1 i} + b_2}\right]$ *is given by the following expressions:*

- *For $a_1/b_1 = 0/1$, $a_2/b_2 = 1/0$ and for $a_1/b_1 = 1/2$, $a_2/b_2 = 0/1$:*

$$P\left(\frac{0}{1}, \frac{1}{N/i}\right) = \sum_{j=1}^{I_N\left(\frac{1}{N/i}\right)} \left| F_N(j) - \frac{j}{|F_N|} \right| = O(\log(N)\delta_B(\log N)) \ ,$$

$$P\left(\frac{1}{2}, \frac{N/(2i)}{N/i + 1}\right) = \sum_{j=I_N\left(\frac{1}{2}\right)}^{I_N\left(\frac{N/(2i)}{N/i+1}\right)} \left| F_N(j) - \frac{j}{|F_N|} \right| = O(\log(N)\delta_B(\log N)) \ ,$$

with $0 < B < A$. The same result holds for $a_1/b_1 = 1/2$, $a_2/b_2 = 1/1$.

- *For $b_1 > 2$ and $b_2 < b_1$:*

$$\sum_{j=I_N\left(\frac{a_1}{b_1}\right)}^{I_N\left(\frac{a_1 \frac{N}{b_1 i} + a_2}{b_1 \frac{N}{b_1 i} + b_2}\right)} \left| F_N(j) - \frac{j}{|F_N|} \right| \leq \left| \frac{a_1}{b_1} - \frac{I_N\left(\frac{a_1}{b_1}\right)}{|F_N|} \right| O(iN) + O(i\delta_B(i)) \ ,$$

which cannot be further developed as no general expression for $I_N(a_1/b_1)$ is known.

Proof. By virtue of Theorem 1 the partial Franel sum under study is written as

$$P\left(\frac{a_1}{b_1}, \frac{a_1\frac{N}{b_1 i}+a_2}{b_1\frac{N}{b_1 i}+b_2}\right) = \sum_{i'=1}^{i-1} \sum_{q=\frac{N}{b_1(i'+1)}+1}^{\frac{N}{b_1 i'}} \sum_{n=2}^{|F'_{i'}|} \left| \frac{k(a_1q+a_2)-a_1h}{k(b_1q+b_2)-b_1h} - \frac{I_N\left(\frac{k(a_1q+a_2)-a_1h}{k(b_1q+b_2)-b_1h}\right)}{|F_N|} \right| ,$$

where the sum over n runs over the elements h/k in $F'_{i'}$, approximately $n = I_{i'}(h/k) + O(i')$. By virtue of Theorem 1 and Corollary 4

$$\begin{aligned} I_N\left(\frac{k(a_1q+a_2)-a_1h}{k(b_1q+b_2)-b_1h}\right) &= I_N\left(\frac{a_1q+a_2}{b_1q+b_2}\right) + sI_{i'}\left(\frac{h}{k}\right) + O(i') \\ &= I_N\left(\frac{a_1}{b_1}\right) + s\frac{3}{\pi^2}\frac{N^2}{b_1^2 q} - s\frac{3}{\pi^2}q\left\{\frac{N}{b_1 q}\right\}^2 + s\frac{3}{\pi^2}\left\lfloor\frac{N}{b_1 q}\right\rfloor^2 \frac{h}{k} + O(N\delta_A(i')) , \end{aligned}$$

where we have used $i' = \left\lfloor \frac{N}{qb_1} \right\rfloor$. Furthermore

$$\frac{I_N\left(\frac{k(a_1q+a_2)-a_1h}{k(b_1q+b_2)-b_1h}\right)}{|F_N|} = \frac{I_N\left(\frac{a_1}{b_1}\right)}{|F_N|} + \frac{s}{b_1^2 q} - \frac{sq}{N^2}\left\{\frac{N}{b_1 q}\right\}^2 + \frac{s}{N^2}\left\lfloor\frac{N}{b_1 q}\right\rfloor^2 \frac{h}{k} + O\left(\frac{\delta_A(i')}{N}\right) .$$

