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The Generalization of Faulhaber's Formula to Sums of Arbitrary Complex Powers

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Abstract

In this paper we present a generalization of Faulhaber's formula to sums of arbitrary complex powers $m \in \mathbb{C}$. These summation formulas for sums of the form $\sum_{k=1}^{\lfloor x \rfloor} k^m$ and $\sum_{k=1}^{n} k^m$, where $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$, are based on a series acceleration involving Stirling numbers of the first kind. While it is well-known that the corresponding expressions obtained from the Euler-Maclaurin summation formula diverge, our summation formulas are all very rapidly convergent.

1 Introduction

For two natural numbers $m, n \in \mathbb{N}_0$, the Faulhaber formula [1], given by

$$\sum_{k=0}^{n} k^m = \frac{1}{m+1} \sum_{k=0}^{m} (-1)^k \binom{m+1}{k} B_k n^{m-k+1}, \qquad (1)$$

where the B_k's are the Bernoulli numbers, provides a very efficient way to compute the sum of the m-th powers of the first n natural numbers. This formula was found by Jacob Bernoulli around 1700 and was first proved by Carl Gustav Jacobi in 1834.

We prove a rapidly convergent exact generalization of Faulhaber's formula to finite sums of the form $\sum_{k=1}^{\lfloor x \rfloor} k^m$ and $\sum_{k=1}^{n} k^m$ for all exponents $m \in \mathbb{C}$. Our key tool is the so-called

Weniger transformation [2, (4.1)] found by J. Weniger, transforming an inverse power series into an inverse factorial series [2, (1.1)]. This transformation of inverse power series was first found by Oskar Schlömilch around 1850 [3, 4, 5] based on earlier works of James Stirling in 1730 [6].

In an expanded form, one of our summation formulas for the sum $\sum_{k=1}^{n} \sqrt{k}$, where $n \in \mathbb{N}$, looks like

$$\begin{aligned}\sum_{k=1}^{n} \sqrt{k} &= \frac{2}{3}n^{3/2} + \frac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \sqrt{n}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(2l-3)!!}{2^l(l+1)!}B_{l+1}S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)} \\ &= \frac{2}{3}n^{3/2} + \frac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \frac{\sqrt{n}}{24(n+1)} + \frac{\sqrt{n}}{24(n+1)(n+2)} \\ &\quad + \frac{53\sqrt{n}}{640(n+1)(n+2)(n+3)} + \frac{79\sqrt{n}}{320(n+1)(n+2)(n+3)(n+4)} + \cdots ,\end{aligned} \tag{2}$$

where the B_l's are the Bernoulli numbers and $S_k^{(1)}(l)$ denotes the Stirling numbers of the first kind.

The above identity (2) is deduced by setting the variable $x := n \in \mathbb{N}$ into the more general formula

$$\begin{aligned}&\sum_{k=1}^{\lfloor x \rfloor} \sqrt{k} \\ &= \frac{2}{3}x^{3/2} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) - \sqrt{x}B_1(\{x\}) + \sqrt{x}\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}(-1)^l\frac{(2l-3)!!}{2^l(l+1)!}S_k^{(1)}(l)B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)} \\ &= \frac{2}{3}x^{3/2} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \left(\frac{1}{2} - \{x\}\right)\sqrt{x} + \frac{\left(\frac{1}{4}\{x\}^2 - \frac{1}{4}\{x\} + \frac{1}{24}\right)\sqrt{x}}{(x+1)} \\ &\quad + \frac{\left(\frac{1}{24}\{x\}^3 + \frac{3}{16}\{x\}^2 - \frac{11}{48}\{x\} + \frac{1}{24}\right)\sqrt{x}}{(x+1)(x+2)} + \frac{\left(\frac{1}{64}\{x\}^4 + \frac{3}{32}\{x\}^3 + \frac{21}{64}\{x\}^2 - \frac{7}{16}\{x\} + \frac{53}{640}\right)\sqrt{x}}{(x+1)(x+2)(x+3)} \\ &\quad + \frac{\left(\frac{1}{128}\{x\}^5 + \frac{19}{256}\{x\}^4 + \frac{109}{384}\{x\}^3 + \frac{29}{32}\{x\}^2 - \frac{977}{768}\{x\} + \frac{79}{320}\right)\sqrt{x}}{(x+1)(x+2)(x+3)(x+4)} + \cdots ,\end{aligned} \tag{3}$$

where this time the $B_l(\{x\})$'s are the fractional Bernoulli polynomials and $x \in \mathbb{R}^+$ is a positive real number. All other formulas in this article have a similar shape, when we expand them.

We have searched our resulting formulas in the literature and on the internet. We could find only two of them, namely equation (46) and its analogues for the sums $\sum_{k=1}^{n} \frac{1}{k^m}$ with $m \in \mathbb{N}_{\geq 2}$, which were already known to Stirling in 1730 [3, 7], and equation (45), which was obtained by Gregorio Fontana around 1780 [3, 8]. Both of these formulas were originally found in another form without the use of Bernoulli numbers.

We believe that all other generalized Faulhaber formulas presented in this article are new and that our method to obtain them has not been recognized before.

2 Definitions and basic facts

As usual, we denote the floor of x by $\lfloor x \rfloor$ and the fractional part of x by $\{x\}$. The symbol $\mathbb{N} := \{1, 2, 3, 4, \ldots\}$ denotes the set of natural numbers and $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ represents the set of positive real numbers. We also set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. Moreover, we define $\mathbb{H}^+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and let $\zeta(s)$ denote the Riemann zeta function at the point $s \in \mathbb{C} \setminus \{1\}$. For a complex number $z = re^{i\varphi} \in \mathbb{C}$, we let $|z| = r \in \mathbb{R}_0^+$ denote its absolute value and by $\varphi = \arg(z) \in (-\pi, \pi]$ its argument or phase. We define for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ the secant function by $\sec(\theta) := \frac{1}{\cos(\theta)}$. For $z \in \mathbb{C}$, the notation $z \to \infty$ means that $|z| \to \infty$.

The double factorial function for $n \in \mathbb{N}_0$ is defined by $n!! := \prod_{k=0}^{\lfloor (n-1)/2 \rfloor} (n - 2k)$.

Definition 1. (Pochhammer symbol [2, p. 1429]) We define the *Pochhammer symbol* (or rising factorial function) $(z)_k$ by

$$(z)_k := z(z+1)(z+2)(z+3)\cdots(z+k-1) = \frac{\Gamma(z+k)}{\Gamma(z)}, \tag{4}$$

where $\Gamma(z)$ is the gamma function [9, (5.2.1), p. 136] defined as the meromorphic continuation of the integral

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt \ \text{ for all } z \in \mathbb{C} \text{ with } \mathrm{Re}(z) > 0 \tag{5}$$

to the whole complex plane $\mathbb{C}$.

