Journal of Integer Sequences, Vol. 25 (2022),
Article 22.9.6

# Fixes of Permutations Acting on Monotone Boolean Functions

Andrzej Szepietowski
Institute of Informatics
Faculty of Mathematics,Physics and Informatics
University of Gdańsk
80-308 Gdańsk
Poland
matszp@inf.ug.edu.pl

**Abstract**

We present a few algorithms and methods to count fixes of permutations acting on monotone Boolean functions.

## 1 Introduction

Let $B$ denote the set $\{0,1\}$ and $B^n$ the set of $n$-element sequences of $B$. A *Boolean function with $n$ variables* is any function from $B^n$ into $B$. There are $2^n$ elements in $B^n$ and $2^{2^n}$ Boolean functions with $n$ variables. There is the order relation in $B$ (namely: $0 \leq 0$, $0 \leq 1$, $1 \leq 1$) and the partial order in $B^n$: for any two elements: $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in $B^n$,

$$x \leq y \quad \text{if and only if} \quad x_i \leq y_i \quad \text{for all } 1 \leq i \leq n.$$

The function $h : B^n \to B$ is *monotone* if

$$x \leq y \Rightarrow h(x) \leq h(y).$$

Let $D_n$ denote the set of monotone functions with $n$ variables and let $d_n$ denote $|D_n|$. Known values of $d_n$, for $n = 0, \ldots, 8$ are presented in the table at the end of this paper. The values $d_n$ for $n \leq 4$ were published by Dedekind [6], Church [4, 5] gave the values $d_5$ and $d_7$, Ward [14] the value $d_6$, and the last known value $d_8$ was published by Wiedemann [15]. Dedekind numbers were also considered in [1, 2, 3, 7, 13].

We have the partial order in $D_n$ defined as follows:

$$g \leq h \quad \text{if and only if} \quad g(x) \leq h(x) \quad \text{for all } x \in B^n.$$

We represent the elements of $D_n$ as strings of bits of length $2^n$. Two elements of $D_0$ will be represented as 0 and 1; any element $g \in D_1$ can be represented as a concatenation $g(0) * g(1)$, where $g(0), g(1) \in D_0$ and $g(0) \leq g(1)$. Hence, $D_1 = \{00, 01, 11\}$. Each element $g \in D_2$ is a concatenation (string) of four bits: $g(00) * g(10) * g(01) * g(11)$ which can be represented as a concatenation $g_0 * g_1$, where $g_0, g_1 \in D_1$ and $g_0 \leq g_1$. Hence, $D_2 = \{0000, 0001, 0011, 0101, 0111, 1111\}$. Similarly any element $g \in D_n$ can be represented as a concatenation $g_0 * g_1$, where $g_0, g_1 \in D_{n-1}$ and $g_0 \leq g_1$.

Let $S_n$ denote the set of permutations on $\{1, \dots, n\}$. Every permutation $\pi \in S_n$ defines the permutation on $B^n$ by $\pi(x) = x \circ \pi$ (we treat each element $x \in B^n$ as a function $x : \{1, \dots, n\} \to \{0, 1\}$). Note that $x \leq y$ if and only if $\pi(x) \leq \pi(y)$. The permutation $\pi$ also generates the permutation on $D_n$. Namely, by $\pi(g) = g \circ \pi$. Note that $\pi(g)$ is monotone if $g$ is monotone. Two functions $f, g \in D_n$ are *equivalent* if there is a permutation $\pi \in S_n$ such that $f = \pi(g)$. By $R_n$ we denote the set of equivalence classes and by $r_n$ we denote the number of the equivalence classes. Known values of $r_n$ (for $n \leq 8$) are given in the table at the end of this paper. The number of the equivalence classes can be computed by Burnside's lemma; see [10, §38]. Namely,

$$r_n = \frac{1}{n!} \sum_{\pi \in S_n} |\operatorname{Fix}(\pi, D_n)|,$$

where $\operatorname{Fix}(\pi, D_n)$ is the set of fixes of the permutation $\pi$ acting on $D_n$. A function $f \in D_n$ is a *fix* of $\pi$ if $\pi(f) = f$.

In 1985 and 1986 Liu and Hu [8, 9] used Burnside's lemma to compute $r_n$ for all $n \leq 7$. Recently, Pawelski [11] computed $r_8$.

In this paper we propose a new framework to study monotone Boolean functions and present a few algorithms and methods to count fixes of permutations acting on $D_n$. The main contributions of the paper are Theorem 4 and Lemma 6 which give formulas for the set of fixes of the composition $\pi \circ \rho$ of two permutations, provided $\pi$ and $\rho$ satisfy certain conditions. A special case of Lemma 6 was used by Pawelski [11] to count and generate fixes of several permutations acting on $D_n$. For completeness, in Sections 6.5 and 6.6, we present a method which was used by Pawelski [11] to compute fixes of the permutation $(12)(34)(56)(78)$ acting on $D_8$.

# 2 Posets

A *poset (partially ordered set)* $(S, \leq)$ consists of a set $S$ (called the *carrier*) together with a binary relation (partial order) $\leq$ which is reflexive, transitive and antisymmetric. For two posets $(S, \leq)$ and $(T, \leq)$ by $S \times T$ we denote the cartesian product with the order defined as follows: $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$. For two disjoint posets $(S, \leq)$ and $(T, \leq)$ by $S + T$ we denote the disjoint union (sum) with the order defined as follows:

$$x \leq y \quad \text{iff} \quad (x, y \in S \quad \text{and} \quad x \leq y) \quad \text{or} \quad (x, y \in T \quad \text{and} \quad x \leq y).$$

Given two posets $(S, \leq)$ and $(T, \leq)$ a function $f : S \to T$ is *monotone*, if $x \leq y$ implies $f(x) \leq f(y)$. By $T^S$ we denote the poset of all monotone functions from $S$ to $T$ with the partial order defined as follows:

$$f \leq g \quad \text{if and only if} \quad f(x) \leq g(x) \text{ for all } x \in S.$$

In this paper we use the following notation:

- $A_n$ denotes an antichain of order $n$, i.e. a poset of $n$ elements, where no two distinct elements are related. We only deal with antichains with the carrier being a finite subset of natural numbers.
- $B$ denotes the poset of two bits $\{0, 1\}$ ordered by $0 \leq 0$, $0 \leq 1$, $1 \leq 1$.
- $B^n$ denotes the poset $B^{A_n}$ of all (monotone) functions from $A_n$ into $B$. Note that each function from $A_n$ to $B$ is monotone. The poset $B^n$ is isomorphic to
  - the poset of all subsets of $\{1, \ldots, n\}$ ordered by the inclusion,
  - the poset of all $n$-strings of bits, where $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ iff $x_i \leq y_i$ for all $i$.
- $D_n$ denotes the poset $B^{B^n}$ of all monotone Boolean functions from $B^n$ into $B$, which are called monotone Boolean functions of $n$ variables.
- $P_n$ denotes the path (or chain) $P_n = \{p_1 < \cdots < p_n\}$. Note that $B^{P_n} = P_{n+1}$.