The Farey element inside the partial Franel sum is approximated as

$$\begin{aligned} \frac{k(a_1q+a_2)-a_1h}{k(b_1q+b_2)-b_1h} &= \frac{k(a_1q+a_2)-a_1h}{k(b_1q+b_2)-b_1h} - \frac{a_1}{b_1} + \frac{a_1}{b_1} \\ &= \frac{s}{b_1^2 q}\frac{1}{1+\frac{b_2}{qb_1}-\frac{h}{qk}} + \frac{a_1}{b_1} \\ &= \frac{s}{b_1^2 q}\left(1-\frac{b_2}{qb_1}+\frac{h}{qk}\right) + \frac{a_1}{b_1} + O(1/q^3) , \end{aligned}$$

where we have used $(b_1a_2 - a_1b_2) = s$. The partial Franel sum under study becomes

$$\sum_{i'=1}^{i-1} \sum_{q=\frac{N}{b_1(i'+1)}+1}^{\frac{N}{b_1 i'}} |F'_{i'}| \left| \frac{a_1}{b_1} - \frac{I_N\left(\frac{a_1}{b_1}\right)}{|F_N|} - \frac{sb_2}{b_1^3 q^2} + \frac{sq}{N^2}\left\{\frac{N}{b_1 q}\right\}^2 + O\left(\frac{\delta_A(i')}{N}\right) \right| ,$$

where the terms proportional to $1/q$ and h/k have canceled out leaving a negligible residue. The sum over n has been evaluated just by multiplying by $|F'_{i'}|$ as the dependency on h/k disappeared. Evaluating the asymptotics of the sums of the individual terms within the absolute value gives

$$\sum_{i'=1}^{i-1} \sum_{q=\frac{N}{b_1(i'+1)}+1}^{\frac{N}{b_1 i'}} |F'_{i'}| \quad = \quad O\left(iN\right) \ ,$$

$$\sum_{i'=1}^{i-1} \sum_{q=\frac{N}{b_1(i'+1)}+1}^{\frac{N}{b_1 i'}} \frac{|F'_{i'}|}{q^2} \quad = \quad O\left(\frac{i^4}{N}\right) \ ,$$

$$\sum_{i'=1}^{i-1} \sum_{q=\frac{N}{b_1(i'+1)}+1}^{\frac{N}{b_1 i'}} |F'_{i'}| \frac{q}{N^2} \quad = \quad O(\log i) \ ,$$

$$\sum_{i'=1}^{i-1} \sum_{q=\frac{N}{b_1(i'+1)}+1}^{\frac{N}{b_1 i'}} |F'_{i'}| \frac{\delta_A(i')}{N} \quad = \quad O\left(i\delta_B(i)\right) \ ,$$

with $0 < B < A$. Keeping the two dominant terms gives

$$P\left(\frac{a_1}{b_1}, \frac{a_1 \frac{N}{b_1 i} + a_2}{b_1 \frac{N}{b_1 i} + b_2}\right) \leq \left| \frac{a_1}{b_1} - \frac{I_N\left(\frac{a_1}{b_1}\right)}{|F_N|} \right| O(iN) + O(i\delta_B(i)) \ ,$$

which is the searched result for $b_1 > 2$. For $1 \leq b_1 \leq 2$ we have

$$\frac{0}{1} - \frac{I_N\left(\frac{0}{1}\right)}{|F_N|} = O(1/N^2) \ , \quad \frac{1}{2} - \frac{I_N\left(\frac{1}{2}\right)}{|F_N|} = O(1/N^2) \ , \quad \frac{1}{1} - \frac{I_N\left(\frac{1}{1}\right)}{|F_N|} = 0 \ ,$$

and $O(i\delta_B(i))$ dominates. The theorem is demonstrated. □

Appendix

Table 1 illustrates the bijections between the first 90 elements in F_{60} and other Farey sequences of lower order.