Definition 2. (Stirling numbers of the first kind [2, (A.2), p. 1437] and [10, A008275]) Let $k, l \in \mathbb{N}_0$ be two non-negative integers such that $k \geq l \geq 0$. We set the *Stirling numbers of the first kind* $S_k^{(1)}(l)$ as the connecting coefficients in the identity

$$(z)_k = (-1)^k \sum_{l=0}^{k} (-1)^l S_k^{(1)}(l) z^l, \tag{6}$$

where $(z)_k$ is the rising factorial function. Furthermore, we set $S_k^{(1)}(l) = 0$ if $k, l \in \mathbb{N}_0$ with $l > k$.

Definition 3. (Binomial coefficients [11]) We introduce the *binomial coefficient* $\binom{z}{s}$ for all $z \in \mathbb{C}$ and $s \in \mathbb{C}$ by [11, (5) and (11), pp. 8–9]

$$\binom{z}{s} := \frac{\Gamma(z+1)}{\Gamma(s+1)\Gamma(z-s+1)}. \tag{7}$$

Moreover, for $z \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$ we have the following asymptotic expansion as $k \to \infty$ [11, (18), p. 2 and p. 35]

$$\binom{z}{k} = \frac{(-1)^k}{\Gamma(-z)k^{z+1}} + O\left(\frac{1}{k^{z+2}}\right) \text{ for all } z \in \mathbb{C} \text{ and } k \in \mathbb{N}. \tag{8}$$

Definition 4. (Bernoulli polynomials and Bernoulli numbers [1, 12, 13, 14]) We define for $n \in \mathbb{N}_0$ the *n-th Bernoulli polynomial* $B_n(x)$ via the following exponential generating function [1] as

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \;\; \forall t \in \mathbb{C} \text{ with } |t| < 2\pi. \tag{9}$$

We also define the *n-th Bernoulli number* B_n as the value of the n-th Bernoulli polynomial $B_n(x)$ at the point $x = 0$, that is

$$B_n := B_n(0). \tag{10}$$

For all $n \in \mathbb{N}_0$ we have the explicit formula [12, Proposition 23.2, p. 86]

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{11}$$

For all $0 \le y \le 1$ it is valid that [13, Corollary B.4, (B.21), p. 500]

$$|B_1(y)| \le \frac{1}{2} \text{ and that } |B_n(y)| \le \frac{2\zeta(n)n!}{(2\pi)^n} \text{ for all } n \in \mathbb{N}_{\ge 2}. \tag{12}$$

We have [14, (1.10), p. 282]

$$(-1)^k B_k(1-y) = B_k(y) \text{ for all } k \in \mathbb{N}_0 \text{ and } 0 \le y \le 1. \tag{13}$$

Definition 5. (Digamma function [9, pp. 136–138]) We set the *digamma function* $\psi(z)$ to

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \text{ for all } z \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}. \tag{14}$$

Therefore, $\psi(z)$ is an analytic function for all $z \in \mathbb{C} \setminus (-\infty, 0]$. For all $z \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$, we have the identity [9, (5.5.2), p. 138]

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{15}$$

and for all $n \in \mathbb{N}$ we have the formula [9, (5.4.14), p. 137]

$$\sum_{k=1}^{n} \frac{1}{k} = \psi(n+1) + \gamma. \tag{16}$$

Definition 6. (Hurwitz zeta function [9, p. 607]) We define the *Hurwitz zeta function* $\zeta(s,z)$ for all complex numbers $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and all $z \in \mathbb{C} \setminus \{0,-1,-2,-3,\ldots\}$ by

$$\zeta(s,z) := \sum_{k=0}^{\infty} \frac{1}{(k+z)^s}. \tag{17}$$

The function $\zeta(s,z)$ extends to an analytic function on $\mathbb{C} \setminus \{0,-1,-2,-3,\ldots\}$ in the z-variable and for every $z \notin \{0,-1,-2,-3,\ldots\}$ to a meromorphic function in $s \in \mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$. For all $s \in \mathbb{C} \setminus \{1\}$ and all $z \in \mathbb{C} \setminus \{0,-1,-2,-3,\ldots\}$ it satisfies the identity

$$\zeta(s,z+1) = \zeta(s,z) - \frac{1}{z^s}. \tag{18}$$

For all $m \in \mathbb{C} \setminus \{1\}$ and all $n \in \mathbb{N}$ we have the formula [15, (1.2), p. 2]

$$\sum_{k=1}^{n} k^m = \zeta(-m) - \zeta(-m, n+1). \tag{19}$$

3 The structure of inverse factorial series expansions

In this section we study the structure of inverse factorial series expansions for analytic functions possessing an asymptotic series expansion by applying a theorem of G. N. Watson [16, Theorem 2, p. 45]. The main result of this section is Theorem 10, from which we later deduce convergent inverse factorial series expansions for the functions $\zeta(s, z+1-y)$ and $\psi(z+1-y)$, where $0 \leq y \leq 1$.

For this procedure, we need the following variant of a result found by J. Weniger [2, (4.1), p. 1433].

Lemma 7. *(finite Weniger transformation [2]) For every finite inverse power series* $\sum_{k=1}^{n} \frac{a_k}{z^k}$*, where the* a_k*'s are any complex numbers and* $n \in \mathbb{N}$*, the following transformation formula holds:*

$$\sum_{k=1}^{n} \frac{a_k}{z^k} = \sum_{k=1}^{n} \frac{(-1)^k \sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) a_l}{(z+1)(z+2)\cdots(z+k)} + O\left(\frac{1}{z^{n+1}}\right) \quad \text{as } z \to \infty. \tag{20}$$

Moreover, we have

$$\sum_{k=1}^{n} \frac{a_k}{z^k} = \sum_{k=1}^{\infty} \frac{(-1)^k \sum_{l=1}^{n} (-1)^l S_k^{(1)}(l) a_l}{(z+1)(z+2)\cdots(z+k)}. \tag{21}$$

Proof. For $l \in \mathbb{N}$ we have ([2, (A.14), p. 1438], [17, (6), p. 78])

$$\begin{aligned}
\frac{1}{z^l} &= \sum_{k=0}^{\infty} \frac{(-1)^k S_{k+l}^{(1)}(l)}{(z+1)(z+2)\cdots(z+k+l)} \\
&= \sum_{k=l}^{\infty} \frac{(-1)^{k-l} S_k^{(1)}(l)}{(z+1)(z+2)\cdots(z+k)} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-l} S_k^{(1)}(l)}{(z+1)(z+2)\cdots(z+k)} \\
&= \sum_{k=1}^{n} \frac{(-1)^{k-l} S_k^{(1)}(l)}{(z+1)(z+2)\cdots(z+k)} + O\left(\frac{1}{z^{n+1}}\right) \quad \text{as } z \to \infty,
\end{aligned}$$

where we used in the third step that $S_k^{(1)}(l) = 0$ for $k < l$.

