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Another Proof of Zagier's Matrix Conjecture

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Abstract

We prove a conjecture of Zagier about the inverse of a $(K-1)\times(K-1)$ matrix $A=A_K$ using elementary methods. This formula allows one to express the product of single zeta values $\zeta(2r)\zeta(2K+1-2r)$ for $1\leq r\leq K-1$ in terms of the double zeta values $\zeta(2r,2K+1-2r)$ for $1\leq r\leq K-1$ and $\zeta(2K+1)$.

Dedicated to Professor Don Zagier on the Occasion of His 70th Birthday.

1 Introduction

This paper addresses a conjecture of Zagier [5]. For positive integers $k_1,\ldots,k_n$, with $k_n\geq 2$, define the multiple zeta value $\zeta(k_1,k_2,\ldots,k_n)$ by

$$\zeta\left(k_1,k_2,\ldots,k_n\right)=\sum_{1\leq m_1<\cdots<m_n}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}}. \tag{1}$$

Here $k=k_1+k_2+\cdots+k_n$ is called the *weight* of this multiple zeta value.

When $n=1$, we have the classical Riemann zeta value

$$\zeta(k)=\sum_{m=1}^{\infty}\frac{1}{m^k}.$$

When $n = 2$, the double sum

$$\zeta(k_1, k_2) = \sum_{m=2}^{\infty} \frac{1}{m^{k_2}} \sum_{j=1}^{m-1} \frac{1}{j^{k_1}} \tag{2}$$

was considered by Euler.

Let $H(0) = 1$ and define

$$H(n) = \zeta(\underbrace{2, 2, \ldots, 2}_{n}) \quad \text{for } n \geq 1.$$

It is well known that for $n \geq 0$,

$$H(n) = \frac{\pi^{2n}}{(2n+1)!}.$$

When a and b are nonnegative integers, define

$$H(a, b) = \zeta(\underbrace{2, \ldots, 2}_{a}, 3, \underbrace{2, \ldots, 2}_{b}).$$

Zagier [5] derived the formula

$$H(a, b) = 2 \sum_{r=1}^{K} (-1)^r \left(\binom{2r}{2a+2} - \left(1 - \frac{1}{2^{2r}}\right) \binom{2r}{2b+1} \right) H(K - r)\zeta(2r + 1), \tag{3}$$

where $K = a + b + 1$. Here for a real number n and a nonnegative integer k, we define the generalized binomial coefficient $\binom{n}{k}$ by

$$\binom{n}{k} = \begin{cases} \dfrac{n(n-1)\cdots(n-k+1)}{k!}, & \text{if } k \geq 1; \\ 1, & \text{if } k = 0. \end{cases}$$

In particular, if n is an integer and $n < k$, then $\binom{n}{k} = 0$.

The formula (3) expresses $H(a, b)$ as a rational linear combination of the $H(K - r)\zeta(2r + 1)$. It can be used to prove that the following two sets

$$\mathscr{B}_1 = \{H(a, K - 1 - a) \,|\, 0 \leq a \leq K - 1\},$$
$$\mathscr{B}_2 = \{H(K - r)\zeta(2r + 1) \,|\, 1 \leq r \leq K\},$$

span the same vector space over $\mathbb{Q}$.

On the other hand, Euler found that all double zeta values of odd weight can be expressed as rational linear combinations of products of Riemann zeta values. The following explicit formula was derived in [5]. For $K \geq 2$ and $1 \leq r \leq K-1$, we have

$$\zeta(2r, 2K+1-2r) = -\frac{1}{2}\zeta(2K+1) + \sum_{s=1}^{K-1} A_{r,s}\zeta(2s)\zeta(2K+1-2s), \tag{4}$$

where

$$A_{r,s} = \binom{2K-2s}{2r-1} + \binom{2K-2s}{2K-2r}. \tag{5}$$

Zagier [5] used an elementary argument to show that the $(K-1)\times(K-1)$ matrix

$$A = A_K = (A_{r,s})$$

has nonzero determinant, and thus it is invertible. Using the fact that both $\zeta(2s)$ and $H(2s)$ are rational multiples of π^{2s}, this shows that the set

$$\mathscr{B}_3 = \{\zeta(2r, 2K+1-2r) \,|\, 1 \leq r \leq K-1\} \cup \{\zeta(2K+1)\}$$

spans the same $\mathbb{Q}$-vector space as the set $\mathscr{B}_2$.

