# Proof of a Conjecture of Stanley About Stern's Array

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**Abstract**

Stanley, building on work of Stern, defined an array of numbers as follows: the first row of the array is $\cdots 0001000 \cdots$. Each successive row is obtained by copying down the previous row and inserting, between each pair of numbers from the previous row, the sum of that pair of numbers. Let $s_n^r$ be the sum of the $r$-th powers of the elements in the $n$-th row of the array. Stanley showed that, for each positive integer $r$, the sequence $s_n^r$ obeys a homogeneous linear recurrence in $n$ of length $r/2 + O(1)$. Numerical evidence, however, suggested that $s_n^r$ obeys shorter recurrences, of length $r/3 + O(1)$. We prove Stanley's conjecture.

## 1 Introduction and main results

Richard Stanley [2], following a construction of Stern [3], introduced an array of numbers defined as follows: We start with the sequence $\cdots 0001000 \cdots$. We copy down this sequence and, in between each two consecutive elements of this sequence, we insert the sum of those two elements to obtain $\cdots 0000011100000 \cdots$; the new elements are shown in red. We repeat

this process to obtain the array shown below (with 0's omitted).

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & 1 & & & 1 & & & 1 & & & & \\
 & & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1
\end{array}
$$

We label the nonzero elements of the $n$-th row as $\left\langle {n \atop k} \right\rangle$ for $1 \le k \le 2^n - 1$. So we have the recurrence:

$$
\left\langle {n \atop 2k} \right\rangle = \left\langle {n-1 \atop k} \right\rangle \quad \text{and} \quad \left\langle {n \atop 2k+1} \right\rangle = \left\langle {n-1 \atop k} \right\rangle + \left\langle {n-1 \atop k+1} \right\rangle
$$

with initial conditions

$$
\left\langle {1 \atop 1} \right\rangle = 1 \quad \text{and} \quad \left\langle {1 \atop k} \right\rangle = 0 \text{ for } k \ne 1.
$$

This triangular array is [A337277](https://oeis.org) in the *On-Line Encyclopedia of Integer Sequences* [1].

For binomial coefficients, we have the easy identities $\sum_k \binom{n}{k} = 2^n$ and $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$, but there are no simple formulas for $\sum_k \binom{n}{k}^r$ for $r \ge 2$. In contrast, Stanley showed that $\sum_k \left\langle {n \atop k} \right\rangle^r$ obeys a homogeneous linear recurrence for all $r$.

We repeat Stanley's argument. For any function $f(x, y)$ from $\mathbb{R}^2 \to \mathbb{R}$, put

$$
S_n(f) = \sum_k f\left( \left\langle {n \atop k} \right\rangle, \left\langle {n \atop k+1} \right\rangle \right).
$$

Define $(\sigma^* f)(x, y) = f(x+y, y)$ and $(\tau^* f)(x, y) = f(x, x+y)$. Then we have

$$
\begin{aligned}
S_n(f) &= \sum_i f\left( \left\langle {n \atop 2i} \right\rangle, \left\langle {n \atop 2i+1} \right\rangle \right) + \sum_j f\left( \left\langle {n \atop 2j+1} \right\rangle, \left\langle {n \atop 2j+2} \right\rangle \right) \\
&= \sum_i f\left( \left\langle {n-1 \atop i} \right\rangle, \left\langle {n-1 \atop i} \right\rangle + \left\langle {n-1 \atop i+1} \right\rangle \right) \\
&\quad + \sum_j f\left( \left\langle {n-1 \atop j} \right\rangle + \left\langle {n-1 \atop j+1} \right\rangle, \left\langle {n-1 \atop j+1} \right\rangle \right) \\
&= S_{n-1}(\sigma^* f) + S_{n-1}(\tau^* f) \\
&= S_{n-1}(\sigma^* f + \tau^* f).
\end{aligned}
$$

If $f$ is a homogenous polynomial of degree $r$, then $\sigma^* f$ and $\tau^* f$ are also such polynomials. Thus $f \mapsto \sigma^* f + \tau^* f$ is a linear endomorphism of the vector space of degree $r$ polynomials, given by some $(r+1) \times (r+1)$ matrix $\Phi$. Since $S_n(f)$ is a linear function of $g$, it follows that, for any degree $r$ polynomial $f$, the sequence $S_n(f)$ obeys the same linear recurrence as the sequence of matrix powers $\Phi^n$.