i'	q	$h/k \in F_{i'}$	$u/l \in F_N$	$I_N(\frac{u}{l})$	i'	q	$h/k \in F_{i'}$	$u/l \in F_N$	$I_N(\frac{u}{l})$	i'	q	$h/k \in F_{i'}$	$u/l \in F_N$	$I_N(\frac{u}{l})$
-	-	-	$\frac{0}{1}$	1	1	32	$\frac{1}{1}$	$\frac{1}{31}$	31	3	18	$\frac{1}{3}$	$\frac{3}{53}$	61
1	60	$\frac{0}{1}$	$\mathbf{\frac{1}{60}}$	2	1	31	$\frac{1}{1}$	$\mathbf{\frac{1}{30}}$	32	3	18	$\frac{1}{2}$	$\frac{2}{35}$	62
1	60	$\frac{1}{1}$	$\frac{1}{59}$	3	2	30	$\frac{1}{2}$	$\frac{2}{59}$	33	3	18	$\frac{2}{3}$	$\frac{3}{52}$	63
1	59	$\frac{1}{1}$	$\frac{1}{58}$	4	2	30	$\frac{1}{1}$	$\frac{1}{29}$	34	3	18	$\frac{1}{1}$	$\frac{1}{17}$	64
1	58	$\frac{1}{1}$	$\frac{1}{57}$	5	2	29	$\frac{1}{2}$	$\frac{2}{57}$	35	3	17	$\frac{1}{3}$	$\frac{3}{50}$	65
1	57	$\frac{1}{1}$	$\frac{1}{56}$	6	2	29	$\frac{1}{1}$	$\frac{1}{28}$	36	3	17	$\frac{1}{2}$	$\frac{2}{33}$	66
1	56	$\frac{1}{1}$	$\frac{1}{55}$	7	2	28	$\frac{1}{2}$	$\frac{2}{55}$	37	3	17	$\frac{2}{3}$	$\frac{3}{49}$	67
1	55	$\frac{1}{1}$	$\frac{1}{54}$	8	2	28	$\frac{1}{1}$	$\frac{1}{27}$	38	3	17	$\frac{1}{1}$	$\frac{1}{16}$	68
1	54	$\frac{1}{1}$	$\frac{1}{53}$	9	2	27	$\frac{1}{2}$	$\frac{2}{53}$	39	3	16	$\frac{1}{3}$	$\frac{3}{47}$	69
1	53	$\frac{1}{1}$	$\frac{1}{52}$	10	2	27	$\frac{1}{1}$	$\frac{1}{26}$	40	3	16	$\frac{1}{2}$	$\frac{2}{31}$	70
1	52	$\frac{1}{1}$	$\frac{1}{51}$	11	2	26	$\frac{1}{2}$	$\frac{2}{51}$	41	3	16	$\frac{2}{3}$	$\frac{3}{46}$	71
1	51	$\frac{1}{1}$	$\frac{1}{50}$	12	2	26	$\frac{1}{1}$	$\frac{1}{25}$	42	3	16	$\frac{1}{1}$	$\mathbf{\frac{1}{15}}$	72
1	50	$\frac{1}{1}$	$\frac{1}{49}$	13	2	25	$\frac{1}{2}$	$\frac{2}{49}$	43	4	15	$\frac{1}{4}$	$\frac{4}{59}$	73
1	49	$\frac{1}{1}$	$\frac{1}{48}$	14	2	25	$\frac{1}{1}$	$\frac{1}{24}$	44	4	15	$\frac{1}{3}$	$\frac{3}{44}$	74
1	48	$\frac{1}{1}$	$\frac{1}{47}$	15	2	24	$\frac{1}{2}$	$\frac{2}{47}$	45	4	15	$\frac{1}{2}$	$\frac{2}{29}$	75
1	47	$\frac{1}{1}$	$\frac{1}{46}$	16	2	24	$\frac{1}{1}$	$\frac{1}{23}$	46	4	15	$\frac{2}{3}$	$\frac{3}{43}$	76