We will use the following lemma which is a part of the folklore and can be easily proved.

**Lemma 1.** *For three posets $R, S, T$,*
*(1) If $S$ and $T$ are disjoint, then the poset $R^{S+T}$ is isomorphic to $R^S \times R^T$.*
*(2) The poset $R^{S \times T}$ is isomorphic to $(R^S)^T$ and to $(R^T)^S$.*

As a corollary we have the following lemma. Similar lemmas in other formulations were used by Wiedemann [15], by Fidytek, Mostowski, Somla and Szepietowski [7], and by Campo [2] in order to compute $d_n = |D_n|$.

**Lemma 2.**

*(a)* $\quad A_{k+m} = A_k + A_m$

*(b)* $\quad B^{k+m} = B^k \times B^m$

*(c)* $\quad D_{k+m} = (D_k)^{B^m}$

*Proof.*

(a) is obvious.

(b) $B^{k+m} = B^{A_{k+m}} = B^{A_k + A_m} = B^{A_k} \times B^{A_m} = B^k \times B^m$.

(c) $D_{k+m} = B^{B^{k+m}} = B^{B^k \times B^m} = (B^{B^k})^{B^m} = (D_k)^{B^m}$.

□

## 3 Arrays

Let $M(S)$ denote the array of the poset $S$. For $i, j \in S$, we have $M(S)[i,j] = 1$ if $i \leq j$, and $M(S)[i,j] = 0$ otherwise. For example, for the poset $D_1 = \{00 < 01 < 11\}$, its array

$$M(D_1) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The poset $D_1$ is equal (isomorphic) to the poset of the path $P_3 = \{a < b < c\}$.

The elements of $M(S)$ describe monotone functions from the poset $B = \{0, 1\}$ to $S$. If $M(S)[i,j] = 1$ then there exists a monotone function with $f(0) = i$ and $f(1) = j$. Thus, if we add the elements of $M(S)$ we obtain $|S^B|$—the number of monotone functions from $B$ to $S$. For example,

$$\text{Sum}(M(D_1)) = 6 = |D_1^B| = |(B^B)^B| = |B^{B \times B}| = |B^{B^2}| = |D_2| = d_2,$$

where $\text{Sum}(M(D_1))$ denotes the sum of all elements of the array $M(D_1)$. Similarly, for every $n \geq 2$, we have

$$\text{Sum}(M(D_n)) = |D_n^B| = |(B^{B^n})^B| = |B^{B^n \times B}| = |B^{B^{n+1}}| = |D_{n+1}| = d_{n+1}.$$

Consider the product $M(S)^2 = M(S) \times M(S)$. Then $M(S)^2[i,j] = |\{k \in S : i \leq k \leq j\}|$ which is the number of elements in the interval $[i,j] \subset S$. Moreover, the elements of $M(S)^2$ are connected to monotone functions from the path $P_3 = \{a < b < c\}$ to $S$. Indeed, $M(S)^2[i,j]$ is equal to the number of monotone functions with $f(a) = i$ and $f(c) = j$, or in other words to the number of elements which can be chosen for the value of $f(b)$. Hence, $\text{Sum}(M(S)^2) = |S^{P_3}|$. For example,

$$M(D_1)^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$M(D_1)^2[1,3] = 3$ is equal to the number of elements in the interval $[00,11] = \{00,01,11\}$. Furthermore, $\mathrm{Sum}(M(D_1)^2) = 10$ is equal to $|D_1^{P_3}|$—the number of monotone functions from $P_3$ to $D_1$, and to $|(B^B)^{P_3}|$, and to $|B^{B\times P_3}|$. Moreover, the squares of the elements of $M(S)^2$ are connected to monotone functions from the cube $B^2 = \{00,01,10,11\}$ to $S$. Indeed, $(M(S)^2[i,j])^2 = M(S)^2[i,j]\cdot M(S)^2[i,j]$ is equal to the number of monotone functions with $f(00) = i$ and $f(11) = j$. Note that we can choose $M(S)^2[i,j]$ elements for the value of $f(01)$ and $M(S)^2[i,j]$ elements for the value of $f(10)$. Since these two values can be chosen independently, we have $(M(S)^2[i,j])^2$ monotone functions with $f(00) = i$ and $f(11) = j$. Hence, $\mathrm{SumSq}(M(S)^2) = |S^{B^2}|$, where $\mathrm{SumSq}(M(S)^2)$ denotes the sum of squares of all elements of the array $M(S)^2$. For example, $\mathrm{SumSq}(M(D_1)^2) = 20$ is equal to $|D_1^{B^2}| = |(B^B)^{B^2}| = |B^{B\times B^2}| = |B^{B^3}| = d_3$. Similarly, for every $n \geq 2$, we have

$$\mathrm{SumSq}(M(D_n)^2) = |D_n^{B^2}| = |(B^{B^n})^{B^2}| = |B^{B^n\times B^2}| = |B^{B^{n+2}}| = d_{n+2}.$$

This fact was used in another formulation by Fidytek, Mostowski, Somla and Szepietowski [7].

Consider the product $M(S)^3 = M(S) \times M(S) \times M(S)$. The elements of $M(S)^3$ are connected to monotone functions from the path $P_4 = (a < b < c < d)$ to $S$. Indeed, $M(S)^3[i,j]$ is equal to the number of pairs $x, y$, such that $i \leq x \leq y \leq j$, and to the number of monotone functions in $S^{P_4}$ with $f(a) = i$ and $f(d) = j$. Hence, $\mathrm{Sum}(M(S)^3) = |S^{P_4}|$. For example,

$$M(D_1)^3 = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{Sum}(M(D_1)^3) = 15$ is equal to $|D_1^{P_4}|$—the number of monotone functions from $P_4$ to $D_1$, and to $|(B^B)^{P_4}|$, and to $|B^{B\times P_4}|$.

# 4 Symmetries

Let $S_n$ denote the set of permutations on $\{1,\ldots,n\}$. Every permutation $\pi \in S_n$ defines the permutation on $B^n$ by $\pi(x) = x \circ \pi$. Here we treat elements $x \in B^n$ as functions $x : \{1,\ldots,n\} \to \{0,1\}$. Note that $x \leq y$ if and only if $\pi(x) \leq \pi(y)$.