Zagier [5] formulated three conjectures about the matrix A_K. The first one is about its determinant, the second is about its LU-decomposition, and the third is about its inverse. The main objective of this paper is to prove the third conjecture, which states a pair of conjectural formulas for A^{-1}.

Let P and Q be the $(K-1)\times(K-1)$ matrices with entries

$$\begin{aligned} P_{s,r} &= \frac{2}{2s-1}\sum_{n=0}^{2K-2s}\binom{2r-1}{2K-2s-n+1}\binom{n+2s-2}{n}B_n, \\ Q_{s,r} &= -\frac{2}{2s-1}\sum_{n=0}^{2K-2s}\binom{2K-2r}{2K-2s-n+1}\binom{n+2s-2}{n}B_n. \end{aligned} \tag{6}$$

Here B_n are the Bernoulli numbers.

Zagier conjectured that both P and Q are the inverse of A. This implies that P and Q must be the same matrix. This conjecture was proved by Ma [3] using generating functions. In this work, we use a different approach, which is interesting in its own right.

In Section 2, we prove that $P = Q$. In Section 3, we prove that they indeed give the inverse of A.

2 The equality of the two conjectural formulas

Recall that the Bernoulli numbers B_n are defined by the generating function [2, p. 12]:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.$$

One can compute B_n recursively by $B_0 = 1$, and

$$B_n = -n! \sum_{k=0}^{n-1} \frac{B_k}{k!(n-k+1)!}, \quad \text{for } n \geq 1.$$

It is an easy consequence of the definitions that $B_1 = -1/2$, and for any odd integer $n > 1$ we have $B_n = 0$.

Let K be an integer greater than or equal to 2, and let $P = (P_{s,r})$ and $Q = (Q_{s,r})$ be the $(K-1) \times (K-1)$ matrices with entries $P_{s,r}$ and $Q_{s,r}$ given by (6). Our goal is to prove that $P_{s,r} = Q_{s,r}$ for all $1 \leq r, s \leq K-1$, using a generating function technique that is quite different from that used by Ma [3]. In Zagier's paper [5], the summations of n in (6) are taken to be until the term $n = 2K - 2s + 1$. Since $s \leq K - 1$, we find that $2K - 2s + 1$ is an odd integer greater than 2, and so $B_{2K-2s+1} = 0$. Thus the summations can be taken to be until $n = 2K - 2s$ only.

We begin with the following lemma.

Lemma 1. *Let s be a positive integer. Define the function $f_s(t)$ by*

$$f_s(t) = \frac{t^{2s-1}}{e^t - 1}.$$

If m is a positive integer, we have the following relation that relates the derivatives of f_s up to order m:

$$\begin{aligned} e^t f_s^{(m)}(t) &= \sum_{p=0}^{m} (-1)^{m-p} \binom{m}{p} f_s^{(p)}(t) \\ &\quad + \sum_{p=0}^{\min(m,2s-1)} (-1)^{m-p} \binom{m}{p} \frac{(2s-1)!}{(2s-1-p)!} t^{2s-1-p}. \end{aligned} \tag{7}$$

Proof. By the definition of $f_s(t)$, we have

$$f_s(t) = e^{-t} f_s(t) + t^{2s-1} e^{-t}.$$

Differentiate both sides m times and apply the Leibniz rule. This gives

$$f_s^{(m)}(t) = \sum_{p=0}^{m} \binom{m}{p} f_s^{(p)}(t) \frac{d^{m-p}}{dt^{m-p}} e^{-t} + \sum_{p=0}^{m} \binom{m}{p} \frac{d^p}{dt^p} t^{2s-1} \frac{d^{m-p}}{dt^{m-p}} e^{-t}.$$