1	46	$\frac{1}{1}$	$\frac{1}{45}$	17	2	23	$\frac{1}{2}$	$\frac{2}{45}$	47	4	15	$\frac{3}{4}$	$\frac{4}{57}$	77
1	45	$\frac{1}{1}$	$\frac{1}{44}$	18	2	23	$\frac{1}{1}$	$\frac{1}{22}$	48	4	15	$\frac{1}{1}$	$\frac{1}{14}$	78
1	44	$\frac{1}{1}$	$\frac{1}{43}$	19	2	22	$\frac{1}{2}$	$\frac{2}{43}$	49	4	14	$\frac{1}{4}$	$\frac{4}{55}$	79
1	43	$\frac{1}{1}$	$\frac{1}{42}$	20	2	22	$\frac{1}{1}$	$\frac{1}{21}$	50	4	14	$\frac{1}{3}$	$\frac{3}{41}$	80
1	42	$\frac{1}{1}$	$\frac{1}{41}$	21	2	21	$\frac{1}{2}$	$\frac{2}{41}$	51	4	14	$\frac{1}{2}$	$\frac{2}{27}$	81
1	41	$\frac{1}{1}$	$\frac{1}{40}$	22	2	21	$\frac{1}{1}$	$\mathbf{\frac{1}{20}}$	52	4	14	$\frac{2}{3}$	$\frac{3}{40}$	82
1	40	$\frac{1}{1}$	$\frac{1}{39}$	23	3	20	$\frac{1}{3}$	$\frac{3}{59}$	53	4	14	$\frac{3}{4}$	$\frac{4}{53}$	83
1	39	$\frac{1}{1}$	$\frac{1}{38}$	24	3	20	$\frac{1}{2}$	$\frac{2}{39}$	54	4	14	$\frac{1}{1}$	$\frac{1}{13}$	84
1	38	$\frac{1}{1}$	$\frac{1}{37}$	25	3	20	$\frac{2}{3}$	$\frac{3}{58}$	55	4	13	$\frac{1}{4}$	$\frac{4}{51}$	85
1	37	$\frac{1}{1}$	$\frac{1}{36}$	26	3	20	$\frac{1}{1}$	$\frac{1}{19}$	56	4	13	$\frac{1}{3}$	$\frac{3}{38}$	86
1	36	$\frac{1}{1}$	$\frac{1}{35}$	27	3	19	$\frac{1}{3}$	$\frac{3}{56}$	57	4	13	$\frac{1}{2}$	$\frac{2}{25}$	87
1	35	$\frac{1}{1}$	$\frac{1}{34}$	28	3	19	$\frac{1}{2}$	$\frac{2}{37}$	58	4	13	$\frac{2}{3}$	$\frac{3}{37}$	88
1	34	$\frac{1}{1}$	$\frac{1}{33}$	29	3	19	$\frac{2}{3}$	$\frac{3}{55}$	59	4	13	$\frac{3}{4}$	$\frac{4}{49}$	89
1	33	$\frac{1}{1}$	$\frac{1}{32}$	30	3	19	$\frac{1}{1}$	$\frac{1}{18}$	60	4	13	$\frac{1}{1}$	$\mathbf{\frac{1}{12}}$	90

Table 1: Correspondence between elements in $F_{i'}$, with $0 < i' < 5$ and first 90 elements in F_N, given by $u/l = h/(hq-k)$ with $N/(i'+1) < q \le N/i'$ and $N = \mathrm{lcm}(1,2,3,4,5) = 60$. Note that the images of elements 1/1 and 0/1 of adjacent maps are equal and only the 1/1 case is shown on the table. The illustrated maps originate from the map M in Theorem 1 with $a_1/b_1 = 0/1$ and $a_2/b_2 = 1/0$.

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