**Example 3.** Consider two permutations: $\pi_1 = (12)$ and $\pi_2 = (123)$, both acting on $B^3$:

| $x$ | 000 | 100 | 010 | 110 | 001 | 101 | 011 | 111 |
|---|---|---|---|---|---|---|---|---|
| $\pi_1(x)$ | 000 | 010 | 100 | 110 | 001 | 011 | 101 | 111 |
| $\pi_2(x)$ | 000 | 001 | 100 | 101 | 010 | 011 | 110 | 111 |

Each permutation $\pi$ acting on $B^n$ generates a permutation on $D_n = B^{B^n}$. Namely, $\pi(g) = g \circ \pi$. Note that $\pi(g)$ is monotone if $g$ is monotone. Two functions $f, g \in D_n$ are *equivalent* if there is a permutation $\pi \in S_n$ such that $f = \pi(g)$. By $R_n$ we denote the set of equivalence classes. The number of the equivalence classes denoted by $r_n$ can be computed by Burnside's lemma; see [10, §38]. Known values of $r_n$ (for $n \le 8$) are given in the table at the end of this paper. By the lemma, the number of equivalence classes in $D_n$ is equal to

$$r_n = \frac{1}{n!} \sum_{\pi \in S_n} |\operatorname{Fix}(\pi, D_n)|,$$

where $\operatorname{Fix}(\pi, D_n)$ is the set of fixes of the permutation $\pi$ acting on $D_n$. A function $f \in D_n$ is a *fix* of $\pi$ if $\pi(f) = f$. Since conjugate group elements have the same number of fixed points, $|\operatorname{Fix}(\pi, D_n)|$ depends only on the cycle type of $\pi$. By the *cycle type* of a permutation $\pi$, we mean the data giving the lengths of disjoint cycles whose product is $\pi$ and the number of cycles of each length. Let $k(n)$ denote the number of cycle types of elements of $S_n$ and assume that $\pi_1, \dots, \pi_{k(n)}$ is an (arbitrarily chosen) sequence of representatives of all cycle types in $S_n$. For $1 \le i \le k(n)$, let $\mu_i$ be the number of elements of $S_n$ of the same cycle type as $\pi_i$ (see the tables in Section 8). Then

$$r_n = \frac{1}{n!} \sum_{i=1}^{k(n)} \mu_i \cdot |\operatorname{Fix}(\pi_i, D_n)|.$$

In this paper we present a few algorithms and methods to count fixes of permutations acting on $D_n$.

Consider a permutation $\pi \in S_n$ and suppose that $\pi$ when acting on $B^n$ is a product of disjoint cycles $\pi = C_1 \circ \cdots \circ C_r$, then a monotone function $f : B^n \to B$ is a fix of $\pi$ if and only if $f$ is constant on every cycle $C_i$. Let $\operatorname{Cycl}(\pi, B^n)$ denote the set of cycles $\{C_1, \dots, C_r\}$, and let $\le$ be the partial order defined in the following way: $C_i \le C_j$ if and only if there exist $x \in C_i$ and $y \in C_j$ such that $x \le y$ (we identify each cycle with the set of its elements). Hence, the poset $\operatorname{Fix}(\pi, D_n)$ is isomorphic to the poset $B^{\operatorname{Cycl}(\pi, B^n)}$ of monotone functions from $\operatorname{Cycl}(\pi, B^n)$ to $B = \{0, 1\}$ and we can represent fixes in $\operatorname{Fix}(\pi, D_n)$ as sequences of bits of length $|\operatorname{Cycl}(\pi, B^n)|$.

For the identity permutation $e$, each $x \in B^n$ forms a cycle of length 1, hence $\operatorname{Cycl}(e, B^n) = B^n$ and $\operatorname{Fix}(e, D_n) = D_n$. For $n = 1$, we have $D_1 = R_1$ and $r_1 = d_1 = 3$. For $n = 2$, we have two permutations: the identity $e$ with $\operatorname{Fix}(e, D_2) = D_2$ and the inversion $(12)$ with three cycles $C_1 = (00)$, $C_2 = (10, 01)$, and $C_3 = (11)$ which form the path $P_3 = \{C_1 < C_2 < C_3\}$. There are four monotone functions from the path $P_3$ to $B$ and four fixes in $\operatorname{Fix}((12), D_2)$. By Burnside's lemma, we have

$$r_2 = \frac{1}{2}(|\operatorname{Fix}(e, D_2)| + |\operatorname{Fix}((12), D_2)|) = \frac{1}{2}(6 + 4) = 5.$$

Indeed, there are five equivalence classes in $D_2$; namely,

$$R_2 = \{\{0000\}, \{0001\}, \{0101, 0011\}, \{0111\}, \{1111\}\}.$$

## 5 Main results

**Theorem 4.** *Consider a partition of the antichain $A_n = \{1, \ldots, n\}$ into two disjoint antichains $A_k = \{1, \ldots, k\}$ and $A_m = \{k+1, \ldots n\}$, where $n = k + m$; and two permutations: one $\pi$ acting on $A_k$ and $\rho$ acting on $A_m$. Suppose that each cycle of $\pi$ has a length which is coprime with the length of every cycle of $\rho$ then*

$$\mathrm{Fix}(\pi \circ \rho, D_n) = \mathrm{Fix}(\pi, D_k)^{\mathrm{Cycl}(\rho, B^m)} = \mathrm{Fix}(\rho, D_m)^{\mathrm{Cycl}(\pi, B^k)}.$$

*Proof.* The cube $B^n$ is isomorphic to the cartesian product $B^n = B^k \times B^m$. Suppose that we have two cycles: one $C_r$ of $\pi$ acting on $B^k$ and the other $C_s$ of $\rho$ acting on $B^m$. The lengths of the two cycles are coprime, so the product $C_r \times C_s$ is a cycle of $\pi \circ \rho$ acting on $B^n$. Furthermore, each cycle of $\pi$ has a length which is coprime with the length of every cycle of $\rho$ so

$$\mathrm{Cycl}(\pi \circ \rho, B^n) = \mathrm{Cycl}(\pi, B^k) \times \mathrm{Cycl}(\rho, B^m)$$

and

$$\begin{aligned}\mathrm{Fix}(\pi \circ \rho, D_n) &= B^{\mathrm{Cycl}(\pi \circ \rho, B^n)} = B^{\mathrm{Cycl}(\pi, B^k) \times \mathrm{Cycl}(\rho, B^m)} \\ &= \mathrm{Fix}(\pi, D_k)^{\mathrm{Cycl}(\rho, B^m)} = \mathrm{Fix}(\rho, D_m)^{\mathrm{Cycl}(\pi, B^k)}.\end{aligned}$$

□

**Example 5.** Consider the partition $A_5 = A_3 + A_2$, with $A_3 = \{1, 2, 3\}$ and $A_2 = \{4, 5\}$, and two permutations: $\pi = (123)$ and $\rho = (45)$. There are four cycles of $(123)$ acting on $B^3$. Namely, $C_1 = (000)$, $C_2 = (100, 001, 010)$, $C_3 = (011, 110, 101)$, and $C_4 = (111)$. They are of length 1 or 3 and they form the chain $P_4 = (C_1 < C_2 < C_3 < C_4)$. There are three cycles of $(45)$ acting on $B^2$. Namely, $c_1 = (00)$, $c_2 = (10, 01)$, and $c_3 = (11)$. They are of length 1 or 2 and they form the chain $P_3 = (c_1 < c_2 < c_3)$.