Since

$$\frac{d^p}{dt^p} t^{2s-1} = 0 \qquad \text{if } p > 2s-1,$$

we find that

$$f_s^{(m)}(t) = \sum_{p=0}^{m} (-1)^{m-p} \binom{m}{p} e^{-t} f_s^{(p)}(t) + \sum_{p=0}^{\min(m,2s-1)} (-1)^{m-p} \binom{m}{p} e^{-t} \frac{(2s-1)!}{(2s-1-p)!} t^{2s-1-p}.$$

Multiplying both sides by e^t gives (7). $\square$

Now we can prove the main theorem in this section.

Theorem 2. *If K, r, s are integers with $K \geq 2$ is an integer and $r, s < K$, then*

$$\begin{aligned} &\frac{(2s-2)!}{(2K-2r)!} \sum_{n=0}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n \\ &= -\frac{(2s-2)!}{(2K-2r)!} \sum_{n=0}^{2K-2s} \binom{2r-1}{2K-2s-n+1} \binom{n+2s-2}{n} B_n. \end{aligned} \tag{8}$$

In particular, this implies that the matrices P and Q defined by (6) are equal.

Proof. The left hand side of (8) can be rewritten as

$$\sum_{n=\max(0,2r-2s+1)}^{2K-2s} \frac{1}{(2K-2s-n+1)!(2s-2r+n-1)!} \frac{(n+2s-2)!}{n!} B_n, \tag{9}$$

and the right hand side of (8) can be rewritten as

$$-\sum_{n=\max(0,2K-2s-2r+2)}^{2K-2s} \frac{(2r-1)!}{(2K-2s-n+1)!(2r+2s-2K+n-2)!} \frac{(n+2s-2)!}{(2K-2r)!} \frac{B_n}{n!}. \tag{10}$$

The equality (8) is proved by taking $m = 2r-1$ in the equation (7) and comparing the coefficients of t^{2K-2r} on both sides. Namely, we want to compare the coefficients of t^{2K-2r} on both sides of the equation

$$\begin{aligned} e^t f_s^{(2r-1)}(t) = &-\sum_{p=0}^{2r-1} (-1)^p \binom{2r-1}{p} f_s^{(p)}(t) \\ &- \sum_{p=0}^{\min(2r-1,2s-1)} (-1)^p \binom{2r-1}{p} \frac{(2s-1)!}{(2s-1-p)!} t^{2s-1-p}. \end{aligned} \tag{11}$$

Notice that

$$f_s(t) = t^{2s-2} \cdot \frac{t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^{n+2s-2}.$$

Therefore, we get

$$f_s^{(p)}(t) = \sum_{n=\max(0,p-2s+2)}^{\infty} B_n \frac{(n+2s-2)!}{(n+2s-2-p)!n!} t^{n+2s-2-p}.$$

First we consider the coefficient of t^{2K-2r} in the left hand side of (11), namely, the coefficient of t^{2K-2r} in

$$e^t f_s^{(2r-1)}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{n=\max(0,2r-2s+1)}^{\infty} B_n \frac{(n+2s-2)!}{(n+2s-2r-1)!n!} t^{n+2s-2r-1}. \tag{12}$$

It is given by the expression (9).