Therefore, we obtain that

$$\begin{aligned}
\sum_{k=1}^{n} \frac{a_k}{z^k} = \sum_{l=1}^{n} \frac{a_l}{z^l} &= \sum_{l=1}^{n} a_l \sum_{k=1}^{n} \frac{(-1)^{k-l} S_k^{(1)}(l)}{(z+1)(z+2)\cdots(z+k)} + O\left(\frac{1}{z^{n+1}}\right) \\
&= \sum_{k=1}^{n} \frac{(-1)^k \sum_{l=1}^{n} (-1)^l S_k^{(1)}(l) a_l}{(z+1)(z+2)\cdots(z+k)} + O\left(\frac{1}{z^{n+1}}\right) \\
&= \sum_{k=1}^{n} \frac{(-1)^k \sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) a_l}{(z+1)(z+2)\cdots(z+k)} + O\left(\frac{1}{z^{n+1}}\right) \quad \text{as } z \to \infty,
\end{aligned}$$

which is the first claimed formula (20).

The second formula (21) follows from the calculation

$$\begin{aligned}
\sum_{k=1}^{n} \frac{a_k}{z^k} = \sum_{l=1}^{n} \frac{a_l}{z^l} &= \sum_{l=1}^{n} a_l \sum_{k=1}^{\infty} \frac{(-1)^{k-l} S_k^{(1)}(l)}{(z+1)(z+2)\cdots(z+k)} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k \sum_{l=1}^{n} (-1)^l S_k^{(1)}(l) a_l}{(z+1)(z+2)\cdots(z+k)},
\end{aligned}$$

because we can always interchange a finite summation with an infinite summation. □

Lemma 8. *(Uniqueness of inverse factorial series expansions) If for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ a function f has the absolutely convergent series expansion*

$$f(z) = \sum_{k=1}^{\infty} \frac{b_k}{(z+1)(z+2)\cdots(z+k)}$$

and the asymptotic expansion

$$f(z) = \sum_{k=1}^{n} \frac{c_k}{(z+1)(z+2)\cdots(z+k)} + O\left(\frac{1}{z^{n+1}}\right) \quad \text{as } z \to \infty,$$

then we have $c_k = b_k$ for all $k \in \mathbb{N}$ and the absolutely convergent series expansion

$$f(z) = \sum_{k=1}^{\infty} \frac{c_k}{(z+1)(z+2)\cdots(z+k)}.$$

Proof. For all $n \in \mathbb{N}$ we have $\sum_{k=n+1}^{\infty} \frac{|b_k|\cdot|z|^n}{|(z+1)||(z+2)|\cdots|(z+k)|} \to 0$ as $z \to \infty$, because this infinite series is convergent for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ by assumption and consists of the monotone decreasing positive terms $\frac{|b_k|\cdot|z|^n}{|(z+1)||(z+2)|\cdots|(z+k)|} \to 0$ as $z \to \infty$.

From the above observation and the given absolutely convergent inverse factorial series expansion of $f(z)$, we deduce for all $n \in \mathbb{N}$ that

$$\begin{aligned}
&\left|\sum_{k=n}^{\infty} \frac{b_k}{(z+1)(z+2)\cdots(z+k)}\right| \\
&\leq \sum_{k=n}^{\infty} \frac{|b_k|}{|(z+1)||(z+2)|\cdots|(z+k)|} \\
&= \frac{|b_n|}{|(z+1)||(z+2)|\cdots|(z+n)|} + \frac{1}{|z|^n} \sum_{k=n+1}^{\infty} \frac{|b_k|\cdot|z|^n}{|(z+1)||(z+2)|\cdots|(z+k)|} \\
&= O\left(\frac{1}{z^n}\right) \quad \text{as } z \to \infty,
\end{aligned}$$

which means that $\lim_{z\to\infty}(f(z)) = 0$ and we have

$$z^m \sum_{k=n}^{\infty} \frac{b_k}{(z+1)(z+2)\cdots(z+k)} \longrightarrow 0 \ \text{ as } z \to \infty \text{ for all } m \in \{0, 1, 2, \ldots, n-1\}.$$

The result now follows by induction on $n \in \mathbb{N}$ via a repeated application of the above limit.

For $n = 1$, we get

$$f(z) = \frac{b_1}{z+1} + \sum_{k=2}^{\infty} \frac{b_k}{(z+1)(z+2)\cdots(z+k)} = \frac{c_1}{z+1} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty,$$

which implies by multiplying both sides with $z+1$ and letting $z \to \infty$ that $c_1 = b_1$.

Similarly, for $n = 2$, we get using $c_1 = b_1$ that

$$f(z) - \frac{c_1}{z+1} = \frac{b_2}{(z+1)(z+2)} + \sum_{k=3}^{\infty} \frac{b_k}{(z+1)(z+2)\cdots(z+k)} = \frac{c_2}{(z+1)(z+2)} + O\left(\frac{1}{z^3}\right),$$

which implies by multiplying both sides with $(z+1)(z+2)$ and letting $z \to \infty$ that $c_2 = b_2$.

In general, we can induct from $n-1$ to n using that $c_k = b_k$ for all $k \in \{1, 2, 3, \ldots, n-1\}$ by the identity

$$\begin{aligned}
&f(z) - \sum_{k=1}^{n-1} \frac{c_k}{(z+1)(z+2)\cdots(z+k)} \\
&= \frac{b_n}{(z+1)(z+2)\cdots(z+n)} + \sum_{k=n+1}^{\infty} \frac{b_k}{(z+1)(z+2)\cdots(z+k)} \\
&= \frac{c_n}{(z+1)(z+2)\cdots(z+n)} + O\left(\frac{1}{z^{n+1}}\right) \quad \text{as } z \to \infty,
\end{aligned}$$

again by multiplying both sides with $(z+1)(z+2)\cdots(z+n)$ and letting $z \to \infty$ to conclude that $c_k = b_k$ holds also for $k = n$. This proves that $c_k = b_k$ for all $k \in \mathbb{N}$. $\square$

The key to our generalized Faulhaber formulas is the following theorem.

