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The Greatest Common Divisor of Shifted Fibonacci Numbers

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Abstract

Let F_n and L_n denote the Fibonacci and Lucas numbers, respectively. We generalize the well-known formula $\gcd(F_{6n+3}+1, F_{6n+6}+1) = L_{3n+2}$ if n is even. We find conditions on a, s, and t so that the functions $n \mapsto \gcd(F_n + s, F_{n+a} + s)$ and $n \mapsto \gcd(F_n + s, F_{n+1} + t)$ are unbounded. Finally, we generalize this for three shifted Fibonacci numbers.

1 Introduction

The Fibonacci sequence $(F_n)_{n\geq 0}$ and Lucas sequence $(L_n)_{n\geq 0}$ are defined by the recursions

$$F_0 = 0, F_1 = 1; F_n = F_{n-1} + F_{n-2}, n \geq 2;$$

and

$$L_0 = 2, L_1 = 1; L_n = L_{n-1} + L_{n-2}, n \geq 2.$$

Notice that F_n and L_n can be extended to integer indices by the recursion. In 1971, Dudley and Tucker [3] showed that $\gcd(F_n + s, F_{n+1} + s)$ is unbounded for $s = \pm 1$. In 2011, Chen [1] determined $\gcd(F_n + s, F_{n+1} + s)$ for $s \in \{\pm 1, \pm 2\}$ and proved that $\gcd(F_n + s, F_{n+1} + s)$ is bounded for $s \neq \pm 1$. In 2018, Rahn and Kreh [7] determined $\gcd(F_n + s, F_{n+1} + s)$ for $s = \pm 3$. In 2016, Spilker [8] proved that $\gcd(F_n + s, F_{n+1} + s)$ divides $s^2 + (-1)^n$. If $s \neq \pm 1$ then the function $n \mapsto \gcd(F_n + s, F_{n+1} + s)$ is periodic. In 2020, Chen and Pan [2] considered general second-order linear homogeneous recurrence functions W_n. They proved a criterion of periodicity of $\gcd(W_n + s, W_{n+k} + t)$. They transformed problems about the $\gcd(W_n + s, W_{n+k} + t)$ with arbitrarily k into those with $k = 1$ and calculated on this way some $\gcd(W_n + s, W_{n+1} + t)$.

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0$ denote the set of non-negative integers, and $\mathbb{Z}$ denote the set of integers. In this article, we generalize these ideas to compute

$$\gcd(F_{2an+2b+i} + (-1)^b, F_{2an+2b+i+k} + (-1)^b F_{k-1})$$

in Theorem 2. Then in Theorem 7, we investigate necessary and sufficien conditions on $a \in \mathbb{N}, s \in \mathbb{Z}$, so that the function

$$\begin{aligned} D_{a,s} : \mathbb{N}_0 &\to \mathbb{N} \\ n &\mapsto \gcd(F_n + s, F_{n+a} + s) \end{aligned}$$

is bounded. In Theorem 8, we treat the same problem for $C_{s,t}(n) := \gcd(F_n + s, F_{n+1} + t)$. If $D_{a,s}(n)$ and $C_{s,t}(n)$ are bounded, they are periodic (Theorem 9 and Theorem 10). Finally, we prove that the function

$$\begin{aligned} E_{a,s} : \mathbb{N}_0 &\to \mathbb{N} \\ n &\mapsto \gcd(F_n + s, F_{n+a} + s, F_{n+2a} + s) \end{aligned}$$

is bounded without exceptions.

2 Results

Firstly, we have a theorem on the gcd of Fibonacci numbers with shift 1 and with general shift $k \geq 1$.

Theorem 1. *Let $a, b, i \in \mathbb{Z}$ with $i = 2j + 1$ and set*

$$D := \gcd(F_{2an+2b+i} + (-1)^b, F_{2an+2b+i+1} + (-1)^b).$$

Then

$$D = \begin{cases} F_{an+b+j}, & \text{if } an + j \text{ is odd;} \\ L_{an+b+j}, & \text{if } an + j \text{ is even} \end{cases}$$

for all $n \in \mathbb{N}_0$.