Furthermore, consider the two cycles $C_2 = (100, 001, 010\}$ of $\pi$ acting on $B^{\{1,2,3\}}$ and $c_2 = \{10, 01\}$ of $\rho$ acting on $B^{\{4,5\}}$. Their cartesian product

$$C_2 \times c_2 = \{10010, 00101, 01010, 10001, 00110, 01001\}$$

forms one cycle of $\pi \circ \rho$ acting on $B^5 = B^3 \times B^2$. The poset of cycles of $\pi \circ \rho$ acting on $B^5$ is the cartesian product $P_4 \times P_3$ and

$$|\,\mathrm{Fix}((123)(45), D_5)| = |B^{P_4 \times P_3}| = |P_5^{P_3}| = |P_4^{P_4}|.$$

In order to compute $\mathrm{Fix}((123)(45), D_5)$, let us consider the arrays

$$M(P_4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(P_4)^3 = \begin{pmatrix} 1 & 3 & 6 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{Fix}((123)(45), D_5) = |P_4^{P_4}| = \mathrm{Sum}(M(P_4)^3) = 35$. We can also use the arrays

$$M(P_5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(P_5)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$|\mathrm{Fix}((123)(45), D_5)| = |P_5^{P_3}| = \mathrm{Sum}(M(P_5)^2) = 35$.

The following lemma is a direct consequence of the theorem. In the special case with $m = 1$ the lemma was used by Pawelski [11] to count and generate fixes of several permutations acting on $D_n$.

**Lemma 6.** *Suppose that a permutation $\pi$ is acting on $B^k$ and $n > k$. Then when we consider $\pi$ acting on $B^n$,*

$$\mathrm{Fix}(\pi, D_n) = (\mathrm{Fix}(\pi, D_k))^{B^m} = D_m^{\mathrm{Cycl}(\pi, B^k)}$$

*where $m = n - k$.*

*Proof.* The permutation $\pi$ is acting on $\{1, \ldots, k\}$ and on $B^k$. We can say that $\pi$ also acts on $\{1, \ldots, n\}$ and on $B^n$, by identifying $\pi$ with $\pi \circ e$. Every cycle in $\mathrm{Cycl}(e, B^m)$ has length 1, hence, $\mathrm{Cycl}(e, B^m) = B^m$ and

$$\mathrm{Cycl}(\pi \circ e, B^n) = \mathrm{Cycl}(\pi, B^k) \times \mathrm{Cycl}(e, B^m) = \mathrm{Cycl}(\pi, B^k) \times B^m$$

and

$$\mathrm{Fix}(\pi, D_n) = \mathrm{Fix}(\pi \circ e, D_n) = B^{\mathrm{Cycl}(\pi, B^k) \times B^m} = (\mathrm{Fix}(\pi, D_k))^{B^m} = D_m^{\mathrm{Cycl}(\pi, B^k)}.$$

□

# 6 Applications

In this section we present a few examples that illustrate concepts from Section 3 and applications of Theorem 4 and Lemma 6.

Consider the poset $D_2 = B^{B^2} = \{0000, 0001, 0011, 0101, 0111, 1111\}$ and its array

$$M(D_2) = \left(\begin{array}{cc|cc|cc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Consider the array

$$M(D_2)^2 = \left(\begin{array}{cc|cc|cc} 1 & 2 & 3 & 3 & 5 & 6 \\ 0 & 1 & 2 & 2 & 4 & 5 \\ \hline 0 & 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Now $\mathrm{Sum}(M(D_2)^2) = 50$, which is equal to $|D_2^{P_3}| = |(B^{B^2})^{P_3}| = |B^{B^2 \times P_3}| = |\operatorname{Fix}((12), D_4)|$; see Lemma 6 and Section 3. Similarly, $\mathrm{SumSq}(M(D_2)^2) = 168$, which is equal to $|D_2^{B^2}| = |(B^{B^2})^{B^2}| = |B^{B^2 \times B^2}| = |B^{B^4}| = |D_4| = d_4$. Furthermore,

$$M(D_2)^3 = \left(\begin{array}{cc|cc|cc} 1 & 3 & 6 & 6 & 14 & 20 \\ 0 & 1 & 3 & 3 & 9 & 14 \\ \hline 0 & 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 & 3 & 6 \\ \hline 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Now $\mathrm{Sum}(M(D_2)^3) = 105$, which is equal to $|D_2^{P_4}| = |(B^{B^2})^{P_4}| = |B^{B^2 \times P_4}| = |\operatorname{Fix}((123), D_5)|$.

## 6.1 Permutation (12)

Consider the permutation $\pi = (12)$. When $\pi$ acts on $B^2$, we have three cycles $\mathrm{Cycl}((12), B^2) = P_3$ and $\operatorname{Fix}((12), D_2) = B^{P_3} = P_4$. When (12) acts on $B^3$, then $\mathrm{Cycl}((12), B^3) = P_3 \times B$ and $\operatorname{Fix}((12), D_3) = B^{P_3 \times B}$; see Lemma 6. By Section 3, the number of fixes can be computed either by

$$|\operatorname{Fix}((12), D_3)| = |B^{P_3 \times B}| = |(B^{P_3})^B| = |P_4^B| = \mathrm{Sum}(M(P_4)) = 10,$$

or by

$$|\operatorname{Fix}((12), D_3)| = |B^{P_3 \times B}| = |(B^B)^{P_3}| = |D_1^{P_3}| = \mathrm{Sum}(M(D_1)^2) = 10.$$