**Example 1.** We consider cubic polynomials. A basis for the space of homogenous cubics in two variables is $x^3$, $x^2y$, $xy^2$, $y^3$, and we have

$$
\begin{aligned}
\sigma^*(x^3) \ + \tau^*(x^3) \ &= (x+y)^3 \ + x^3 \quad\quad\; &= 2x^3+3x^2y+3xy^2+ \ y^3 \\
\sigma^*(x^2y)+\tau^*(x^2y) &= (x+y)^2y + x^2(x+y) = \ &x^3+2x^2y+2xy^2+ \ y^3 \\
\sigma^*(xy^2)+\tau^*(xy^2) &= (x+y)y^2 + x(x+y)^2 = \ &x^3+2x^2y+2xy^2+ \ y^3 \\
\sigma^*(y^3) \ + \tau^*(y^3) \ &= y^3 \quad\quad\quad\; + (x+y)^3 \ = \ &x^3+3x^2y+3xy^2+2y^3.
\end{aligned}
$$

So $f \mapsto \sigma^*(f)+\tau^*(f)$ acts on this four-dimensional vector space by the matrix

$$
\Phi = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 \\ 3 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}.
$$

We say that a sequence $T_n$ obeys a homogenous linear recurrence of length $\ell$ if there are constants $a_1$, $a_2$, $\ldots$, $a_\ell$ such that $T_n = \sum_{j=1}^{\ell} a_j T_{n-j}$. Since the matrix $\Phi$ in the above example is $4 \times 4$, the matrix powers $\Phi^n$ obey a homogenous linear recurrence of length 4, and thus the $S_n(f)$ do likewise for any homogenous cubic polynomial $f$. More generally, if $f$ is a homogenous polynomial of degree $r$, this argument shows that the sequence $S_n(f)$ will obey a recurrence of length $r+1$.

This is an elegant argument, but it falls far short of the truth. Computation shows that

$$
\begin{aligned}
(S_n(x^3))_{n\geq 1} \ &= (S_n(y^3))_{n\geq 1} \ = (1, \ 3, \ 21, \ 147, \ \ldots) \\
(S_n(x^2y))_{n\geq 1} &= (S_n(xy^2))_{n\geq 1} = (0, \ 2, \ 14, \ 98, \ \ldots).
\end{aligned}
$$

We actually have the length-1 recurrence $S_n(f) = 7S_{n-1}(f)$ (for $n \geq 2$)!

Why do these recurrences have length 1 rather than 4? There are two reasons. The first is that the operator $\Phi$ commutes with the map $f(x,y) \mapsto f(y,x)$, and we have $S_n(f(x,y)) = S_n(f(y,x))$. So $\Phi$ passes to the 2-dimensional quotient vector space where we identify $f(x,y)$ and $f(y,x)$. Writing $[f]$ for the class of a polynomial in this quotient, we have $\Phi([x^3]) = 3[x^3]+6[x^2y]$ and $\Phi([x^2y]) = 2[x^3]+4[x^2y]$, so $\Phi$ acts by the matrix $\Phi_{\text{sym}} = \left[\begin{smallmatrix} 3 & 2 \\ 6 & 4 \end{smallmatrix}\right]$. Thus $S_n(f)$ obeys a recurrence of length 2. In general, this argument shows that, for $f$ a homogenous polynomial of degree $r$, the sequence $S_n(f)$ will obey a recurrence of length $\lceil\frac{r+1}{2}\rceil$. (For the reader who doesn't like quotient vector spaces, we can instead note that $\Phi$ preserves the subspace of symmetric polynomials, and that $S_n(f(x,y)) = S_n(\frac{f(x,y)+f(y,x)}{2})$, so we can use the matrix of $\Phi$ on this $\lceil\frac{r+1}{2}\rceil$-dimensional subspace.)

The second reason is that the characteristic polynomial of $\Phi_{\text{sym}}$ is $x^2 - 7x$, so we have $\Phi_{\text{sym}}^n = 7\Phi_{\text{sym}}^{n-1}$ for $n \geq 2$, and thus $S_n(f) = 7S_{n-1}(f)$. The 0-eigenvalue makes the recurrence shorter!