Theorem 9. *(Watson's Transformation Theorem [16, Theorem 2, p. 45]) Let $f(z)$ be a function of $z \in \mathbb{C}$ which is analytic when $\mathrm{Re}(z) > 0$; and let $f(z)$ be also analytic in the region D of the complex plane defined by*

$$D := \left\{ z \in \mathbb{C} : |z| > \gamma \text{ and } |\arg(z)| \le \frac{\pi}{2} + \alpha + 3\delta \right\},$$

where $\gamma \ge 0$ is a finite number, $\alpha > 0$, $\delta > 0$ and $\alpha + 3\delta < \frac{\pi}{2}$.

In the region D let $f(z)$ possess the asymptotic expansion

$$f(z) = \sum_{k=0}^{n} \frac{a_k}{z^k} + R_n(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \frac{a_4}{z^4} + \cdots + \frac{a_n}{z^n} + R_n(z),$$

where

$$|a_n| < A\rho^n n! \quad \text{and} \quad |R_n(z) z^{n+1}| < B\sigma^n n!$$

with some constants A, B, ρ and σ, which are independent of n.

Let $M \le M_0$ be any positive real number, where M_0 is the largest positive root of the equation

$$e^{-\frac{2\cos(\alpha)}{\rho M_0}} - 2\cos\left(\frac{\sin(\alpha)}{\rho M_0}\right) \cdot e^{-\frac{\cos(\alpha)}{\rho M_0}} + 1 - p^2 = 0,$$

where

$$1 < p < 1 + e^{-\pi \cot(\alpha)}.$$

Then the function $f(z)$ can be expanded into the absolutely convergent series

$$f(z) = b_0 + \sum_{k=1}^{\infty} \frac{b_k}{(Mz+w+1)(Mz+w+2)\cdots(Mz+w+k)},$$

when $\mathrm{Re}(z) > 0$ and $w \in \mathbb{C}$ with $\mathrm{Re}(w) \ge 0$.

Proof. The proof of this theorem is given in Watson's paper [16]. □

Theorem 10. *(Structure of inverse factorial series expansions)*
Let $f(z)$ be a function of $z \in \mathbb{C}$ which is analytic when $\mathrm{Re}(z) > 0$*; and let $f(z)$ be also analytic in the region D of the complex plane defined by*

$$D := \{z \in \mathbb{C} : |z| > 0 \textit{ and } |\arg(z)| \leq \pi - \varepsilon, \textit{ where } \varepsilon > 0 \textit{ is arbitrarily small}\}.$$

In the region D let $f(z)$ possess the asymptotic expansion

$$f(z) = \sum_{k=1}^{n} \frac{a_k}{z^k} + R_n(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots + \frac{a_n}{z^n} + R_n(z),$$

where

$$|a_n| < A\rho^n n! \quad \textit{and} \quad |R_n(z)z^{n+1}| < B\sigma^n n!$$

with some constants A, B, $\rho < \frac{3}{\pi}$ and σ, which are independent of n.

Then the function $f(z)$ is equal to the absolutely convergent inverse factorial series

$$f(z) = \sum_{k=1}^{\infty} \frac{(-1)^k \sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) a_l}{(z+1)(z+2)\cdots(z+k)} \tag{22}$$

for $\mathrm{Re}(z) > 0$.

Proof. Let $f(z)$ and the region D be as described in the above Theorem 10. Because of the conditions on the function $f(z)$ and the region D, we can choose in Theorem 9 the variables $\gamma := 0$, $\alpha := \frac{\pi}{2} - 4\varepsilon$, $\delta := \varepsilon$ and $p := 1 + \varepsilon$ for $\varepsilon > 0$ arbitrarily small by [16, beginning of p. 85]. We have then that $M_0 = \frac{3}{\pi\rho} - \varepsilon$ for some arbitrarily small number $\varepsilon > 0$ and because $\rho < \frac{3}{\pi}$, we obtain that $M_0 > 1$. According to Watson's Theorem 9 with $M := 1 < M_0$, $w := 0$ and $a_0 = b_0 = 0$, we know that we can expand the function $f(z)$ into an absolutely convergent series of the form

$$f(z) = \sum_{k=1}^{\infty} \frac{b_k}{(z+1)(z+2)\cdots(z+k)}$$

for some constants $b_k \in \mathbb{C}$ and all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$.

On the other hand, we have by applying a finite Weniger transformation (20) to the asymptotic expansion of $f(z)$ that

$$f(z) = \sum_{k=1}^{n} \frac{(-1)^k \sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) a_l}{(z+1)(z+2)\cdots(z+k)} + O\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty$$

also holds.

Comparing the two expressions above for $f(z)$ by using Lemma 8 with

$$c_k := (-1)^k \sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) a_l,$$

we conclude that there is an absolutely convergent series

$$f(z) = \sum_{k=1}^{\infty} \frac{(-1)^k \sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) a_l}{(z+1)(z+2)\cdots(z+k)}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. □

4 The convergent inverse factorial series expansions for $\zeta(s, z+1-y)$ and $\psi(z+1-y)$

In this section, in Theorem 13 we deduce the convergent inverse factorial series expansions for the functions $\zeta(s, z+1-y)$ and $\psi(z+1-y)$, where $0 \leq y \leq 1$.

For this, we need the following lemma.

Lemma 11. *(Euler-Maclaurin summation formula [13, Theorem B.5, pp. 500–501]) Suppose that $n \in \mathbb{N}$ is a positive integer and that the function $f(t)$ has continuous derivatives through the n-th order on the interval $[a, b]$ where a and b are real numbers with $a < b$. Then we have*

$$\begin{aligned} \sum_{a<k\leq b} f(k) = \int_a^b f(t)dt + \sum_{k=1}^{n} (-1)^k \frac{B_k(\{b\})}{k!} f^{(k-1)}(b) - \sum_{k=1}^{n} (-1)^k \frac{B_k(\{a\})}{k!} f^{(k-1)}(a) \\ + \frac{(-1)^{n+1}}{n!} \int_a^b f^{(n)}(t) B_n(\{t\}) dt. \end{aligned} \tag{23}$$

Proof. The proof of this Lemma 11 is given in [13, p. 501]. □

From the above Lemma 11, it follows the next lemma.