In a similar way we count fixes when (12) acts on $B^4$, then $\text{Cycl}((12), B^4) = P_3 \times B^2$ and $\text{Fix}((12), D_4) = B^{P_3 \times B^2}$. The number of fixes can be computed either by

$$|\,\text{Fix}((12), D_4)| = |B^{P_3 \times B^2}| = |(B^{P_3})^{B^2}| = |P_4^{B^2}| = \text{SumSq}(M(P_4)^2) = 50,$$

or by

$$|\,\text{Fix}((12), D_4)| = |B^{P_3 \times B^2}| = |(B^{B^2})^{P_3}| = |D_2^{P_3}| = \text{Sum}(M(D_2)^2) = 50.$$

## 6.2 Permutation (123)

Consider the permutation $\pi = (123)$. When $\pi$ acts on $B^3$, we have four cycles $\text{Cycl}((123), B^3) = P_4$ and $\text{Fix}((123), D_3) = B^{P_4} = P_5$. When (123) acts on $B^4$, then $\text{Cycl}((123), B^4) = P_4 \times B$ and $\text{Fix}((123), D_4) = B^{P_4 \times B}$, see Lemma 6. By Section 3, the number of fixes can be computed either by

$$|\,\text{Fix}((123), D_4)| = |B^{P_4 \times B}| = |(B^{P_4})^{B}| = |P_5^{B}| = \text{Sum}(M(P_5)) = 15,$$

or by

$$|\,\text{Fix}((123), D_4)| = |B^{P_4 \times B}| = |(B^{B})^{P_4}| = |D_1^{P_4}| = \text{Sum}(M(D_1)^3) = 15.$$

In a similar way we can count fixes when (123) acts on $B^5$. Then we have $\text{Cycl}((123), B^5) = P_4 \times B^2$ and $\text{Fix}((123), D_5) = B^{P_4 \times B^2}$. The number of fixes can be computed either by

$$|\,\text{Fix}((123), D_5)| = |B^{P_4 \times B^2}| = |(B^{P_4})^{B^2}| = |P_5^{B^2}| = \text{SumSq}(M(P_5)^2) = 105,$$

or by

$$|\,\text{Fix}((123), D_5)| = |B^{P_4 \times B^2}| = |(B^{B^2})^{P_4}| = |D_2^{P_4}| = \text{Sum}(M(D_2)^3) = 105.$$

## 6.3 Permutation (1234)

The permutation (1234) acting on $B^4$ has six cycles: $C_0 = \{0000\}$, $C_1 = \{1000, 0001, 0010, 0100\}$, $C_2 = \{1100, 1001, 0011, 0110\}$, $C_3 = \{1010, 0101\}$, $C_4 = \{1110, 1101, 1011, 0111\}$, and $C_5 = \{1111\}$. They are of length 1, 2, or 4 and ordered by $C_0 < C_1 < C_2, C_3 < C_4 < C_5$. There are 8 fixes of (1234) acting on $D_4$ with the array

$$M(\text{Fix}((1234), D_4)) = \left(\begin{array}{ccc|cc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\mathrm{Sum}(M(\mathrm{Fix}((1234), D_4))) = 35 = |\,\mathrm{Fix}((1234), D_5))|$, see Lemma 6 and Section 3. Furthermore,

$$M(\mathrm{Fix}((1234), D_4))^2 = \left(\begin{array}{ccc|cc|ccc} 1 & 2 & 3 & 4 & 4 & 6 & 7 & 8 \\ 0 & 1 & 2 & 3 & 3 & 5 & 6 & 7 \\ 0 & 0 & 1 & 2 & 2 & 4 & 5 & 6 \\ \hline 0 & 0 & 0 & 1 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

and $\mathrm{SumSq}(M(\mathrm{Fix}((1234), D_4))^2) = 494 = |\,\mathrm{Fix}((1234), D_6))|$. Also

$$M(\mathrm{Fix}((1234), D_4))^3 = \left(\begin{array}{ccc|cc|ccc} 1 & 3 & 6 & 10 & 10 & 20 & 27 & 35 \\ 0 & 1 & 3 & 6 & 6 & 14 & 20 & 27 \\ 0 & 0 & 1 & 3 & 3 & 9 & 14 & 20 \\ \hline 0 & 0 & 0 & 1 & 0 & 3 & 6 & 10 \\ 0 & 0 & 0 & 0 & 1 & 3 & 6 & 10 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

and $\mathrm{Sum}(M(\mathrm{Fix}((1234), D_4))^3) = 294 = |\,\mathrm{Fix}((1234)(567), D_7))|$.

## 6.4 Permutation (12345)

The permutation (12345) acting on $B^5$ has eight cycles:

- $C_0 = \{00000\}$,
- $C_1 = \{10000, 00001, 00010, 00100, 01000\}$,
- $C_2 = \{11000, 10001, 00011, 00110, 01100\}$,
- $C_3 = \{10100, 01001, 10010, 00101, 01010\}$,
- $C_4 = \{11100, 11001, 10011, 00111, 01110\}$,
- $C_5 = \{10110, 01101, 11010, 10101, 01011\}$,
- $C_6 = \{11110, 11101, 11011, 10111, 01111\}$,
- $C_7 = \{11111\}$.

They are of length 1 or 5 and ordered by $C_0 < C_1 < C_2, C_3 < C_4, C_5 < C_6 < C_7$. There are 11 fixes of (12345) acting on $D_5$ with the array

$$M(\mathrm{Fix}((12345), D_5)) = \left(\begin{array}{ccc|cc|c|cc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right).$$

We have $\mathrm{Sum}(M(\mathrm{Fix}((12345), D_5))) = 64 = |\,\mathrm{Fix}((12345), D_6))|$ and

$$M(\mathrm{Fix}((12345), D_5))^2 = \left(\begin{array}{ccc|cc|c|cc|ccc}
1 & 2 & 3 & 4 & 4 & 6 & 7 & 7 & 9 & 10 & 11 \\
0 & 1 & 2 & 3 & 3 & 5 & 6 & 6 & 8 & 9 & 10 \\
0 & 0 & 1 & 2 & 2 & 4 & 5 & 5 & 7 & 8 & 9 \\
\hline
0 & 0 & 0 & 1 & 0 & 2 & 3 & 3 & 5 & 6 & 7 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 5 & 6 & 7 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 5 & 6 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right).$$

By Theorem 4 and Section 3, we have $\mathrm{Sum}(M(\mathrm{Fix}((12345), D_5))^2) = 264 = |\,\mathrm{Fix}((12345)(67), D_7))|$ and by Lemma 6, we have

$$\mathrm{SumSq}(M(\mathrm{Fix}((12345), D_5))^2) = 1548 = |\,\mathrm{Fix}((12345), D_7))|.$$

Also

$$M(\mathrm{Fix}((12345), D_5))^3 = \left(\begin{array}{ccc|cc|c|cc|ccc}
1 & 3 & 6 & 10 & 10 & 20 & 27 & 27 & 43 & 53 & 64 \\
0 & 1 & 3 & 6 & 6 & 14 & 20 & 20 & 34 & 43 & 53 \\
0 & 0 & 1 & 3 & 3 & 9 & 14 & 14 & 26 & 34 & 43 \\
\hline
0 & 0 & 0 & 1 & 0 & 3 & 6 & 6 & 14 & 20 & 27 \\
0 & 0 & 0 & 0 & 1 & 3 & 6 & 6 & 14 & 20 & 27 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 9 & 14 & 20 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 6 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 & 10 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

and $\mathrm{Sum}(M(\mathrm{Fix}((12345), D_5))^3) = 870 = |\,\mathrm{Fix}((12345)(678), D_8))|$; see Theorem 4.