The term on the right hand side of (11) can be written as $T_1 + T_2$, where

$$\begin{aligned} T_1 &= -\sum_{p=0}^{2r-1} (-1)^p \binom{2r-1}{p} f_s^{(p)}(t) \\ &= -\sum_{p=0}^{2r-1} (-1)^p \frac{(2r-1)!}{p!(2r-1-p)!} \sum_{n=\max(0,p-2s+2)}^{\infty} B_n \frac{(n+2s-2)!}{(n+2s-2-p)!n!} t^{n+2s-2-p} \\ &= -\sum_{n=0}^{\infty} \sum_{p=0}^{\min(2r-1,n+2s-2)} (-1)^p \frac{(2r-1)!}{p!(2r-1-p)!} B_n \frac{(n+2s-2)!}{(n+2s-2-p)!n!} t^{n+2s-2-p}, \\ T_2 &= -\sum_{p=0}^{\min(2r-1,2s-1)} (-1)^p \frac{(2r-1)!}{p!(2r-1-p)!} \frac{(2s-1)!}{(2s-1-p)!} t^{2s-1-p}. \end{aligned}$$

The term T_2 contains a term in t^{2K-2r} if and only if $2s-1 \geq 2K-2r$, or equivalently, $r+s \geq K+1$. In this case the coefficient of t^{2K-2r} in T_2 is

$$\frac{(2r-1)!}{(2r+2s-2K-1)!(2K-2s)!} \frac{(2s-1)!}{(2K-2r)!}. \tag{13}$$

For the term T_1, the coefficient of t^{2K-2r} is

$$-\sum_{n=\max(0,2K-2s-2r+2)}^{2K-2s} (-1)^n \frac{(2r-1)!}{(2K-2s-n+1)!(2r+2s-2K+n-2)!} \frac{(n+2s-2)!}{(2K-2r)!} \frac{B_n}{n!}. \tag{14}$$

When $r+s \leq K$, we find that $2K-2s-2r+2 \geq 2$. Hence, the sum over n in (14) does not contain the $n=1$ term. Since $B_n = 0$ when $n > 2$ is an odd integer, we find that (14) is equal to (10). Since there is no contribution from T_2 to the term t^{2K-2r} when $r+s \leq K$, this proves that when $r+s \leq K$, the coefficient of t^{2K-2r} in the right hand side of (11) is (10).

When $r+s \geq K+1$, the $n=1$ term is present in (14). Using the fact that $B_1 = -\frac{1}{2}$, we find that this term is given by

$$-\frac{1}{2} \frac{(2r-1)!}{(2r+2s-2K-1)!(2K-2s)!} \frac{(2s-1)!}{(2K-2r)!},$$

which is $-1/2$ of the term (13). Summing the coefficients of t^{2K-2r} from T_1 and T_2, we find that the sum is equal to (10). Therefore, when $r+s \geq K+1$, the coefficient of t^{2K-2r} in the right hand side of (11) is (10).

Thus, we have shown that the coefficient of t^{2K-2r} in the left hand side of (11) is (9), and the coefficient of t^{2K-2r} in the right hand side of (11) is (10). This completes the proof of the theorem.

□

As we mentioned before, this theorem was proved by Ma [3] using a different method, with the help of Carlitz's Bernoulli number identity [1, 4]. Our proof uses the generating function of the Bernoulli numbers directly.

Remark 3. Carlitz's identity says that for any nonnegative integers m and n, we have

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k}. \tag{15}$$

Prodinger [4] gave a short proof using an exponential generating function of two variables. Here we show that this identity can be derived directly from (7) by setting $s=1$. Namely, we consider the generating function of the Bernoulli numbers

$$f(t) = \frac{t}{e^t - 1}.$$

Since (15) is symmetric in m and n, it is sufficient to consider the case $0 \leq m < n$. In this case, $n \geq 1$. Equation (7) says that

$$e^t f^{(m)}(t) = (-1)^m t + (-1)^{m-1} m + \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f^{(k)}(t).$$

This gives

$$\sum_{l=0}^{\infty} \frac{t^l}{l!} \sum_{k=m}^{\infty} \frac{B_k}{(k-m)!} t^{k-m} = (-1)^m t + (-1)^{m-1} m + \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{l=k}^{\infty} \frac{B_l}{(l-k)!} t^{l-k}.$$

Compare the coefficients of t^n on both sides, we have

$$\sum_{k=m}^{m+n} \frac{1}{(k-m)!(m+n-k)!} B_k = (-1)^m \delta_{n,1} + \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{B_{n+k}}{n!}.$$