Lemma 12. *(Asymptotic series expansions for $\zeta(s, z+h)$ and $\psi(z+h)$ with $0 \leq h \leq 1$) Let $n \in \mathbb{N}_0$ and let $s \in \mathbb{C} \setminus \{1\}$ such that $\mathrm{Re}(s) > -n$. We for $z \in \mathbb{C}$ with $|\arg(z)| < \pi$ and $0 \leq h \leq 1$ the asymptotic series expansions*

$$\zeta(s, z+h) = \frac{z^{1-s}}{s-1} + \frac{z^{1-s}}{s-1} \sum_{k=1}^{n} \binom{1-s}{k} \frac{B_k(h)}{z^k} + O_n(z) \tag{24}$$

with

$$|O_n(z)| = \left| \binom{1-s}{n+2} \frac{n+2}{s-1} \int_0^\infty \frac{B_{n+1}(\{x-h\}) - (-1)^{n+1} B_{n+1}(h)}{(x+z)^{n+s+1}} dx \right| \\ \leq \frac{2(n+2)}{|s-1|} \left| \binom{1-s}{n+2} \right| \frac{|B_{n+1}| \sec^{n+\mathrm{Re}(s)+1}\left(\frac{1}{2}\arg(z)\right)}{(n+\mathrm{Re}(s))|z|^{n+\mathrm{Re}(s)}} \max\left\{1, e^{\mathrm{Im}(s)\arg(z)}\right\} \tag{25}$$

and

$$\psi(z+h) = \log(z) - \sum_{k=1}^{n} \frac{(-1)^k B_k(h)}{kz^k} + U_n(z) \tag{26}$$

with

$$|U_n(z)| = \left| \int_0^\infty \frac{(-1)^{n+1} B_{n+1}(h) - B_{n+1}(\{x-h\})}{(x+z)^{n+2}} dx \right| \leq \frac{2\,|B_{n+1}| \sec^{n+2}\left(\frac{1}{2}\arg(z)\right)}{(n+1)|z|^{n+1}}. \tag{27}$$

Proof. Let $0 < h \leq 1$ and let $z \in \mathbb{C}$ with $|\arg(z)| < \pi$. Setting $a := -h$, $b := N$ and $f(x) := \frac{1}{(z+h+x)^s}$ with $\frac{d^n f(x)}{dx^n} = \frac{d^n}{dx^n}\left(\frac{1}{(z+h+x)^s}\right) = -\frac{(n+1)!}{s-1}\binom{1-s}{n+1}\frac{1}{(z+h+x)^{n+s}}$ into Lemma 11, we obtain for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ that

$$\zeta(s, z+h) = \sum_{k=0}^{\infty} \frac{1}{(z+h+k)^s} = \lim_{N\to\infty} \left(\sum_{-h<k\leq N} \frac{1}{(z+h+k)^s} \right)$$

$$= \int_{-h}^{\infty} \frac{1}{(z+h+x)^s} dx + \lim_{N\to\infty} \left(\sum_{k=1}^{n} (-1)^k \frac{B_k(\{N\})}{k!} \frac{d^{k-1}\left(\frac{1}{(z+h+x)^s}\right)}{dx^{k-1}} \Bigg|_{x=N} \right)$$

$$- \sum_{k=1}^{n} (-1)^k \frac{B_k(\{-h\})}{k!} \frac{d^{k-1}\left(\frac{1}{(z+h+x)^s}\right)}{dx^{k-1}} \Bigg|_{x=-h} + \frac{(-1)^{n+1}}{n!} \int_{-h}^{\infty} \frac{d^n\left(\frac{1}{(z+h+x)^s}\right)}{dx^n} \Bigg|_{x=t} B_n(\{t\}) dt$$

$$= \frac{z^{1-s}}{s-1} + \frac{z^{1-s}}{s-1} \sum_{k=1}^{n} (-1)^k \binom{1-s}{k} \frac{B_k(\{1-h\})}{z^k} + (-1)^n \binom{1-s}{n+1} \frac{n+1}{s-1} \int_0^\infty \frac{B_n(\{x-h\})}{(x+z)^{n+s}} dx.$$

Now replacing n by $n+1$, we get

$$\zeta(s, z+h) = \frac{z^{1-s}}{s-1} + \frac{z^{1-s}}{s-1} \sum_{k=1}^{n+1} (-1)^k \binom{1-s}{k} \frac{B_k(\{1-h\})}{z^k} \\ + (-1)^{n+1} \binom{1-s}{n+2} \frac{n+2}{s-1} \int_0^\infty \frac{B_{n+1}(\{x-h\})}{(x+z)^{n+s+1}} dx.$$

This is equivalent to

$$\zeta(s, z+h) = \frac{z^{1-s}}{s-1} + \frac{z^{1-s}}{s-1} \sum_{k=1}^{n} (-1)^k \binom{1-s}{k} \frac{B_k(1-h)}{z^k} \\ + (-1)^{n+1} \binom{1-s}{n+2} \frac{n+2}{s-1} \int_0^\infty \frac{B_{n+1}(\{x-h\}) - B_{n+1}(1-h)}{(x+z)^{n+s+1}} dx.$$

We use the relation (13) and deduce equation (24), which extends $\zeta(s, z+h)$ analytically to the whole punctured complex s-plane $\mathbb{C} \setminus \{1\}$. Therefore, the equation (24) is also true for all $s \in \mathbb{C} \setminus \{1\}$. By using the identity (18), we see that the formula (24) is also true for $h = 0$. The bound (25) for the error term $O_n(z)$ follows from [14, p. 294] and [15, p. 6]. This proves the first part about $\zeta(s, z+h)$ of the above Lemma 12.

Now, we prove the second part for $\psi(z+h)$. For $z \in \mathbb{C}$ with $|\arg(z)| < \pi$, $n \geq 2$ and $0 \leq h \leq 1$ we have the following series expansion [14, Ex. 4.4, p. 295]:

$$\ln\left(\Gamma(z+h)\right) = \left(z + h - \frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \sum_{k=2}^{n} \frac{(-1)^k B_k(h)}{k(k-1)z^{k-1}} - \frac{1}{n}\int_0^\infty \frac{B_n\left(\{x-h\}\right)}{(x+z)^n}dx.$$

Replacing n by $n+1$ again, we get

$$\begin{aligned}
\ln\left(\Gamma(z+h)\right) &= \left(z + h - \frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \sum_{k=2}^{n+1} \frac{(-1)^k B_k(h)}{k(k-1)z^{k-1}} \\
&\quad - \frac{1}{n+1}\int_0^\infty \frac{B_{n+1}\left(\{x-h\}\right)}{(x+z)^{n+1}}dx \\
&= \left(z + h - \frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \sum_{k=2}^{n} \frac{(-1)^k B_k(h)}{k(k-1)z^{k-1}} \\
&\quad + \frac{1}{n+1}\int_0^\infty \frac{(-1)^{n+1}B_{n+1}(h) - B_{n+1}\left(\{x-h\}\right)}{(x+z)^{n+1}}dx.
\end{aligned}$$

Differentiating this identity with respect to the variable z, we get equation (26). The estimate (27) for the error term $U_n(z)$ follows from [14, p. 294 and Ex. 4.2, p. 295]. $\square$

We get the following theorem.