Theorem 2. *Let $a, b, i, k \in \mathbb{Z}$ with $i = 2j - 1, k \geq 1$ and set*

$$D := \gcd(F_{2an+2b+i} + (-1)^b, F_{2an+2b+i+k} + (-1)^b F_{k-1}).$$

Then

$$D = \begin{cases} F_{an+b+j} \gcd(L_{an+b+j-1}, F_k), & \text{if } an + j \text{ is odd;} \\ L_{an+b+j} \gcd(F_{an+b+j-1}, F_k), & \text{if } an + j \text{ is even,} \end{cases}$$

for all $n \in \mathbb{N}_0$.

Corollary 3. *Let $a, b, i \in \mathbb{Z}$ with $i = 2j - 1$. Then*

$$\gcd(F_{2an+2b+i} + (-1)^b, F_{2an+2b+i+1}) = \begin{cases} F_{an+b+j}, & \text{if } an + j \text{ is odd;} \\ L_{an+b+j}, & \text{if } an + j \text{ is even,} \end{cases}$$

for all $n \in \mathbb{N}_0$.

Corollary 4. *Let $a, b, i, j \in \mathbb{Z}$ with $i = 2j - 1$. Then*

$$\gcd(F_{2an+2b+i} + (-1)^b, F_{2an+2b+i+2} + (-1)^b) = \begin{cases} F_{an+b+j}, & \text{if } an + j \text{ is odd;} \\ L_{an+b+j}, & \text{if } an + j \text{ is even,} \end{cases}$$

for all $n \in \mathbb{N}_0$.

Corollary 5. *Let $a, b, i, j \in \mathbb{Z}$ with $i = 2j - 1$. Then*

$$\gcd(F_{2an+2b+i} + (-1)^b, F_{2an+2b+i+3} + (-1)^b) = \begin{cases} \lambda F_{an+b+j}, & \text{if } an + j \text{ is odd;} \\ \lambda L_{an+b+j}, & \text{if } an + j \text{ is even,} \end{cases}$$

for all $n \in \mathbb{N}_0$, where

$$\lambda := \begin{cases} 2, & \text{if } an + b + j \equiv 1 \pmod 3; \\ 1, & \text{otherwise.} \end{cases}$$

Corollary 6. *Let $a, b, i \in \mathbb{Z}$ with $i = 2j - 1$. Then*

$$\gcd(F_{2an+2b+i} + (-1)^b, F_{2an+2b+i+4} + (-1)^b 2) = \begin{cases} \lambda F_{an+b+j}, & \text{if } an + j \text{ is odd;} \\ \delta L_{an+b+j}, & \text{if } an + j \text{ is even,} \end{cases}$$

for $n \in \mathbb{N}_0$, where

$$\lambda := \begin{cases} 3, & \text{if } an + b + j \equiv 3 \pmod 4; \\ 1, & \text{otherwise,} \end{cases}$$

and

$$\delta := \begin{cases} 3, & \text{if } an + b + j \equiv 1 \pmod 4; \\ 1, & \text{otherwise.} \end{cases}$$

The next theorem characterizes the boundedness of $D_{a,s}(n) := \gcd(F_n + s, F_{n+a} + s)$.

Theorem 7. *Let $a \in \mathbb{N}, s \in \mathbb{Z}$, and set*

$$\mathcal{S} := \{(1,1), (1,-1), (3,1), (3,-1)\} \cup \{(4k+2, F_{2k+1}) : k \geq 0\} \cup \{(4k+2, -F_{2k+1}) : k \geq 0\}.$$

Then $D_{a,s}(n)$ is bounded on $\mathbb{N}_0$ if and only if $(a, s) \notin \mathcal{S}$.

Let $e_{s,t}(n) := s^2 + st - t^2 + (-1)^n$ and $e^*_{s,t} := e_{s,t}(0)e_{s,t}(1)$. We characterize the boundedness of

$$C_{s,t}(n) := \gcd(F_n + s, F_{n+1} + t).$$

Theorem 8. *Let $s, t \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The following assertions on $C_{s,t}(n)$ are equivalent:*

a) *$n \mapsto C_{s,t}(n)$ is unbounded on $\mathbb{N}_0$;*

b) *$(s, t) \in \mathcal{R} := \{\pm(F_{j-1}, F_j), \pm(F_j, -F_{j-1}) : j \in \mathbb{N}_0\}$;*

c) *$e^*(s, t) = 0$.*

The next two theorems show the boundedness of $D_{a,s}(n)$ and $C_{s,t}(n)$ respectively imply periodicity.