If $n=1$, the last sum contains the term $(-1)^m B_1$, which can be combined with $(-1)^m \delta_{n,1}$ to yield $(-1)^{m-1} B_1$. Since $B_k = 0$ when k is odd and greater than 2, we find that

$$\sum_{k=m}^{m+n} \frac{n!}{(k-m)!(m+n-k)!} B_k = \sum_{k=0}^{m} (-1)^{m-n} \binom{m}{k} B_{n+k}.$$

Multiplying $(-1)^n$ on both sides and shifting the summation variables on the left hand side, we obtain

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k},$$

which is Carlitz's identity.

3 The proof of the conjecture

In this section, we prove that the inverse of the matrix $A = A_K$ is the matrix P.

Theorem 4 (Zagier's conjecture)**.** *Let $K > 1$ be an integer, and let $A = A_K$ be the $(K-1) \times (K-1)$ matrix defined by*

$$A_{r,s} = \binom{2K-2s}{2r-1} + \binom{2K-2s}{2K-2r}. \tag{16}$$

The inverse of A is the matrix P defined by one of the following two formulas that are equal.

$$\begin{aligned} P_{s,r} &= \frac{2}{2s-1} \sum_{n=0}^{2K-2s} \binom{2r-1}{2K-2s-n+1} \binom{n+2s-2}{n} B_n, \\ &= -\frac{2}{2s-1} \sum_{n=0}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n. \end{aligned} \tag{17}$$

Proof. Define the $(K-1) \times (K-1)$ matrices B and C by

$$B_{r,s} = \binom{2K-2s}{2r-1}, \qquad C_{r,s} = \binom{2K-2s}{2K-2r},$$

so that $A = B + C$. Notice that $B_{r,s} = 0$ if $r + s > K$, and $C_{r,s} = 0$ if $r < s$.

Our strategy of the proof is to show that the matrix $PA = PB + PC$ is indeed the identity matrix, by showing that if s and s' are positive integers less than K, then

$$(PB)_{s,s'} + (PC)_{s,s'} = \begin{cases} 1, & \text{if } s = s'; \\ 0, & \text{if } s \neq s'. \end{cases} \tag{18}$$

We use the following two elementary identities of binomial coefficients. If $n \geq 1$, then

$$\begin{aligned} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} &= 2^{n-1}, \\ \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} &= 2^{n-1}. \end{aligned} \tag{19}$$

First we compute $(PB)_{s,s'}$ using the first formula in (17) for $P_{s,r}$. The cases where $s \le s'$ and $s > s'$ are considered separately.

If $s \le s'$, then for $r \le K-s'$, we have $r \le K-s$. Hence, $2K-2s-2r+2 \ge 2$. Therefore,

$$(PB)_{s,s'} = \frac{2}{2s-1} \sum_{r=1}^{K-s'} \binom{2K-2s'}{2r-1} \sum_{n=2K-2s-2r+2}^{2K-2s} \binom{2r-1}{2K-2s-n+1} \binom{n+2s-2}{n} B_n$$
$$= \frac{2}{2s-1} \sum_{n=2s'-2s+2}^{2K-2s} \sum_{r=K-s-\frac{n}{2}+1}^{K-s'} \binom{2K-2s'}{2r-1} \binom{2r-1}{2K-2s-n+1} \binom{n+2s-2}{n} B_n.$$

Notice that the summation over n only contains terms with n even, since $2s'-2s+2 \ge 2$.