Theorem 13. *(Inverse factorial series expansions for $\zeta(s, z+1-y)$ and $\psi(z+1-y)$) Let $0 \leq y \leq 1$ and let $s \in \mathbb{C} \setminus \{1\}$. For $z \in \mathbb{H}^+$ and all $a \in \mathbb{N}_0$ we have the absolutely convergent inverse factorial series expansions*

$$\begin{aligned}
\zeta(s, z+1-y) &= \frac{z^{1-s}}{s-1} + \frac{z^{1-s}}{s-1}\sum_{k=1}^{a}(-1)^k \binom{1-s}{k}\frac{B_k(y)}{z^k} \\
&\quad + \frac{z^{1-s-a}}{s-1}\sum_{k=1}^{\infty}(-1)^{k+a}\frac{\sum_{l=1}^{k}\binom{1-s}{l+a}S_k^{(1)}(l)B_{l+a}(y)}{(z+1)(z+2)\cdots(z+k)}
\end{aligned} \tag{28}$$

and

$$\psi(z+1-y) = \log(z) - \sum_{k=1}^{a}\frac{B_k(y)}{kz^k} + \frac{1}{z^a}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l}{l+a}S_k^{(1)}(l)B_{l+a}(y)}{(z+1)(z+2)\cdots(z+k)}. \tag{29}$$

Proof. Let $s \in \mathbb{C} \setminus \{1, 0, -1, -2, -3, \ldots\}$ be a fixed complex number and let $z \in \mathbb{C} \setminus (-\infty, 0]$ with $|\arg(z)| \leq \pi - \varepsilon < \pi$ for some arbitrarily small, but fixed $\varepsilon > 0$. Setting $h := 1 - y$ for $0 \leq y \leq 1$ into the identities (24) and (26), by using the relation (13) and by exchanging n with $n + a$ we deduce that

$$\zeta(s, z + 1 - y) = \frac{z^{1-s}}{s-1} + \frac{z^{1-s}}{s-1} \sum_{k=1}^{n+a} (-1)^k \binom{1-s}{k} \frac{B_k(y)}{z^k} + O_{n+a}(z) \tag{30}$$

and that

$$\psi(z + 1 - y) = \log(z) - \sum_{k=1}^{n+a} \frac{B_k(y)}{k z^k} + U_{n+a}(z), \tag{31}$$

where $O_{n+a}(z)$ and $U_{n+a}(z)$ are as in the previous Lemma 12 with $h := 1 - y$.

We can write the equations (30) and (31) in the form

$$\begin{aligned} \zeta(s, z + 1 - y) = {} & \frac{z^{1-s}}{s-1} + \frac{z^{1-s}}{s-1} \sum_{k=1}^{a} (-1)^k \binom{1-s}{k} \frac{B_k(y)}{z^k} \\ & + \frac{z^{1-s-a}}{s-1} \sum_{k=1}^{n} (-1)^{k+a} \binom{1-s}{k+a} \frac{B_{k+a}(y)}{z^k} + O_{n+a}(z) \end{aligned} \tag{32}$$

and

$$\psi(z + 1 - y) = \log(z) - \sum_{k=1}^{a} \frac{B_k(y)}{k z^k} - \frac{1}{z^a} \sum_{k=1}^{n} \frac{B_{k+a}(y)}{(k+a) z^k} + U_{n+a}(z). \tag{33}$$

In the following calculations, we use the fact that the function $g(k) := 2^k$ grows faster than any polynomial $p(k)$ as $k \to \infty$.

From equation (32) we get (28) for $s \in \mathbb{C} \setminus \{1, 0, -1, -2, -3, \ldots\}$ by applying Theorem 10 with $R_n(z) := (s-1) z^{s+a-1} O_{n+a}(z)$ to the analytic function $f_1(z)$ defined by

$$\begin{aligned} f_1(z) : &= (s-1) z^{s+a-1} \left[\zeta(s, z + 1 - y) - \frac{z^{1-s}}{s-1} - \frac{z^{1-s}}{s-1} \sum_{k=1}^{a} (-1)^k \binom{1-s}{k} \frac{B_k(y)}{z^k} \right] \\ &= \sum_{k=1}^{n} (-1)^{k+a} \binom{1-s}{k+a} \frac{B_{k+a}(y)}{z^k} + (s-1) z^{s+a-1} O_{n+a}(z) \end{aligned}$$

on $z \in \mathbb{C} \setminus (-\infty, 0]$ with $|\arg(z)| \leq \pi - \varepsilon < \pi$. Because of the relation $|x^s| = x^{\mathrm{Re}(s)}$ for $x \in \mathbb{R}_0^+$ and the use of the identity (8), we have

$$\begin{aligned} \left| \binom{1-s}{k+a} B_{k+a}(y) \right| &\leq \frac{2\zeta(k+a)(k+a)!(k+a)^{\mathrm{Re}(s)-2}}{(2\pi)^{k+a} \, |\Gamma(s-1)|} + O\left(\frac{2\zeta(k+a)(k+a)!(k+a)^{\mathrm{Re}(s)-3}}{(2\pi)^{k+a}} \right) \\ &< \frac{C_1(a) k!}{\pi^k} \end{aligned}$$

and with $A_1 := \max\left\{1, e^{\operatorname{Im}(s)\arg(z)}\right\}$, $A_2 := \max\left\{1, e^{-\operatorname{Im}(s)\arg(z)}\right\}$, as well as $\operatorname{Re}(s) > -n$, we get that