Theorem 9. *Let $a \in \mathbb{N}$ and $s \in \mathbb{Z}$. If $(a, s) \notin \mathcal{S}$, then the function $D_{a,s}(n)$ is simply periodic on $\mathbb{N}_0$, which means there exists a positive integer p such that $D_{a,s}(n + p) = D_{a,s}(n)$ for all $n \geq 0$. A period $p \leq c^2$ can be chosen such that*

$$F_p \equiv 0 \pmod c \text{ and } F_{p+1} \equiv 1 \pmod c \tag{1}$$

where $c := |F_a^4 - s^4(L_a - 1 + (-1)^{a+1})^2|$.

Theorem 10. *If $(s, t) \notin \mathcal{R}$, then the function $C_{s,t}(n)$ is simply periodic on $\mathbb{N}_0$. A period $p \leq c^2$ can be chosen by (1), where $c := |(s^2 + st - t^2)^2 - 1|$.*

A theorem on three shifted Fibonacci numbers is given in the following last main result.

Theorem 11. *Let $a \in \mathbb{N}$ and $s \in \mathbb{Z}$ and set*

$$E_{a,s}(n) := \gcd(F_n + s, F_{n+a} + s, F_{n+2a} + s),$$

for all $n \in \mathbb{N}_0$. Then we have

a) *$E_{a,s}(n)$ divides d where*

$$d := \begin{cases} s(L_a - 1 + (-1)^{a+1}), & \text{if } s \neq 0; \\ F_a, & \text{if } s = 0. \end{cases}$$

b) *The function $n \mapsto E_{a,s}(n)$ is simply periodic.*

3 Auxiliary results

In the following proofs we need some well-known facts on the sequences $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$, which we have collected in the following lemma:

Lemma 12 ([6])**.** *Let $n, m \in \mathbb{Z}$. Then we have*

a) $\gcd(F_n, F_{n+1}) = 1$;

b) $\gcd(L_n, L_{n+1}) = 1$;

c) $F_{-n} = (-1)^{n+1} F_n$;

d) $L_{-n} = (-1)^n L_n$;

e) $\gcd(F_m, F_n) = F_{\gcd(m,n)}$;

f) $F_m \mid F_n$ *if and only if* $m \mid n$;

g) $F_{2n} = F_n L_n$;

h) $L_n = F_{n+1} + F_{n-1}$;

i) $5F_n = L_{n+1} + L_{n-1}$;

j) $F_{m+n} = F_m L_n + (-1)^{n+1} F_{m-n}$, $m \geq n$;

k) $F_{m+n} = F_n L_m + (-1)^n F_{m-n}$, $m \geq n$;

l) $5F_n = L_m L_{n-m+1} + L_{m-1} L_{n-m}$;

m) $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$;

n) $F_{m+2n} F_m - F_{m+n}^2 = (-1)^{m+1} F_n^2$;

o) $L_{4n+2} - 2 = L_{2n+1}^2$;

p) $5F_{2n}^2 = L_{4n} - 2$.

4 Proofs of Theorems 1 and 2 and Corollaries 3–6

Proof of Theorem 1. Using the property $\gcd(u, v) = \gcd(u, v - u)$, observe that

$$D = \gcd(F_{2an+2b+i} + (-1)^b, F_{2an+2b+2j}).$$

By Lemma 12.g, we have

$$F_{2an+2b+2j} = F_{an+b+j} L_{an+b+j}.$$

If $an + j$ is odd, then by Lemma 12.k, it follows that

$$F_{2an+2b+i} + (-1)^b = F_{an+b+j} L_{an+b+j+1}.$$

This implies

$$D = F_{an+b+j} \gcd(L_{an+b+j+1}, L_{an+b+j}),$$

and hence by Lemma 12.b, we have $D = F_{an+b+j}$.

If $an + j$ is even, then

$$F_{2an+2b+i} + (-1)^b = F_{an+b+j+1} L_{an+b+j}.$$

and

$$D = L_{an+b+j} \gcd(F_{an+b+j+1}, F_{an+b+j}) = L_{an+b+j}.$$

□

Proof of Theorem 2. Using Lemma 12.m, we deduce that

$$F_{2an+2b+i+k} = F_k F_{2an+2b+i+1} + F_{k-1} F_{2an+2b+i}.$$

Thus, by $\gcd(u, v) = \gcd(u, v - F_{k-1} u)$, we have

$$D = \gcd(F_{2an+2b+i} + (-1)^b, F_k F_{2an+2b+i+1}).$$

Using Lemma 12.g, we deduce that

$$F_{2an+2b+i+1} = F_{an+b+j} L_{an+b+j}.$$

Now we consider two cases: $an + j$ is odd or $an + j$ is even.