Stanley conjectured that the analogous operators $\Phi_{\text{sym}}$ had repeated eigenvalues of 0 for $r$ odd, and of $\pm 1$ for $r$ even. The latter phenomenon makes the minimal polynomial of $\Phi_{\text{sym}}$ of lower degree than the characteristic polynomial, and thus also shortens the length of the

recurrence satisfied by $S_n(f)$. After introducing some general notation, we state Stanley's conjectures, and prove them.

Fix a nonnegative integer $r$. Let $V$ be the vector space of degree $r$ homogeneous polynomials in $x$ and $y$. For a $2 \times 2$ integer matrix $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we define

$$(\gamma^* f)(x, y) = f(ax + cy, \ bx + dy).$$

We put $\sigma = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\tau = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, so this is consistent with our previous notation, and we note that $(\alpha\beta)^*(f) = \beta^* \alpha^* f$. We put $\Phi = \sigma^* + \tau^*$, a linear map $V \to V$.

Let $V_{\mathrm{sym}}$ be the quotient of $V$ by $f(x, y) \equiv f(y, x)$. The map $\Phi$ passes to the quotient $V_{\mathrm{sym}}$, and we denote this quotient map by $\Phi_{\mathrm{sym}}$. Write $m(M, \lambda)$ for the multiplicity of $\lambda$ as an eigenvalue of the matrix $M$.

Our main theorem involves many periodic functions of $r$, where $r$ is restricted to either even or odd numbers. If $r$ is restricted to even values, we write $[a_1, a_2, \ldots, a_p]_r$ for the periodic function $f$ with $f(0) = a_1$, $f(2) = a_2$, $f(4) = a_3$, $\ldots$, $f(2p) = a_1$, $\ldots$, et cetera; if $r$ is restricted to odd values, we write $[a_1, a_2, \ldots, a_p]_r$ for the periodic function $f$ with $f(1) = a_1$, $f(3) = a_2$, $f(5) = a_3$, $\ldots$, $f(2p+1) = a_1$, $\ldots$, et cetera.

**Theorem 2.** *The operators $\Phi$ and $\Phi_{\mathrm{sym}}$ are diagonalizable over $\mathbb{R}$. For $r$ odd, we have*

$$m(\Phi, 0) \geq \tfrac{r}{3} + \left[-\tfrac{1}{3}, 1, \tfrac{1}{3}\right]_r, \qquad m(\Phi_{\mathrm{sym}}, 0) \geq \tfrac{r}{6} + \left[-\tfrac{1}{6}, \tfrac{1}{2}, \tfrac{1}{6}\right]_r.$$

*For $r$ even, we have*

$$m(\Phi, 1) \geq \tfrac{r}{6} + \left[-1, \tfrac{2}{3}, \tfrac{1}{3}, 0, -\tfrac{1}{3}, \tfrac{4}{3}\right]_r \qquad m(\Phi_{\mathrm{sym}}, 1) \geq \tfrac{r}{12} + \left[-1, -\tfrac{1}{6}, -\tfrac{1}{3}, -\tfrac{1}{2}, -\tfrac{2}{3}, \tfrac{1}{6}\right]_r$$

$$m(\Phi, -1) \geq \tfrac{r}{6} + \left[0, -\tfrac{1}{3}, \tfrac{4}{3}, -1, \tfrac{2}{3}, \tfrac{1}{3}\right]_r \qquad m(\Phi_{\mathrm{sym}}, -1) \geq \tfrac{r}{12} + \left[0, -\tfrac{1}{6}, \tfrac{2}{3}, -\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}\right]_r$$

$$m(\Phi, 1) + m(\Phi, -1) \geq \tfrac{r}{3} + \left[-1, \tfrac{1}{3}, \tfrac{5}{3}\right]_r \quad m(\Phi_{\mathrm{sym}}, 1) + m(\Phi_{\mathrm{sym}}, -1) \geq \tfrac{r}{6} + \left[-1, -\tfrac{1}{3}, \tfrac{1}{3}\right]_r.$$

The parts of this theorem related to $\Phi_{\mathrm{sym}}$ were conjectured by Stanley, except that Stanley only conjectured diagonalizability over $\mathbb{C}$. Stanley did not consider $\Phi$, but we think it is natural to consider this operator both because it is easier to compute with and because it would occur naturally if one considered the Stern recurrence with asymmetric initial conditions. Computations show that the lower bounds are equalities for all $0 < r \leq 100$.

