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Generalization of Greatest Common Divisor of Shifted Fibonacci Numbers

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Abstract

Let F_n denote the n'th Fibonacci number and let k be a positive integer. We find necessary and sufficient conditions on s and t so that the function $n \mapsto \gcd(F_n + s, F_{n+k} + t)$ is unbounded.

1 Introduction

The Fibonacci sequence $(F_n)_{n\geq 0}$ and Lucas sequence $(L_n)_{n\geq 0}$ are defined by the recursions

$$F_0 = 0, F_1 = 1; F_n = F_{n-1} + F_{n-2}, n \geq 2;$$

and

$$L_0 = 2, L_1 = 1; L_n = L_{n-1} + L_{n-2}, n \geq 2.$$

In 1971, Dudley and Tucker [3] showed that $\gcd(F_n + s, F_{n+1} + s)$ is unbounded for $s = \pm 1$. In 2011, Chen [1] determined $\gcd(F_n + s, F_{n+1} + s)$ for $s \in \{\pm 1, \pm 2\}$ and proved that

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$\gcd(F_n + s, F_{n+1} + s)$ is bounded for $s \neq \pm 1$. In 2018, Rahn and Kreh [7] determined $\gcd(F_n+s, F_{n+1}+s)$ for $s = \pm 3$. In 2016, Spilker [8] proved that $\gcd(F_n+s, F_{n+1}+s)$ divides $s^2 + (-1)^n$. In 2020, Chen and Pan [2] considered general second-order linear homogeneous recurrence functions W_n. They proved a criterion of periodicity of $\gcd(W_n + s, W_{n+k} + t)$. They transformed also problems about the $\gcd(W_n+s, W_{n+k}+t)$ with arbitrarily k into those with $k = 1$ and calculated on this way some $\gcd(W_n + s, W_{n+1} + t)$. In 2022, Hieu, Spilker, and Thang [5] investigated the necessary and sufficient conditions on $k \in \mathbb{N}, s, t \in \mathbb{Z}$, so that the function $\gcd(F_n + s, F_{n+k} + s)$ and the function $\gcd(F_n + s, F_{n+1} + t)$ are unbounded. They proposed the open problem to characterize the function $\gcd(F_n + s, F_{n+k} + t)$ for $k \in \mathbb{N}, s, t \in \mathbb{Z}$. Let $\mathbb{N}_0$ denote the set of non-negative integers. In this article, given a positive integer k, we investigate necessary and sufficient conditions on $s, t \in \mathbb{Z}$ (depend on k), so that the function

$$\begin{aligned} B^k_{s,t} : \mathbb{N}_0 &\to \mathbb{N} \\ n &\mapsto \gcd(F_n + s, F_{n+k} + t) \end{aligned}$$

is bounded (Theorem 1). If $B^k_{s,t}$ is bounded, it is periodic (Theorem 2). In more detail, we obtain the following results:

Theorem 1. *Let $s, t \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The following assertions on $B^k_{s,t}(n)$ are equivalent:*

1. *$n \mapsto B^k_{s,t}(n)$ is unbounded on $\mathbb{N}_0$;*
2. *$(s, t) \in \mathcal{R} := \{\pm(F_j, F_{j+k}) : j \in \mathbb{Z}\}$;*
3. *$e^{k*}_{s,t} = 0$,*

where $e^k_{s,t}(n) := t^2 - L_k st + (-1)^k s^2 - (-1)^n F_k^2$ and $e^{k*}_{s,t} := e^k_{s,t}(0)e^k_{s,t}(1)$.

Theorem 2. *If $(s, t) \notin \mathcal{R}$, then the function $B^k_{s,t}(n)$ is simply periodic on $\mathbb{N}_0$, which means there exists a positive integer p such that $B^k_{s,t}(n+p) = B^k_{s,t}(n)$ for all $n \geq 0$. A period $p \leq c^2$ can be chosen where $c := \left|e^{k*}_{s,t}\right|$ such that*

$$F_p \equiv F_0 \pmod{c} \text{ and } F_{p+1} \equiv F_1 \pmod{c}.$$

2 Auxiliary results

The following lemma contains some well-known facts on $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$, which we have used. Notice that F_n and L_n can be extended to integer indices by the recursion.