### 6.5 Permutation (12)(34)

In the following two sections we present a method which was used by Pawelski [11] to compute fixes of the permutation (12)(34)(56)(78) acting on $D_8$. Consider the first partition $A_4 = \{1, 2, 3, 4\} = \{1, 2\} + \{3, 4\}$, and two permutations: $\pi = (12)$ and $\rho = (34)$. Furthermore, consider a partition of $B^4$ into four subcubes:

$$\begin{aligned} B^4_{00} &= \{0000, 1000, 0100, 1100\} = B^2 \times \{00\} \\ B^4_{10} &= \{0010, 1010, 0110, 1110\} = B^2 \times \{10\} \\ B^4_{01} &= \{0001, 1001, 0101, 1101\} = B^2 \times \{01\} \\ B^4_{11} &= \{0011, 1011, 0111, 1111\} = B^2 \times \{11\}. \end{aligned}$$

Each of these subcubes is isomorphic to $B^2$. There are three kinds of cycles of $\pi \circ \rho$ acting on $B^4$:

- (0000), (1000, 0100), (1100). They are contained in $B^4_{00}$ and are isomorphic to the cycles of $\pi$ acting on $B^2$.
- (0011), (1011, 0111), (1111). They are contained in $B^4_{11}$ and are isomorphic to the cycles of $\pi$ acting on $B^2$.
- (0010, 0001), (1010, 0101), (0110, 1001), (1110.1101). Each of the cycles contains two elements $\{x, y\}$ such that $x \in B^4_{10}$, $y \in B^4_{01}$, and $y = \pi \circ \rho(x)$. Moreover, each $x \in B^4_{10}$ belongs to one of these cycles.

Suppose that $f$ is a fix of $\pi \circ \rho$ acting on $D_4$ and consider four restrictions: $f_{00} = f|_{B^4_{00}}$, $f_{10} = f|_{B^4_{10}}$, $f_{01} = f|_{B^4_{01}}$, and $f_{11} = f|_{B^4_{11}}$. They satisfy the following conditions:

1. $f_{00}, f_{11} \in \mathrm{Fix}(\pi, D_2)$. Here we identify $B^2 \times \{00\}$ (and $B^2 \times \{11\}$) with $B^2$ and functions $B^{B^2 \times \{00\}}$ (and $B^{B^2 \times \{11\}}$) with $B^{B^2}$.
2. $f_{10}, f_{01} \in B^{B^2} = D_2$. We identify functions $B^{B^2 \times \{10\}}$ (and $B^{B^2 \times \{01\}}$) with $B^{B^2}$.
3. $f_{10} = \pi(f_{01})$
4. $f_{00} \leq f_{10}, f_{10} \leq f_{11}$.

On the other hand, if for a function $f$, its restrictions $f_{00} = f|_{B^4_{00}}$, $f_{10} = f|_{B^4_{10}}$, $f_{01} = f|_{B^4_{01}}$, $f_{11} = f|_{B^4_{11}}$ satisfy conditions (1–4), then $f$ is a fix of $\pi \circ \rho$ acting on $B^4$.

### 6.6 Permutation (12)(34)(56)(78)

Consider partition $A_{n+2} = \{1, \ldots, n+2\} = \{1, \ldots, n\} + \{n+1, n+2\}$, and two permutations: $\pi$ acting on $\{1, \ldots, n\}$ and $\rho = (n+1, n+2)$. and suppose that cycles of $\pi$ are of length 1 or 2. Consider a partition of $B^{n+2}$ into four subcubes:

- $B^{n+2}_{00} = B^n \times \{00\}$
- $B^{n+2}_{10} = B^n \times \{10\}$
- $B^{n+2}_{01} = B^n \times \{01\}$
- $B^{n+2}_{11} = B^n \times \{11\}$.

There are three kinds of cycles of $\pi \circ \rho$ acting on $B^{n+2}$:

- Those contained in $B^{n+2}_{00}$; isomorphic to the cycles of $\pi$ acting on $B^n$.
- Those contained in $B^{n+2}_{11}$; isomorphic to the cycles of $\pi$ acting on $B^n$.
- Each $x \in B^{n+2}_{10}$ belongs to the cycle with $y = \pi \circ \rho(x) \in B^{n+2}_{01}$.

Suppose that $f$ is a fix of $\pi \circ \rho$ acting on $D_{n+2}$ and consider four restrictions:

- $f_{00} = f|_{B^{n+2}_{00}}$,
- $f_{10} = f|_{B^{n+2}_{10}}$,
- $f_{01} = f|_{B^{n+2}_{01}}$,
- $f_{11} = f|_{B^{n+2}_{11}}$.

They satisfy the following conditions:

1. $f_{00}, f_{11} \in \mathrm{Fix}(\pi, D_n)$
2. $f_{10}, f_{01} \in B^{B^n} = D_n$
3. $f_{10} = \pi(f_{01})$
4. $f_{00} \le f_{10}, f_{10} \le f_{11}$.

On the other hand, if for a function $f$, its restrictions $f_{00} = f|_{B^{n+2}_{00}}$, $f_{10} = f|_{B^{n+2}_{10}}$, $f_{01} = f|_{B^{n+2}_{01}}$, $f_{11} = f|_{B^{n+2}_{11}}$ satisfy conditions (1–4), then $f$ is a fix of $\pi \circ \rho$ acting on $D_{n+2}$.

**Algorithm counting fixes**
Input: posets $D_n = B^{B^n}$ and $\mathrm{Fix}(\pi, D_n)$.
Output: $|\,\mathrm{Fix}(\pi \circ \rho, D_{n+2})|$.

- Sum := 0
- For each $f_{10} \in D_n$:
  - $f_{01} := \pi(f_{10})$;
  - Down $:= |\{g \in \text{Fix}(\pi, D_n) : g \leq f_{10} \& f_{01}\}|$ //the number of possibilities for choosing $f_{00}$
  - Up $:= |\{g \in \text{Fix}(\pi, D_n) : g \geq f_{10} | f_{01}\}|$ //the number of possibilities for choosing $f_{11}$
  - Sum := Sum + Down · Up
- Return $|\text{Fix}(\pi \circ \rho, D_{n+2})| :=$ Sum.