If $an + j$ is odd, then by Lemma 12.j, we have

$$F_{2an+2b+i} + (-1)^b = F_{an+b+j} L_{an+b+j-1} + (-1)^{an+b+j} + (-1)^b = F_{an+b+j} L_{an+b+j-1}.$$

Thus, combining with Lemma 12.b, we have

$$D = F_{an+b+j} \gcd(L_{an+b+j-1}, F_k L_{an+b+j}) = F_{an+b+j} \gcd(L_{an+b+j-1}, F_k).$$

If $an + j$ is even, then by Lemma 12.k, we deduce that

$$F_{2an+2b+i} + (-1)^b = L_{an+b+j} F_{an+b+j-1} + (-1)^{an+b+j-1} + (-1)^b = L_{an+b+j} F_{an+b+j-1}.$$

Thus, combining with Lemma 12.a, we have that

$$D = L_{an+b+j} \gcd(F_{an+b+j-1}, F_k F_{an+b+j}) = L_{an+b+j} \gcd(F_{an+b+j-1}, F_k).$$

□

Proof of Corollary 3 and Corollary 4. Notice that we have

$$\gcd(L_{an+b+j-1}, F_1) = \gcd(F_{an+b+j-1}, F_1) = 1,$$

and

$$\gcd(L_{an+b+j-1}, F_2) = \gcd(F_{an+b+j-1}, F_2) = 1.$$

So we obtain Corollary 3 and Corollary 4 from Theorem 2 by setting $k = 1$ or $k = 2$. □

Proof of Corollary 5. We have

$$\begin{cases} F_m \equiv 0 \pmod{2}, & \text{if and only if } m \equiv 0 \pmod{3}; \\ L_m \equiv 0 \pmod{2}, & \text{if and only if } m \equiv 0 \pmod{3}; \end{cases}$$

and

$$\gcd(F_{an+b+j-1}, F_3) = \begin{cases} 2, & \text{if } an+b+j \equiv 1 \pmod{3}; \\ 1, & \text{otherwise;} \end{cases}$$

and

$$\gcd(L_{an+b+j-1}, F_3) = \begin{cases} 2, & \text{if } an+b+j \equiv 1 \pmod{3}; \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, we conclude that Corollary 5 is the case $k = 3$ in Theorem 2. □

Example 13. If $a = 3, b = 0, i = 3$ and n is even, we get

$$\gcd(F_{6n+3} + 1, F_{6n+6} + 1) = L_{3n+2}.$$

Proof of Corollary 6. The proof is similar to the proof of Corollary 5 by using the fact that $F_4 = 3$ and

$$\begin{cases} F_m \equiv 0 \pmod{3} & \text{if and only if } m \equiv 0 \pmod{4}; \\ L_m \equiv 0 \pmod{3} & \text{if and only if } m \equiv 2 \pmod{4}. \end{cases}$$

□

5 Proof of Theorem 7

Let $D = D_{a,s}(n)$. We need two lemmas.

Lemma 14. *For all $a \in \mathbb{N}, s \in \mathbb{Z}$, and $n \in \mathbb{N}_0$, we have $D_{a,s}(n)$ divides the value $F_a^2 + (-1)^n s^2 (L_a - 1 + (-1)^{a+1})$.*

Proof. We apply Lemma 12.n and get on the one hand

$$F_{n+2a}F_n - F_{n+a}^2 = (-1)^{n+1}F_a^2,$$

hence

$$-sF_{n+2a} - s^2 \equiv (-1)^{n+1}F_a^2 \pmod{D}.$$

On the other hand, we have

$$F_{n+2a} = F_{n+a}L_a + (-1)^{a+1}F_n$$

by Lemma 12.j and

$$F_{n+2a} \equiv -sL_a + (-1)^a s \pmod{D}.$$

Together, this gives us

$$s^2(L_a + (-1)^{a+1} - 1) \equiv (-1)^{n+1}F_a^2 \pmod{D}.$$

□

Lemma 15. *Let* $(a, s) \in \mathbb{Z}^2$ *and* $a > 0$*. Then*

1) $F_a^2 + s^2(L_a - 1 + (-1)^{a+1}) = 0$ *is impossible.*
2) $F_a^2 - s^2(L_a - 1 + (-1)^{a+1}) = 0$ *implies* $(a, s) \in \mathcal{S}$*.*