Lemma 3 ([6])**.** *Let $k \in \mathbb{N}, n, m \in \mathbb{Z}$. Then we have*

(a) $F_n^2 - F_{n-k}F_{n+k} = (-1)^{n-k}F_k^2$;

(b) $F_{n+2k} = L_kF_{n+k} - (-1)^kF_n$;

(c) $F_{\frac{m-n}{2}}$ *divides* $F_m + F_n$ *if* $\dfrac{m-n}{2}$ *is an odd integer;*

(d) $F_{\frac{m-n}{2}}$ *divides* $F_m - F_n$ *if* $\dfrac{m-n}{2}$ *is an even integer;*

(e) $\dfrac{L_k^2}{4} - (-1)^k = \dfrac{5}{4}F_k^2$*;*

(f) $2F_{n+k} = L_kF_n + L_nF_k$.

3 Proof of Theorem 1 and 2

Set $B := B_{s,t}^k(n)$. We need the following lemma.

Lemma 4. *Let* $s, t \in \mathbb{Z}, n \in \mathbb{N}_0$. *Then*

(a) B *divides* $e_{s,t}^k(n)$*;*

(b) B *divides* $e_{s,t}^{k*}$.

Proof. By using Lemma 3 (a) we have:

$$F_{n+k}^2 - F_nF_{n+2k} = (-1)^nF_k^2.$$

By using Lemma 3 (b) we have:

$$F_{n+k}^2 - F_n(L_kF_{n+k} - (-1)^kF_n) = (-1)^nF_k^2.$$

Since $B_{s,t}^k(n) = \gcd(F_n + s, F_{n+k} + t)$, by applying

$$F_{n+k} \equiv -t \pmod{B} \text{ and } F_n \equiv -s \pmod{B},$$

we deduce that

$$t^2 - L_kst + (-1)^ks^2 \equiv (-1)^nF_k^2 \pmod{B}.$$

Hence, $e_{s,t}^k(n) \equiv 0 \pmod{B}$ and $e_{s,t}^{k*} \equiv 0 \pmod{B}$. □

Corollary 5. *If* $s, t \in \mathbb{Z}$ *and* $e_{s,t}^{k*} \neq 0$, *then the function* $n \mapsto B_{s,t}^k(n)$ *is bounded on* $\mathbb{N}_0$.

Proof of Theorem 1. Now, we give a proof of Theorem 1 by the implications (1) ⇒ (3) ⇒ (2) ⇒ (1).

(1) ⇒ (3): by Corollary 5.

(3) $\Rightarrow$ (2): We have

$$t^2 - L_k st + (-1)^k s^2 = \left(t - \frac{L_k}{2}s\right)^2 + \left((-1)^k - \frac{L_k^2}{4}\right)s^2$$
$$= \left(t - \frac{L_k}{2}s\right)^2 - \frac{5}{4}F_k^2 s^2 \ \text{(by Lemma 3 (e))}.$$

Hence

$$\left(t - \frac{L_k}{2}s\right)^2 - \frac{5}{4}F_k^2 s^2 = \pm F_k^2. \tag{1}$$

If s is even, from (1) we have $F_k \mid t - \frac{L_k}{2}s$. Hence,

$$\frac{t - \frac{L_k}{2}s}{F_k} - \frac{\sqrt{5}}{2}s = \left(\frac{t - \frac{L_k}{2}s}{F_k} + \frac{s}{2}\right) - s\frac{\sqrt{5}+1}{2}$$

is integer in the quadratic field $\mathbb{Q}(\sqrt{5})$.

If s is odd, from (1) we have $\dfrac{t - \frac{L_k}{2}s}{F_k} = \dfrac{c}{2}$ where c is odd. Hence,

$$\frac{t - \frac{L_k}{2}}{F_k} - \frac{\sqrt{5}}{2}s = \frac{c-s}{2} + s\frac{\sqrt{5}+1}{2}$$

is integer in the quadratic field $\mathbb{Q}(\sqrt{5})$. By dividing F_k^2 of two sides in (1), we have

$$\pm 1 = \left(\frac{t - \frac{L_k}{2}s}{F_k}\right)^2 - \frac{5}{4}s^2$$
$$= \left(\frac{t - \frac{L_k}{2}s}{F_k} - \frac{\sqrt{5}}{2}s\right)\left(\frac{t - \frac{L_k}{2}s}{F_k} + \frac{\sqrt{5}}{2}s\right).$$