Using the binomial identity

$$\binom{a}{b}\binom{b}{c} = \binom{a}{a-c}\binom{a-c}{b-c},$$

we have

$$\binom{2K-2s'}{2r-1}\binom{2r-1}{2K-2s-n+1} = \binom{2K-2s'}{2s-2s'+n-1}\binom{2s-2s'+n-1}{2r+2s-2K+n-2}.$$

Therefore,

$$(PB)_{s,s'} = \frac{2}{2s-1} \sum_{n=2s'-2s+2}^{2K-2s} \binom{2K-2s'}{2s-2s'+n-1}\binom{n+2s-2}{n} B_n$$
$$\times \sum_{r=K-s-\frac{n}{2}+1}^{K-s'} \binom{2s-2s'+n-1}{2r+2s-2K+n-2}.$$

For $n \ge 2s'-2s+2$, we have $2s-2s'+n-1 \ge 1$. The first formula in (19) implies that

$$\sum_{r=K-s-\frac{n}{2}+1}^{K-s'} \binom{2s-2s'+n-1}{2r+2s-2K+n-2} = \sum_{r=0}^{s-s'+n/2-1} \binom{2s-2s'+n-1}{2r} = 2^{2s-2s'+n-2}. \tag{20}$$

This shows that when $s \le s'$, we have

$$(PB)_{s,s'} = \frac{2}{2s-1} \sum_{n=2s'-2s+2}^{2K-2s} \binom{2K-2s'}{2s-2s'+n-1}\binom{n+2s-2}{n} 2^{2s-2s'+n-2} B_n. \tag{21}$$

When $s > s'$, we get

$$(PB)_{s,s'} = \frac{2}{2s-1} \sum_{n=0}^{2K-2s} \sum_{K-s-\frac{n}{2}+1 \le r \le K-s'} \binom{2K-2s'}{2r-1}\binom{2r-1}{2K-2s-n+1}\binom{n+2s-2}{n} B_n.$$

Splitting out the $n = 1$ term, we have

$$(PB)_{s,s'} = \frac{2}{2s-1} \sum_{\substack{0 \le n \le 2K-2s \\ n \text{ is even}}} \binom{2K-2s'}{2s-2s'+n-1} \binom{n+2s-2}{n} B_n \times \sum_{r=K-s-\frac{n}{2}+1}^{K-s'} \binom{2s-2s'+n-1}{2r+2s-2K+n-2} + \frac{2}{2s-1} \binom{2K-2s'}{2s-2s'} \binom{2s-1}{1} B_1 \sum_{r=K-s+1}^{K-s'} \binom{2s-2s'}{2r+2s-2K-1}.$$

Notice that the second formula in (19) gives

$$\sum_{r=K-s+1}^{K-s'} \binom{2s-2s'}{2r+2s-2K-1} = \sum_{r=0}^{s-s'-1} \binom{2s-2s'}{2r+1} = 2^{2s-2s'-1}.$$

Together with (20), we find that when $s > s'$, we have

$$(PB)_{s,s'} = \frac{2}{2s-1} \sum_{n=0}^{2K-2s} \binom{2K-2s'}{2s-2s'+n-1} \binom{n+2s-2}{n} 2^{2s-2s'+n-2} B_n.$$

Next we compute $(PC)_{s,s'}$. Using the second expression in (17) for $P_{s,r}$, we have

$$(PC)_{s,s'} = -\frac{2}{2s-1} \sum_{r=s'}^{K-1} \binom{2K-2s'}{2K-2r} \sum_{n=\max(0,2r-2s+1)}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n. \tag{22}$$

Using the binomial identity

$$\binom{a}{b}\binom{b}{c} = \binom{a}{a-c}\binom{a-c}{a-b},$$

we have

$$\binom{2K-2s'}{2K-2r}\binom{2K-2r}{2K-2s-n+1} = \binom{2K-2s'}{2s-2s'+n-1}\binom{2s-2s'+n-1}{2r-2s'}.$$

Now we discuss the cases $s < s'$, $s = s'$ and $s > s'$ separately.