Proof. Let (a, s) be a solution of the equation $F_a^2 + s^2(L_a - 1 + (-1)^{a+1}) = 0$. If $a = 1$, then $1 + s^2 = 0$, a contradiction. If $a > 1$, then $L_a - 1 + (-1)^{a+1} \geq L_a - 2 > 0$; hence $F_a = 0$ and $s = 0$, also a contradiction.

Let (a, s) be a solution of the equation $F_a^2 - s^2(L_a - 1 + (-1)^{a+1}) = 0$. Then we have

$$F_a^2 = s^2(L_a - 1 + (-1)^{a+1}).$$

Now we consider two cases for a.

First case: a is odd, that is $F_a^2 = s^2L_a$. If $a = 1$, then $1 = s^2$ and $(a, s) = (1, \pm 1) \in \mathcal{S}$. We can assume $a \geq 3$. Let p a prime divisor of L_a, hence of F_a. Since

$$\gcd(F_a, L_a) = \gcd(F_a, L_a - F_a) = \gcd(F_a, 2F_{a-1}) \in \{1, 2\},$$

we get $p = 2, L_a = 2^{2\alpha}, F_a = |s|\, 2^\alpha, \alpha \geq 1$. Since a is odd and F_a is even, we see from Table 1 that $a \equiv 3 \pmod{6}$ and $F_a \not\equiv 0 \pmod{4}$; hence $\alpha = 1, L_a = 4$ and $(a, s) = (3, \pm 1) \in \mathcal{S}$.

a	3	5	7	9	11
F_a	2	5	13	34	89
$F_a \mod 2$	0	1	1	0	1
$F_a \mod 4$	2	1	3	2	1

Table 1: Residue

Second case: a is even, that is $F_a^2 = s^2(L_a - 2)$. If $a \equiv 0 \pmod 4$, say $a = 4k, k > 0$, then $F_a^2 = s^2 5F_{2k}^2$ by Lemma 12.p and 5 is square, a contradiction. Now we can assume $a \equiv 2 \pmod 4$, say $a = 4k+2, k \geq 0$. Then $L_a - 2 = L_{2k+1}^2$ by Lemma 12.o and

$$F_{2k+1}^2(L_{4k+2} - 2) = F_{2k+1}^2 L_{2k+1}^2 = F_{4k+2}^2$$

by Lemma 12.g. This implies $(a, s) = (4k+2, \pm F_{2k+1}) \in \mathcal{S}$. The proof is completed.

□

Proof of Theorem 7. First, let $(a, s) \in \mathbb{N} \times \mathbb{Z}, (a, s) \notin S$. By Lemma 15, we have

$$F_a^2 + (-1)^n s^2(L_a - 1 + (-1)^{a+1}) \neq 0 \text{ for } n \in \{0, 1\}$$

and by Lemma 14, the function $D_{a,s}(n)$ is bounded.

On the other hand, if $(a, s) \in S$, then we see that $D_{a,s}$ is unbounded, since for $n \in \mathbb{N}$ by Theorem 1, we have

$$D_{1,1}(4n+1) = L_{2n} \text{ and } D_{1,-1}(4n-1) = F_{2n},$$

and by Corollary 5, we have

$$D_{3,1}(4n+1) = \lambda F_{2n+1} \geq F_{2n+1} \text{ and } D_{3,-1}(4n+1) = \lambda L_{2n+1} \geq L_{2n+1}.$$

If $n = 4m + 2k + 3$ then

$$\begin{aligned}
\gcd(F_n + F_{2k+1}, F_{n+4k+2} + F_{2k+1}) &= \gcd(F_n + F_{2k+1}, F_{n+4k+2} - F_n) \\
&= \gcd(F_{4m+2k+3} + F_{2k+1}, F_{4m+6k+5} - F_{4m+2k+3}) \\
&= \gcd(F_{2m+1}L_{2m+2k+2}, F_{4m+4k+4}L_{2k+1}) \\
&\quad (\text{ by Lemma 12.j and 12.k}) \\
&= L_{2m+2k+2} \gcd(F_{2m+1}, F_{2m+2k+2}L_{2k+1}) \text{ (by Lemma 12.g)} .
\end{aligned}$$

Then $L_{2m+2k+2}$ divides $D_{4k+2,F_{2k+1}}(n)$. Hence, $D_{4k+2,F_{2k+1}}(n)$ is unbounded.