Note that for each function $g \in \text{Fix}(\pi, D_n)$ we have

$g \leq f_{10}$ and $g \leq f_{01}$ if and only if $g \leq f_{10} \& f_{01}$

and

$g \geq f_{10}$ and $g \geq f_{01}$ if and only if $g \geq f_{10} | f_{01}$.

A similar algorithm was used by Pawelski [11] in order to count fixes of the permutation (12)(34)(56)(78) acting on $D_8$.

**Example 7.** Consider the algorithm working on the permutation (12)(34) acting on $D_4$. Then $D_2 = \{0000 < 0001 < 0011, 0101 < 0111 < 1111\}$ and $\text{Fix}((12), D_2) = \{0000 < 0001 < 0111 < 1111\}$.

- for $f_{10} = 0000$: $f_{01} = 0000$; $f_{10} \& f_{01} = 0000$; Down $= 1$; $f_{10} | f_{01} = 0000$; Up $= 4$.
- for $f_{10} = 0001$: $f_{01} = 0001$; $f_{10} \& f_{01} = 0001$; Down $= 2$; $f_{10} | f_{01} = 0001$; Up $= 3$.
- for $f_{10} = 0011$: $f_{01} = 0101$; $f_{10} \& f_{01} = 0001$; Down $= 2$; $f_{10} | f_{01} = 0111$; Up $= 2$.
- for $f_{10} = 0101$: $f_{01} = 0011$; $f_{10} \& f_{01} = 0001$; Down $= 2$; $f_{10} | f_{01} = 0111$; Up $= 2$.
- for $f_{10} = 0111$: $f_{01} = 0111$; $f_{10} \& f_{01} = 0111$; Down $= 3$; $f_{10} | f_{01} = 0111$; Up $= 2$.
- for $f_{10} = 1111$: $f_{01} = 1111$; $f_{10} \& f_{01} = 1111$; Down $= 4$; $f_{10} | f_{01} = 1111$; Up $= 1$.

The algorithm returns $|\text{Fix}((12)(34), D_4)| = 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 2 + 2 \cdot 2 + 3 \cdot 2 + 4 \cdot 1 = 28$.

# 7 Generating fixes

In this section we present one more method to generate $\mathrm{Fix}(\pi, D_n)$ fixes of a permutation $\pi$ acting on $D_n$. We start with the poset $\mathrm{Cycl}(\pi, B^n)$ with its array $M(\mathrm{Cycl}(\pi, B^n))$. For example, consider the permutation (12) acting on $B^3$. The poset $\mathrm{Cycl}((12), B^3) = \{a < b < c\} \times \{0 < 1\} = \{a0, b0, c0, a1, b1, c1\}$ has the matrix

$$M(\mathrm{Cycl}((12)), B^3)) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We identify rows of the array with subsets of $\mathrm{Cycl}(\pi, B^n)$ and with functions from $\mathrm{Cycl}(\pi, B^n)$ to $\{0, 1\}$. It is well known that monotone functions from $\mathrm{Cycl}(\pi, B^n)$ to $\{0, 1\}$ may be identified with upsets. A subset $U \subset \mathrm{Cycl}(\pi, B^n)$ is an *upset* if for every $x, y$, we have:

$$\text{if} \quad x \in U \quad \text{and} \quad x \leq y, \quad \text{then} \quad y \in U.$$

Each row in the array $M(\mathrm{Cycl}(\pi, B^n))$ represents the upset $\mathrm{Up}(c) = \{x \in \mathrm{Cycl}(\pi, B^n) : x \geq c\}$. The set of all upsets can be generated in the following way: We start with rows of the array $M(\mathrm{Cycl}(\pi, B^n))$. Then we add the zero vector and the bitwise or $(x|y)$ of every pair $x, y$ already in Fix.

**Algorithm generating** $\mathrm{Fix}(\pi, D_n)$
Input: poset $C = \mathrm{Cycl}(\pi, B^n)$ and its array
Output: $\mathrm{Fix}(\pi, D_n)$.

- Fix $:= \emptyset$
- add zero vector to Fix
- For each $c \in C$:
  - for each $x \in \mathrm{Fix}$ add $x|\,\mathrm{Up}(c)$ to Fix
  - remove repetitions in Fix
- Return $\mathrm{Fix}(\pi, D_n) := \mathrm{Fix}$

For example, the algorithm adds four rows to the array $M(\mathrm{Cycl}((12), B^3))$

$$\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{array}$$

These ten rows form the poset $\mathrm{Fix}((12), D_3)$ with the partial order defined by

$$x \le y \text{ iff } x|y = y.$$

# 8 Tables of fixes

In this section we present tables with numbers of fixes of all permutations acting in $D_n$ for $n = 3, \ldots, 8$. Values for $n \le 6$ are from [8], values for $n = 7, 8$ are from [9, 11].

$n = 3$

| $i$ | $\pi_i$ | $\mu_i$ | $\lvert\mathrm{Fix}(\pi_i, D_3)\rvert$ |
|---|---|---|---|
| 1 | $e$ | 1 | 20 |
| 2 | (12) | 3 | 10 |
| 3 | (123) | 2 | 5 |

$n = 4$

| $i$ | $\pi_i$ | $\mu_i$ | $\lvert\mathrm{Fix}(\pi_i, D_4)\rvert$ |
|---|---|---|---|
| 1 | $e$ | 1 | 168 |
| 2 | (12) | 6 | 50 |
| 3 | (123) | 8 | 15 |
| 4 | (1234) | 6 | 8 |
| 5 | (12)(34) | 3 | 28 |

$n = 5$

| $i$ | $\pi_i$ | $\mu_i$ | $\lvert\mathrm{Fix}(\pi_i, D_5)\rvert$ |
|---|---|---|---|
| 1 | $e$ | 1 | 7 581 |
| 2 | (12) | 10 | 887 |
| 3 | (123) | 20 | 105 |
| 4 | (1234) | 30 | 35 |
| 5 | (12)(34) | 15 | 309 |
| 6 | (12345) | 24 | 11 |
| 7 | (12)(345) | 20 | 35 |