Theorem 2 has the following implications for shortening the recurrence obeyed by $S_n(f)$:

**Corollary 3.** *For $r$ odd, the sequence $S_n(f)$ obeys a homogenous linear recurrence of length $\leq \tfrac{r}{3} + [\tfrac{2}{3}, 0, \tfrac{1}{3}]_r$ for $n$ large. For $r$ even, the sequence $S_n(f)$ obeys a homogenous linear recurrence of length $\leq \tfrac{r}{3} + [4, \tfrac{10}{3}, \tfrac{8}{3}]_r$ for $n$ large. This can be shortened to length $\leq \tfrac{r}{3} + [2, \tfrac{4}{3}, \tfrac{2}{3}]_r$ if we allow recurrences of the form $S_n(f) = \sum_{j=1}^{\ell} a_j S_{n-j}(f) + b + c(-1)^n$.*

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# 2 Diagonalizability of $\Phi$ and $\Phi_{\mathrm{sym}}$

In this section, we will show that $\Phi$ on $V$ is diagonalizable over $\mathbb{R}$. Since $\Phi_{\mathrm{sym}}$ is the action of $\Phi$ on the quotient $V_{\mathrm{sym}}$, this also establishes diagonalizability of $\Phi_{\mathrm{sym}}$. Working explicitly in the basis $x^b y^{r-b}$ for $V$, we have

$$\sigma^*(x^b y^{r-b}) = \sum_a \binom{b}{a} x^a y^{r-a} \text{ and } \tau^*(x^b y^{r-b}) = \sum_a \binom{r-b}{r-a} x^a y^{r-a}$$

so, in this basis, the entries of the matrix $\Phi$ are given by

$$\Phi_{ab} = \binom{b}{a} + \binom{r-b}{r-a}.$$

**Example 4.** Returning to the example of cubic polynomials, we have

$$\Phi = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 \\ 3 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ 3 & 2 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & & 1 & 3 \\ & & & 1 \end{bmatrix}.$$

Conjugate $\Phi$ by the diagonal matrix $D$ with entries $D_{kk} = \sqrt{k!(r-k)!}$. We have

$$(D\Phi D^{-1})_{ab} = \sqrt{\frac{a!(r-a)!}{b!(r-b)!}} \cdot \begin{cases} \binom{b}{a}, & \text{for } a < b; \\ \binom{r-b}{r-a}, & \text{for } a > b; \end{cases}$$

$$= \frac{1}{|a-b|!} \sqrt{\frac{\max(a,b)! \max(r-a,r-b)!}{\min(a,b)! \min(r-a,r-b)!}}.$$

The last expression is symmetric in $a$ and $b$, so $D\Phi D^{-1}$ is a symmetric matrix and thus diagonalizable over $\mathbb{R}$. We conclude that $\Phi$ is likewise diagonalizable over $\mathbb{R}$.

# 3 The case of $r$ odd

In this section and the next, we will want to discuss complex eigenvalues of $\Phi$, so we require $V$ to be the vector space of degree $r$ polynomials with complex coefficients.

Throughout this section, we take $r$ to be odd. Set

$$\rho = \sigma \tau^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let $\omega$ be a primitive 6-th root of unity. Then $\rho$ acts on $\mathbb{C}^2$ with eigenvalues $\omega$ and $\omega^{-1}$. Thus $\rho^*$ acts on $V$ with eigenvalues $\omega^a(\omega^{-1})^{r-a}$ for $0 \le a \le r$ for $0 \le a \le r$; the corresponding

eigenvectors are $(x + \omega y)^a (\omega x + y)^{r-a}$. In particular, the $(-1)$-eigenspace of $\rho^*$ on $V$ corresponds to those values of $a$ for which $a \equiv r - a + 3 \pmod 6$ and has dimension equal to the number of such $a$, which is $\frac{r}{3} + [-\frac{1}{3}, 1, \frac{1}{3}]_r$.