Same argumentation, we obtain the unboundedness of $D_{4k+2,-F_{2k+1}}(n)$. So all $(a, s) \in \mathcal{S}$ generate unbounded functions $D_{a,s}(n)$. □

Remark 16. The same arguments as in the proof of Theorem 7 give the following result on Lucas numbers.

Theorem 17. *Let $a \in \mathbb{N}, s \in \mathbb{Z}$. Then the conditions a) and b) are equivalent:*

a) *The function $D^*_{a,s}(n) = \gcd(L_n + s, L_{n+a} + s)$ is unbounded on $\mathbb{N}_0$;*

b) *$a = 4b, s = \pm L_{2b}$ for $b \in \mathbb{N}$.*

6 Proof of Theorem 8

We need the following lemma.

Lemma 18. *Let $s, t \in \mathbb{Z}, n \in \mathbb{N}_0$. Then*

1) *$C = C_{s,t}(n)$ divides $e_{s,t}(n)$;*

2) *C divides $e^*_{s,t}$.*

Proof. Let $n \in \mathbb{N}_0$ and define $M(n)$ and Q as follows:

$$M(n) := \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} \text{ and } Q := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $M(n+1) = QM(n)$ and $M(n) = Q^n M(0)$. Taking determinants gives

$$\begin{aligned}(-1)^{n+1} = \det M(n) &= F_{n+2}F_n - F_{n+1}^2 \\ &= (F_{n+1} + F_n)F_n - F_{n+1}^2 \\ &\equiv (t+s)s - t^2 \pmod{C}.\end{aligned}$$

Hence $e_{s,t}(n) \equiv 0 \pmod{C}$ and $e^*_{s,t} \equiv 0 \pmod{C}$. □

Corollary 19. *If $s, t \in \mathbb{Z}$ and $e^*_{s,t} \neq 0$, then the function $n \mapsto C_{s,t}(n)$ is bounded on $\mathbb{N}$.*

Proof. Now, we give a proof of Theorem 8 by the circle $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$.
$(a) \Rightarrow (c)$ by Corollary 19.
$(c) \Rightarrow (b)$ We have

$$\begin{aligned}\pm 1 &= s^2 + st - t^2 \\ &= (s + \frac{t}{2})^2 - 5(\frac{t}{2})^2 \\ &= \frac{2s + t + t\sqrt{5}}{2} \cdot \frac{2s + t - t\sqrt{5}}{2}.\end{aligned}$$

These fractions are units in the quadratic field $\mathbb{Q}(\sqrt{5})$; hence by [4, Theorem 257]

$$\frac{2s + t + t\sqrt{5}}{2} = \pm\alpha^j, \text{ where } \alpha := \frac{1+\sqrt{5}}{2}, \ j \in \mathbb{Z}.$$

Since $\alpha^j = \dfrac{L_j + F_j\sqrt{5}}{2}$, we get $t = \pm F_j$ and

$$s = \pm\frac{L_j - t}{2} = \pm\frac{F_{j+1} + F_{j-1} - F_j}{2} = \pm F_{j-1}, j \in \mathbb{Z}.$$

Since $F_{-j} = (-1)^{j+1}F_j$, we get $(s,t) \in \mathcal{R}$.

$(b) \Rightarrow (a)$ Let $i \geq 2$. By Jacobson [5] we have

$$F_{2^i 3} \equiv 0 \pmod{2^{i+1}}; \quad F_{2^i 3-1} \equiv 1 \pmod{2^{i+1}}.$$