$$\begin{aligned}
&\left|(s-1)z^{s+a-1}O_{n+a}(z)\right| \\
&\le 2(n+a+2)A_1 \left|\binom{1-s}{n+a+2}\right| \frac{|B_{n+a+1}|\, e^{-\operatorname{Im}(s)\arg(z)} \sec^{n+a+\operatorname{Re}(s)+1}\left(\frac{1}{2}\arg(z)\right)}{|n+a+\operatorname{Re}(s)| \cdot |z|^{n+1}} \\
&\le \frac{4(n+a+2)A_2}{|n+a+\operatorname{Re}(s)|} \cdot \frac{\zeta(n+a+1)(n+a+1)!(n+a+2)^{\operatorname{Re}(s)-2} \sec^{n+a+\operatorname{Re}(s)+1}\left(\frac{1}{2}\arg(z)\right)}{(2\pi)^{n+a+1}\left|\Gamma(s-1)\right| |z|^{n+1}} \\
&\quad + O\left(\frac{4(n+a+2)A_2}{|n+a+\operatorname{Re}(s)|} \cdot \frac{\zeta(n+a+1)(n+a+1)!(n+a+2)^{\operatorname{Re}(s)-3} \sec^{n+a+\operatorname{Re}(s)+1}\left(\frac{1}{2}\arg(z)\right)}{(2\pi)^{n+a+1}|z|^{n+1}}\right) \\
&< \frac{C_2(a)\sec^n\left(\frac{1}{2}\arg(z)\right) n!}{\pi^n |z|^{n+1}}
\end{aligned}$$

for some positive constants $C_1(a)$, $C_2(a)$ depending on a and independent of n. In the last computation above, we have used the relation $|z^s| = |z|^{\operatorname{Re}(s)} e^{-\operatorname{Im}(s)\arg(z)}$.

The above bound for $|(s-1)z^{s+a-1}O_{n+a}(z)|$ also holds if $\operatorname{Re}(s) \le -n$ by taking $C_2(a)$ large enough, because $\operatorname{Re}(s) \le -n$ is only possible for finitely many n's and in each case we have $|(s-1)z^{s+a-1}O_{n+a}(z)| \le \frac{C(n)}{|z|^{n+1}}$ for all $n \in \mathbb{N}$ and some positive constants $C(n)$.

To get formula (28) also for all $s \in \{0, -1, -2, -3, \ldots\}$, we apply the Weniger transformation formula (21) directly to the function $f_1(z)$ with $n := 1-s-a$ and $O_{n+a}(z) = O_{1-s}(z) = 0$.

Similarly, from equation (33) we obtain the formula (29) by applying Theorem 10 with $R_n(z) := z^a U_{n+a}(z)$ to the analytic function $f_2(z)$ defined by

$$f_2(z) := z^a \left[\log(z) - \psi(z+1-y) - \sum_{k=1}^{a} \frac{B_k(y)}{kz^k}\right] = \sum_{k=1}^{n} \frac{B_{k+a}(y)}{(k+a)z^k} + z^a U_{n+a}(z)$$

on $z \in \mathbb{C}$ with $|\arg(z)| \le \pi - \varepsilon < \pi$, because we have

$$\left|\frac{B_{k+a}(y)}{k+a}\right| \le \frac{2\zeta(k+a)(k+a)!}{(2\pi)^{k+a}(k+a)} < \frac{C_3(a)k!}{\pi^k}$$

and

$$\begin{aligned}
|z^a U_{n+a}(z)| &\le \frac{2\,|B_{n+a+1}| \sec^{n+a+2}\left(\frac{1}{2}\arg(z)\right)}{(n+a+1)|z|^{n+1}} \le \frac{4\zeta(n+a+1)\sec^{n+a+2}\left(\frac{1}{2}\arg(z)\right)(n+a+1)!}{(2\pi)^{n+a+1}(n+a+1)|z|^{n+1}} \\
&< \frac{C_4(a)\sec^n\left(\frac{1}{2}\arg(z)\right) n!}{\pi^n |z|^{n+1}}
\end{aligned}$$

for some positive constants $C_3(a)$, $C_4(a)$ depending on a and independent of n. $\square$

5 The generalized Faulhaber formulas

In this section we prove our generalized versions of Faulhaber's formula, which all converge very rapidly. For their proofs, we use the above Theorem 13.

Theorem 14. *(extended generalized Faulhaber formulas) For every complex number $m \in \mathbb{C} \setminus \{-1\}$ and every positive real number $x \in \mathbb{R}^+$, we have*

$$\sum_{k=1}^{\lfloor x \rfloor} k^m = \frac{1}{m+1} x^{m+1} + \zeta(-m) + \frac{x^{m+1}}{m+1} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \binom{m+1}{l} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}. \tag{34}$$

More generally, for every $x \in \mathbb{R}^+$ and every $a \in \mathbb{N}_0$, we have

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} k^m = {} & \frac{1}{m+1} x^{m+1} + \zeta(-m) + \frac{1}{m+1} \sum_{k=1}^{a} (-1)^k \binom{m+1}{k} B_k(\{x\}) x^{m-k+1} \\ & + \frac{x^{m-a+1}}{m+1} \sum_{k=1}^{\infty} (-1)^{k+a} \frac{\sum_{l=1}^{k} \binom{m+1}{l+a} S_k^{(1)}(l) B_{l+a}(\{x\})}{(x+1)(x+2)\cdots(x+k)} \end{aligned} \tag{35}$$

and for $m = m_1 + im_2 \in \mathbb{C} \setminus \{-1\}$ with $m_1 = \mathrm{Re}(m) \geq -1$ the special case

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} k^m = {} & \frac{1}{m+1} x^{m+1} + \zeta(-m) + \frac{1}{m+1} \sum_{k=1}^{\lfloor m_1+1 \rfloor} (-1)^k \binom{m+1}{k} B_k(\{x\}) x^{m-k+1} \\ & + (-1)^{\lfloor m_1+1 \rfloor} \frac{x^{m-\lfloor m_1+1 \rfloor+1}}{m+1} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \binom{m+1}{l+\lfloor m_1+1 \rfloor} S_k^{(1)}(l) B_{l+\lfloor m_1+1 \rfloor}(\{x\})}{(x+1)(x+2)\cdots(x+k)}. \end{aligned} \tag{36}$$

Moreover, if $m = -1$, for every positive real number $x \in \mathbb{R}^+$ and every $a \in \mathbb{N}_0$ we have

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma - \sum_{k=1}^{a} \frac{B_k(\{x\})}{k x^k} + \frac{1}{x^a} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l}{l+a} S_k^{(1)}(l) B_{l+a}(\{x\})}{(x+1)(x+2)\cdots(x+k)}. \tag{37}$$

In particular, for $x \in \mathbb{R}^+$ we have

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l}{l} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)} \tag{38}$$

and that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma - \frac{B_1(\{x\})}{x} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l}{l+1} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}. \tag{39}$$

Proof. From the formula (28) with the parameters $s := -m$, $z := x$ and $y := \{x\}$, we get

$$\begin{aligned}\sum_{k=1}^{\lfloor x\rfloor} k^m - \zeta(-m) &= -\zeta(-m, x+1-\{x\}) \\ &= \frac{1}{m+1}x^{m+1} + \frac{1}{m+1}\sum_{k=1}^{a}(-1)^k\binom{m+1}{k}B_k(\{x\})x^{m-k+1} \\ &\quad + \frac{x^{m-a+1}}{m+1}\sum_{k=1}^{\infty}(-1)^{k+a}\frac{\sum_{l=1}^{k}\binom{m+1}{l+a}S_k^{(1)}(l)B_{l+a}(\{x\})}{(x+1)(x+2)\cdots(x+k)}\end{aligned}$$

by using the formula (19) with $n := \lfloor x\rfloor = x - \{x\}$ in the first step. This gives the above identity (35) with its special cases (34) and (36).