$n = 6$

| $i$ | $\pi_i$ | $\mu_i$ | $\lvert \mathrm{Fix}(\pi_i, D_6) \rvert$ |
|---|---|---|---|
| 1 | $e$ | 1 | 7 828 354 |
| 2 | (12) | 15 | 160 948 |
| 3 | (123) | 40 | 3 490 |
| 4 | (1234) | 90 | 494 |
| 5 | (12)(34) | 45 | 24 302 |
| 6 | (12345) | 144 | 64 |
| 7 | (123456) | 120 | 44 |
| 8 | (12)(345) | 120 | 490 |
| 9 | (123)(456) | 40 | 562 |
| 10 | (12)(3456) | 90 | 324 |
| 11 | (12)(34)(56) | 15 | 860 |

$n = 7$

| $i$ | $\pi_i$ | $\mu_i$ | $\lvert \mathrm{Fix}(\pi_i, D_7) \rvert$ |
|---|---|---|---|
| 1 | $e$ | 1 | 2 414 682 040 998 |
| 2 | (12) | 15 | 2 208 001 624 |
| 3 | (123) | 40 | 2 068 224 |
| 4 | (1234) | 90 | 60 312 |
| 5 | (12345) | 144 | 1 548 |
| 6 | (123456) | 120 | 766 |
| 7 | (1234567) | 120 | 101 |
| 8 | (12)(34) | 45 | 67 922 470 |
| 9 | (12)(345) | 45 | 59 542 |
| 10 | (12)(3456) | 120 | 26 878 |
| 11 | (12)(34567) | 120 | 264 |
| 12 | (123)(456) | 120 | 69 264 |
| 13 | (123)(4567) | 120 | 294 |
| 14 | (12)(34)(56) | 15 | 12 015 832 860 |
| 15 | (12)(34)(567) | 15 | 10 192 |

| | $i$ | $\pi_i$ | $\mu_i$ | $\lvert \mathrm{Fix}(\pi_i, D_8)\rvert$ |
|---|---|---|---|---|
| $n = 8$ | 1 | $e$ | 1 | 56 130 437 228 687 557 907 788 |
| | 2 | (12) | 28 | 101 627 867 809 333 596 |
| | 3 | (123) | 112 | 262 808 891 710 |
| | 4 | (1234) | 420 | 424 234 996 |
| | 5 | (12345) | 1344 | 531 708 |
| | 6 | (123456) | 3366 | 144 320 |
| | 7 | (1234567) | 5760 | 3 858 |
| | 8 | (12345678) | 5040 | 2 364 |
| | 9 | (12)(34) | 210 | 182 755 441 509 724 |
| | 10 | (12)(345) | 1120 | 401 622 018 |
| | 11 | (12)(3456) | 2520 | 93 994 196 |
| | 12 | (12)(34567) | 4032 | 21 216 |
| | 13 | (12)(345678) | 3360 | 70 096 |
| | 14 | (123)(456) | 1120 | 535 426 780 |
| | 15 | (123)(4567) | 3360 | 25 168 |
| | 16 | (123)(45678) | 2688 | 870 |
| | 17 | (1234)(5678) | 1260 | 3 211 276 |
| | 18 | (12)(34)(56) | 420 | 7 377 670 895 900 |
| | 19 | (12)(34)(567) | 1680 | 16 380 370 |
| | 20 | (12)(34)(5678) | 1260 | 37 834 164 |
| | 21 | (12)(345)(678) | 1120 | 3 607 596 |
| | 22 | (12)(34)(56)(78) | 105 | 2 038 188 253 420 |

# 9 Known values of $d_n$ and $r_n$

| $n$ | $d_n$ | $r_n$ |
|---|---|---|
| 0 | 2 | 2 |
| 1 | 3 | 3 |
| 2 | 6 | 5 |
| 3 | 20 | 10 |
| 4 | 168 | 30 |
| 5 | 7 581 | 210 |
| 6 | 7 828 354 | 16 353 |
| 7 | 2 414 682 040 998 | 490 013 148 |
| 8 | 56 130 437 228 687 557 907 788 | 1 392 195 548 889 993 358 |

These are sequences A000372 and A003182 in the *On-Line Encyclopedia of Integer Sequences* [12].

## References

[1] V. Bakoev, One more way for counting monotone Boolean functions, in *Thirteenth International Workshop on Algebraic and Combinatorial Coding Theory*, Pomorie, Bulgaria (2012), pp. 47–52.

[2] F. a Campo, Relations between powers of Dedekind numbers and exponential sums related to them, *J. Integer Sequences* **21** (2018) Article 18.4.4.

[3] P. De Causmaecker and S. De Wannemacker, On the number of antichains of sets in a finite universe, Arxiv preprint 1407.4288v2 [math.CO], available at `https://arxiv.org/abs/1407.4288`, 2014.

[4] R. Church, Numerical analysis of certain free distributive structures, *Duke Math. J.* **6** (1940) 732–734.

[5] R. Church, Enumeration by rank of the elements of the free distributive lattice with 7 generators, *Notices Amer. Math. Soc.* **12** (1965) 724.

[6] R. Dedekind, Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler, *Festschrift Hoch. Braunschweig u. ges. Werke II* (1897) 103–148.

[7] R. Fidytek, A. W. Mostowski, R. Somla, and A. Szepietowski, Algorithms counting monotone Boolean functions, *Inform. Process. Lett.* **79** (2001), 203–209.

[8] C. C. Liu and S. B. Hu, A mechanical algorithm of equivalent classification for free distributive lattices, *Chinese J. Comput.* **3** (2) (1985), 128–135. In Chinese.

[9] C. C. Liu and S. B. Hu, A note on the problem of computing the number of equivalence classes of free distributive lattices, *J. Wuhan Univ. Natur. Sci. Ed.* (1986), no. 1, 13–17. In Chinese.

[10] G. E. Martin, *Counting: The Art of Enumerative Combinatorics,* Springer, 2001.

[11] B. Pawelski, On the number of inequivalent monotone Boolean functions of 8 variables, *J. Integer Sequences* **25** (2022), Article 22.7.7.

[12] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, 2022. Available at `https://oeis.org`.

[13] T. Stephen and T. Yusun, Counting inequivalent monotone Boolean functions, *Discrete Appl. Math.*, **167** (2014), 15–24.

[14] M. Ward, Note on the order of the free distributive lattice, *Bull. Amer. Math. Soc.* **52** (1946) 423.

[15] D. Wiedemann, A computation of the eighth Dedekind number, *Order* **8** (1991), 5–6.

2020 *Mathematics Subject Classification*: Primary 06E30.
*Keywords:* monotone Boolean function, Dedekind number, equivalence class, fix.

---

(Concerned with sequence A000372 and A003182.)

---

Received May 8 2022; revised versions received September 19 2022; October 26 2022. Published in *Journal of Integer Sequences*, November 20 2022.

---

Return to Journal of Integer Sequences home page.