Hence, by Lemma 12.m

$$\begin{aligned} F_{2^i 3+j} &= F_{2^i 3} F_{j+1} + F_{2^i 3-1} F_j \\ &\equiv F_j \pmod{2^{i+1}} \end{aligned}$$

and $F_{2^i 3-j} \equiv F_{-j} \equiv (-1)^{j+1} F_j \pmod{2^{i+1}}$. We see that the function $n \mapsto C_{s,t}(n)$ is unbounded on $\mathbb{N}$ if $(s,t) = -(F_j, F_{j+1})$ and $(s,t) = (-1)^j (F_j, -F_{j-1}), j \in \mathbb{N}_0$. If $(s,t) = (F_j, F_{j-1}), j \in \mathbb{N}_0$, then

$$\begin{aligned} C_{s,t}(n) &= \gcd(F_n + F_j, F_{n+1} + F_{j+1}) \\ &= \gcd(F_{n-1} + F_{j-1}, F_n + F_j) \\ &= \gcd(F_{n-2} + F_{j-2}, F_{n-1} + F_{j-1}) \\ &= \gcd(F_{n-j}, F_{n-j+1} + 1). \end{aligned}$$

Since $F_{2m} = F_m L_m$ and $F_{2m+1} + 1 = F_m L_{m+1}$ if m is odd, we have for m is odd

$$C_{s,t}(2m + j) = \gcd(F_{2m}, F_{2m+1} + 1) = F_m \gcd(L_m, L_{m+1}) = F_m,$$

so $C_{s,t}(n)$ is unbounded. The last case $(s,t) = (-1)^{j+1}(F_j, -F_{j-1}) = (F_{-j}, F_{-(j-1)}), j \in \mathbb{N}_0$ is proved in a similar way. Thus, the proof of Theorem 8 is completed.

Remark 20. It is an open problem to characterize the general function

$$n \mapsto \gcd(F_n + s, F_{n+a} + t), a \in \mathbb{N}, s, t \in \mathbb{Z}.$$

7 Proof of periodicity (Theorem 9 and 10)

The proof of Theorem 9 and Theorem 10 are completely analogous to the one in [8] with $a = 1$. Set $c_n := F_a^2 + (-1)^n s^2 (L_a - 1 + (-1)^{a+1})$ and $c := |c_0 c_1|$, then $c \neq 0$ and $D_{a,s}(n)$ divides c for $(a, s) \notin \mathcal{S}$. Then there is an integer $1 \leq p \leq c^2$ such that $F_{n+p} \equiv F_n \pmod{c}$ for all $n \in \mathbb{N}$. This p is a period of $D_{a,s}(n)$. Same argumentation, we obtain the proof of Theorem 10. □

A period p that we get by this method is not necessarily the smallest positive one. For example, if $a = 1, s = 2$ in Theorem 9 or $t = s = 2$ in Theorem 10, then we have $c = 15$ and $F_{n+80} \equiv F_n \pmod{15}$. Hence $p = 80$, but 40 is also a period of $D_{1,2}(n)$ and it is the smallest one [1].

One cannot always get the smallest period by this method. For example, if $a = 4, s = -2$ in Theorem 9; hence $c = 319$. Then 10 is the smallest period, but $F_{10} \not\equiv F_0 \pmod{319}$. If $s = 7, t = 2$ in Theorem 10, then we have $c = 3480$ and $p = 840$, but 120 is the smallest period and $F_{120} \not\equiv F_0 \pmod{3480}$.

8 Gcd of three shifted Fibonacci numbers

Proof of Theorem 11. Fix $a > 0, n \geq 0$ and $s \in \mathbb{Z}$, and set $E(n) = E_{a,s}(n)$. Then by Lemma 12.j, we have

$$F_{n+2a} = F_{n+a}L_a + (-1)^{a+1}F_n,$$

which gives us that

$$-s \equiv -sL_a - (-1)^{a+1}s \pmod{E(n)}.$$

So $E(n)$ divides $s(L_a - 1 + (-1)^{a+1})$. If $s = 0$, then by Lemma 12.e we have

$$E(n) = \gcd(F_n, F_{n+a}, F_{n+2a}) = F_{\gcd(n,n+a,n+2a)} = F_{\gcd(n,a)} = \gcd(F_n, F_a).$$

Hence, $E(n)$ divides F_a. Let

$$d := \begin{cases} s(L_a - 1 + (-1)^{a+1}), & \text{if } s \neq 0; \\ F_a, & \text{if } s = 0. \end{cases}$$

Then $d \neq 0$, and we deduce that $E(n)$ divides d. In particular, the function $E(n)$ is bounded. As in the proof of Theorem 9, we conclude that this function is simply periodic. □

Remark 21. If $s = 0$, then $E(a) = \gcd(F_a, F_{2a}, F_{3a}) = F_a$ so arbitrarily large values are possible.

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