Let $W$ be this $(-1)$-eigenspace of $\rho^*$. For $g \in W$, we have $\rho^* g = -g$, so $\sigma^* g = -\tau^* g$. Then $\Phi(g) = \sigma^* g + \tau^* g = 0$. So $\Phi$ acts by $0$ on $W$, and thus $m(\Phi, 0) \geq \frac{r}{3} + [-\frac{1}{3}, 1, \frac{1}{3}]_r$

We now consider the corresponding computation for $\Phi_{\text{sym}}$. Let $W_{\text{sym}}$ be the $(-1)$-eigenspace of $\Phi_{\text{sym}}$ on $V_{\text{sym}}$. In $\Phi_{\text{sym}}$, the eigenvectors $(x + \omega y)^a (\omega x + y)^{r-a}$ and $(x + \omega y)^{r-a} (\omega x + y)^a$ are identified, and no other relations are imposed. So the dimension of $W_{\text{sym}}$ is the number of $a$ for which $a \equiv r - a + 3 \pmod 6$, modulo the relation that $a$ and $r - a$ are equivalent; the number of such $a$ is $\frac{r}{6} + \left[-\frac{1}{6}, \frac{1}{2}, \frac{1}{6}\right]_r$. We have proved the multiplicity bound for $m(\Phi_{\text{sym}}, 0)$.

# 4 The case of $r$ even

Throughout this section, we take $r$ to be even. We continue to use the notation $\rho = \sigma\tau^{-1} = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and also introduce

$$\iota = \sigma\tau^{-1}\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of $\rho$ on $\mathbb{C}^2$ are the primitive sixth roots of unity and the eigenvalues of $\iota$ are $\pm i$. Thus, the eigenvalues of $\rho^*$ on $V$ are cube roots of unity (not necessarily primitive) and the eigenvalues of $\iota^*$ are $\pm 1$. We deduce that, on $V$, we have $(\rho^*)^3 = (\iota^*)^2 = \text{Id}$.

Let $X$ be the subspace of $V$ where $(\rho^*)^2 + (\rho^*) + 1 = 0$, and let $Y^+$ and $Y^-$ be the subspaces of $V$ where $\iota^*$ has eigenvalues $1$ and $-1$ respectively. We have

$$\dim X = \tfrac{2r}{3} + [0, \tfrac{2}{3}, \tfrac{4}{3}]_r \qquad \dim Y^+ = \tfrac{r}{2} + [1, 0]_r \qquad \dim Y^- = \tfrac{r}{2} + [0, 1]_r.$$

We deduce that

$$\dim(X \cap Y^+) \geq \tfrac{2r}{3} + [0, \tfrac{2}{3}, \tfrac{4}{3}]_r + \tfrac{r}{2} + [1, 0]_r - (r + 1) = \tfrac{r}{6} + [0, -\tfrac{1}{3}, \tfrac{4}{3}, -1, \tfrac{2}{3}, \tfrac{1}{3}]_r$$
$$\dim(X \cap Y^-) \geq \tfrac{2r}{3} + [0, \tfrac{2}{3}, \tfrac{4}{3}]_r + \tfrac{r}{2} + [0, 1]_r - (r + 1) = \tfrac{r}{6} + [-1, \tfrac{2}{3}, \tfrac{1}{3}, 0, -\tfrac{1}{3}, \tfrac{4}{3}]_r.$$

Adding these formulas,

$$\dim(X \cap Y^+) + \dim(X \cap Y^-) \geq \tfrac{r}{3} + [-1, \tfrac{1}{3}, \tfrac{5}{3}]_r.$$

For $f$ in $X$, we have

$$\Phi(f) = \sigma^*(f) + \tau^*(f) = \tau^* \left(\rho^* + 1\right)(f) = -\tau^*(\rho^*)^2(f) = -\sigma^*(\tau^{-1})^*\sigma^*(f) = -\iota^*(f).$$

So, if $f \in X \cap Y^\pm$, we have $\Phi(f) = \mp f$. This proves the claimed bounds for $m(\Phi, \pm 1)$.

We now repeat the analysis for $\Phi_{\text{sym}}$. We define $X_{\text{sym}}$ and $Y^\pm_{\text{sym}}$ analogously. A basis for $X$ is those polynomials $(x + \omega y)^a (\omega x + y)^{r-a}$ for which $2a \not\equiv r \pmod 3$. In $X_{\text{sym}}$ the

polynomials $(x + \omega y)^a(\omega x + y)^{r-a}$ and $(x + \omega y)^{r-a}(\omega x + y)^r$ are identified, which cuts the dimension exactly in half. (Note that $(x + \omega y)^{r/2}(\omega x + y)^{r/2}$ has eigenvalue 1, so it doesn't contribute to $X_{\text{sym}}$.) So

$$\dim X_{\text{sym}} = \tfrac{r}{3} + [0, \tfrac{1}{3}, \tfrac{2}{3}]_r.$$

Similar but simpler analysis shows that

$$\dim Y^+_{\text{sym}} = \tfrac{r}{4} + [1, \tfrac{1}{2}]_r \qquad \dim Y^-_{\text{sym}} = \tfrac{r}{4} + [0, \tfrac{1}{2}]_r.$$