When $s = s'$, we have

$$(PC)_{s,s} = -\frac{2}{2s-1} \sum_{r=s}^{K-1} \binom{2K-2s}{2K-2r} \sum_{n=2r-2s+1}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n.$$

In this case, we have an $n = 1$ term when $r = s$. When $r > s$, the summation over $n \geq 2r - 2s + 1$ can be replaced by the summation over $n \geq 2r - 2s + 2$. The term with $r = s$ and $n = 1$ contributes the term 1. Therefore,

$$\begin{aligned}(PC)_{s,s} &= 1 - \frac{2}{2s-1} \sum_{r=s}^{K-1} \binom{2K-2s}{2K-2r} \sum_{n=2r-2s+2}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n \\ &= 1 - \frac{2}{2s-1} \sum_{n=2}^{2K-2s} \sum_{r=s}^{s+n/2-1} \binom{2K-2s}{n-1} \binom{n-1}{2r-2s} \binom{n+2s-2}{n} B_n \\ &= 1 - \frac{2}{2s-1} \sum_{n=2}^{2K-2s} \binom{2K-2s}{n-1} \binom{n+2s-2}{n} 2^{n-2} B_n.\end{aligned}$$

The last equality follows from the first equation in (19). Comparing to the $s = s'$ case in (21) shows that

$$(PB + PC)_{s,s} = 1.$$

Next we consider the case $s < s'$. Since $r \geq s'$ in the summation (22), we find that $2r - 2s + 1 \geq 3$. Therefore,

$$\begin{aligned}(PC)_{s,s'} &= -\frac{2}{2s-1} \sum_{r=s'}^{K-1} \sum_{n=2r-2s+2}^{2K-2s} \binom{2K-2s'}{2s-2s'+n-1} \binom{2s-2s'+n-1}{2r-2s'} \binom{n+2s-2}{n} B_n \\ &= -\frac{2}{2s-1} \sum_{n=2s'-2s+2}^{2K-2s} \binom{2K-2s'}{2s-2s'+n-1} \binom{n+2s-2}{n} B_n \sum_{r=s'}^{s+n/2-1} \binom{2s-2s'+n-1}{2r-2s'} \\ &= -\frac{2}{2s-1} \sum_{n=2s'-2s+2}^{2K-2s} \binom{2K-2s'}{2s-2s'+n-1} \binom{n+2s-2}{n} 2^{2s-2s'+n-2} B_n \\ &= -(PB)_{s,s'}.\end{aligned}$$

Finally, we consider the case $s > s'$. In this case

$$(PC)_{s,s'} = -\frac{2}{2s-1} \sum_{n=0}^{2K-2s} \sum_{s' \leq r \leq s+n/2-1} \binom{2K-2s'}{2K-2r} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n.$$

Splitting out the $n = 1$ term, we have

$$\begin{aligned}(PC)_{s,s'} = &-\frac{2}{2s-1} \sum_{\substack{0 \leq n \leq 2K-2s \\ n \text{ is even}}} \binom{2K-2s'}{2s-2s'+n-1} \binom{n+2s-2}{n} B_n \sum_{r=s'}^{s+n/2-1} \binom{2s-2s'+n-1}{2r-2s'} \\ &- \frac{2}{2s-1} \binom{2K-2s'}{2s-2s'} \binom{2s-1}{1} B_1 \sum_{r=s'}^{s} \binom{2s-2s'}{2r-2s'}\end{aligned}$$

$$= -\frac{2}{2s-1}\sum_{n=0}^{2K-2s}\binom{2K-2s'}{2s-2s'+n-1}\binom{n+2s-2}{n}2^{2s-2s'+n-2}B_n$$
$$= -(PB)_{s,s'}.$$

This completes the proof of (18), and so the assertion of the theorem is proved. □

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References

[1] L. Carlitz, Problem 795, *Math. Mag.* **44** (1971), 107.

[2] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, 1999.

[3] D. Ma, Inverse of some matrix related to double zeta values of odd weight, *J. Number Theory* **166** (2016), 166–180.

[4] H. Prodinger, Carlitz's symmetric Bernoulli number identity, *J. Integer Sequences* **17** (2014), Article 14.4.1.

[5] D. Zagier, Evaluation of the multiple zeta values $\zeta(2,\ldots,2,3,2,\ldots,2)$, *Ann. of Math.* **175** (2012), 977–1000.

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