These fractions are units in the quadratic field $\mathbb{Q}(\sqrt{5})$, hence by [4, Theorem 257]

$$\frac{t - \frac{L_k}{2}s}{F_k} + \frac{\sqrt{5}}{2}s = \pm\alpha^j, \ \text{ where } \ \alpha := \frac{1+\sqrt{5}}{2}, j \in \mathbb{Z}.$$

Since $\alpha^j = \dfrac{L_j + F_j\sqrt{5}}{2}$, we get $s = \pm F_j$ and

$$t = \pm\frac{F_j L_k + F_k L_j}{2} = \pm F_{j+k} \ \text{(by Lemma 3.f)}.$$

This implies $(s, t) \in \mathcal{R}$.

(2) $\Rightarrow$ (1): If $(s,t)=(F_j,F_{j+k})$, we choose n such that $\dfrac{n-j}{2}$ is an odd integer. Then

$$F_{\frac{n-j}{2}} \mid F_n+F_j \text{ and } F_{\frac{n-j}{2}} \mid F_{n+k}+F_{j+k} \ \text{ (by Lemma 3 (c))}.$$

If $(s,t)=-(F_j,F_{j+k})$, we choose n such that $\dfrac{n-j}{2}$ is an even integer, then

$$F_{\frac{n-j}{2}} \mid F_{n+k}-F_{j+k} \text{ and } F_{\frac{n-j}{2}} \mid F_n-F_j \ \text{ (by Lemma 3 (d))}.$$

Hence, $B^k_{s,t}(n)$ is unbounded. The proof of Theorem 1 is completed. □

Proof of Theorem 2. The proof of Theorem 2 is completely analogous to the one in [8] with $k=1$ and $s=t$. Set $e^k_{s,t}(n):=t^2-L_kst+(-1)^ks^2-(-1)^nF_k^2$ and $c:=\left|e^k_{s,t}(0)e^k_{s,t}(1)\right|$. Then $c\neq 0$ and $B^k_{s,t}(n)$ divides c for $(s,t)\notin\mathcal{R}$. Then there is a positive integer $p\le c^2$ such that

$$F_n\equiv F_{n+p} \pmod{c} \text{ for all } n\in\mathbb{N}.$$

We have $B^k_{s,t}(n)=\gcd(F_n+s,F_{n+k}+t)\mid c$ and $F_{n+p}-F_n, F_{n+k+p}-F_{n+k}$ are divisible by c, then

$$B^k_{s,t}(n)\mid((F_{n+p}-F_n)+(F_n+s)) \text{ and } B^k_{s,t}(n)\mid((F_{n+k+p}-F_{n+k})+(F_{n+k}+t)).$$

Thus,

$$B^k_{s,t}(n)\mid(F_{n+p}+s) \text{ and } B^k_{s,t}(n)\mid(F_{n+k+p}+t).$$

This implies $B^k_{s,t}(n)\mid\gcd(F_{n+k}+s,F_{n+p+k}+t)$. Hence, $B^k_{s,t}(n)\mid B^k_{s,t}(n+p)$. Similarly, we have also $B^k_{s,t}(n+p)\mid B^k_{s,t}(n)$. Therefore, $B^k_{s,t}(n+p)=B^k_{s,t}(n)$.

This p is a period of $B^k_{s,t}(n)$. However, a period p that we get by this method is not necessarily the smallest positive one. For example, if we choose $s=t=2$ and $k=1$ then we have $c=15$ and $F_{n+80}\equiv F_n \pmod{15}$. Hence $p=80$ but 40 is also a period of $B^1_{1,2}(n)$ and it is the smallest one. □

Remark 6. If $s=t$, we obtain Theorem 7 in [5]. If $k=1$, we obtain Theorem 8 in [5].

Remark 7. The same arguments as in the proof of Theorem 1 give the following result on Lucas numbers.

Theorem 8. *Let $k\in\mathbb{N}, s,t\in\mathbb{Z}$. The following assertions on $B^{k*}_{s,t}(n)$ are equivalent:*

1. *The function $B^{k*}_{s,t}(n):=\gcd(L_n+s,L_{n+k}+t)$ is unbounded on $\mathbb{N}_0$;*
2. *$(s,t)\in\mathcal{S}:=\{\pm(L_j,L_{j+k}):j\in\mathbb{Z}\}$;*
3. *$t^2-L_kst+(-1)^ks^2=\pm 5F_k^2$.*

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