We have

$$\dim(X_{\text{sym}} \cap Y^+_{\text{sym}}) \geq \tfrac{r}{3} + [0, \tfrac{1}{3}, \tfrac{2}{3}]_r + \tfrac{r}{4} + [1, \tfrac{1}{2}]_r - (\tfrac{r}{2} + 1) = \tfrac{r}{12} + [0, -\tfrac{1}{6}, \tfrac{2}{3}, -\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}]_r$$

$$\dim(X_{\text{sym}} \cap Y^-_{\text{sym}}) \geq \tfrac{r}{3} + [0, \tfrac{1}{3}, \tfrac{2}{3}]_r + \tfrac{r}{4} + [0, \tfrac{1}{2}]_r - (\tfrac{r}{2} + 1) = \tfrac{r}{12} + [-1, -\tfrac{1}{6}, -\tfrac{1}{3}, -\tfrac{1}{2}, -\tfrac{2}{3}, \tfrac{1}{6}]_r$$

Adding these,

$$m(\Phi_{\text{sym}}, 1) + m(\Phi_{\text{sym}}, -1) \geq \tfrac{r}{6} + \left[-1, -\tfrac{1}{3}, \tfrac{1}{3}\right]_r.$$

We have now deduced all claims about $m(\Phi_{\text{sym}}, \pm 1)$.

# 5 Consequences for the length of recurrences

First, we consider the case of $r$ odd. The matrix $\Phi_{\text{sym}}$ is square of size $\frac{r+1}{2}$; let the characteristic polynomial of $\Phi_{\text{sym}}$ be $x^{(r+1)/2} - \sum_{j=1}^{(r+1)/2} a_j x^{(r+1)/2-j}$. So we have

$$\Phi^n_{\text{sym}} = \sum_{j=1}^{(r+1)/2} a_j \Phi^{n-j}_{\text{sym}}$$

. But the last $m(\Phi_{\text{sym}}, 0)$ terms of the sum are 0, so the sum actually only runs up to $j = \frac{r+1}{2} - m(\Phi_{\text{sym}}, 0) \geq \tfrac{r}{3} + [\tfrac{2}{3}, 0, \tfrac{1}{3}]_r$. For any degree $r$ polynomial $f$, the sequence $S_n(f)$ is a linear function of $\Phi^n_{\text{sym}}$, so $S_n(f)$ obeys a homogenous linear recurrence of the same length.

We now consider the case of $r$ even. Write the characteristic polynomial of $\Phi_{\text{sym}}$ in the form $g(x)(x-1)^{m(\Phi_{\text{sym}}, 1)}(x+1)^{m(\Phi_{\text{sym}}, 1)}$. Then $\Phi_{\text{sym}}$ obeys the polynomial $g(x)(x-1)(x+1)$ with degree $\tfrac{r}{2} + 1 - (m(\Phi_{\text{sym}}, 1) + m(\Phi_{\text{sym}}, -1) - 2) = \tfrac{r}{3} + [4, \tfrac{10}{3}, \tfrac{8}{3}]_r$. Hence $\Phi^n_{\text{sym}}$ obeys a homogenous linear recurrence of this length, and $S_n(f)$ does as well.

Moreover, we can change bases to put $\Phi_{\text{sym}}$ into block form $\begin{bmatrix} M & 0 & 0 \\ 0 & \text{Id} & 0 \\ 0 & 0 & -\text{Id} \end{bmatrix}$, so $g(M) = 0$ and let $g(x) = \sum g_j x^{\deg(g)-j}$. Then $\sum g_j \Phi^{n-j}_{\text{sym}}$ will have block form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & p\text{Id} & 0 \\ 0 & 0 & q(-1)^n \end{bmatrix}$ and hence $\sum g_j S_{n-j}(f)$ will be of the form $b + c(-1)^n$ for some constants $b$ and $c$. So $S_n(f)$ obeys a recurrence of the form $S_n(f) = \sum_{j=1}^{\ell} a_j S_{n-j}(f) + b + c(-1)^n$ for $\ell = \deg(g)$.

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# 6 Acknowledgments

# References

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(Concerned with sequence A337277.)

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