Similarly, we now use the formula (29) again with the variables $z := x$ and $y := \{x\}$, and then we get

$$\begin{aligned}\sum_{k=1}^{\lfloor x\rfloor}\frac{1}{k} - \gamma &= \psi(x+1-\{x\}) \\ &= \log(x) - \sum_{k=1}^{a}\frac{B_k(\{x\})}{kx^k} + \frac{1}{x^a}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l}{l+a}S_k^{(1)}(l)B_{l+a}(\{x\})}{(x+1)(x+2)\cdots(x+k)}\end{aligned}$$

by employing the formula (16) with $n := \lfloor x\rfloor = x - \{x\}$ in the first line of the above calculation. This gives the above identity (37) with its special cases (38) and (39). $\square$

By setting $x := n \in \mathbb{N}$ into Theorem 14, we obtain the following corollary.

Corollary 15. *(generalized Faulhaber formulas) For every complex number $m \in \mathbb{C}\setminus\{-1\}$ and every natural number $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}n^{m+1} + \zeta(-m) + \frac{n^{m+1}}{m+1}\sum_{k=1}^{\infty}\frac{(-1)^k\sum_{l=1}^{k}\binom{m+1}{l}B_lS_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)} \tag{40}$$

and more generally when $a \in \mathbb{N}_0$ that

$$\begin{aligned}\sum_{k=1}^{n} k^m &= \frac{1}{m+1}n^{m+1} + \zeta(-m) + \frac{1}{m+1}\sum_{k=1}^{a}(-1)^k\binom{m+1}{k}B_kn^{m-k+1} \\ &\quad + \frac{n^{m-a+1}}{m+1}\sum_{k=1}^{\infty}\frac{(-1)^{k+a}\sum_{l=1}^{k}\binom{m+1}{l+a}B_{l+a}S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)}.\end{aligned} \tag{41}$$

We have again when $m = m_1 + im_2 \in \mathbb{C} \setminus \{-1\}$ with $m_1 = \mathrm{Re}(m) \geq -1$ the special case

$$\begin{aligned}\sum_{k=1}^{n} k^m = \frac{1}{m+1} n^{m+1} + \zeta(-m) + \frac{1}{m+1} \sum_{k=1}^{\lfloor m_1+1 \rfloor} (-1)^k \binom{m+1}{k} B_k n^{m-k+1} \\ + (-1)^{\lfloor m_1+1 \rfloor} \frac{n^{m-\lfloor m_1+1 \rfloor+1}}{m+1} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \binom{m+1}{l+\lfloor m_1+1 \rfloor} B_{l+\lfloor m_1+1 \rfloor} S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)}.\end{aligned} \tag{42}$$

For $m = -1$, for every natural number $n \in \mathbb{N}$ and every $a \in \mathbb{N}_0$ we have

$$\sum_{k=1}^{n} \frac{1}{k} = \log(n) + \gamma - \sum_{k=1}^{a} \frac{B_k}{kn^k} + \frac{1}{n^a} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sum_{l=1}^{k} \frac{(-1)^l}{l+a} B_{l+a} S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)}. \tag{43}$$

In particular, for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k} &= \log(n) + \gamma + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sum_{l=1}^{k} \frac{(-1)^l}{l} B_l S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)} \\ &= \log(n) + \gamma + \frac{1}{2(n+1)} + \frac{5}{12(n+1)(n+2)} + \frac{3}{4(n+1)(n+2)(n+3)} \\ &\quad + \frac{251}{120(n+1)(n+2)(n+3)(n+4)} + \cdots\end{aligned} \tag{44}$$

and

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k} &= \log(n) + \gamma + \frac{1}{2n} + \sum_{k=1}^{\infty} \frac{(-1)^k \sum_{l=1}^{k} \frac{B_{l+1}}{l+1} S_k^{(1)}(l)}{n(n+1)(n+2)\cdots(n+k)} \\ &= \log(n) + \gamma + \frac{1}{2n} - \frac{1}{12n(n+1)} - \frac{1}{12n(n+1)(n+2)} - \frac{19}{120n(n+1)(n+2)(n+3)} \\ &\quad - \frac{9}{20n(n+1)(n+2)(n+3)(n+4)} - \cdots .\end{aligned} \tag{45}$$

For every positive real number $x \in \mathbb{R}^+$ and for every natural number $n \in \mathbb{N}$, we list the following 8 most used generalized Faulhaber summation formulas:

1.) Generalized Faulhaber formula for the partial sums of $\zeta(2)$:

For every natural number $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k^2} &= \zeta(2) - \frac{1}{n} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sum_{l=1}^{k} (-1)^l B_l S_k^{(1)}(l)}{n(n+1)(n+2)\cdots(n+k)} \\
&= \zeta(2) - \frac{1}{n} + \sum_{k=1}^{\infty} \frac{1}{k+1} \cdot \frac{(k-1)!}{n(n+1)(n+2)\cdots(n+k)} \\
&= \zeta(2) - \frac{1}{n} + \frac{1}{2n(n+1)} + \frac{1}{3n(n+1)(n+2)} + \frac{1}{2n(n+1)(n+2)(n+3)} \\
&\quad + \frac{6}{5n(n+1)(n+2)(n+3)(n+4)} + \cdots .
\end{aligned} \tag{46}$$

2.) Extended generalized Faulhaber formula for the partial sums of $\zeta(3)$:

For every real number $x \in \mathbb{R}^+$, we obtain

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^3} = \zeta(3) - \frac{1}{2x^2} + \frac{1}{2x} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l (l+1) S_k^{(1)}(l) B_l(\{x\})}{x(x+1)(x+2)\cdots(x+k)}. \tag{47}$$

3.) Extended generalized Faulhaber formula for the sum of the square roots:

For every real number $x \in \mathbb{R}^+$, we get

$$\sum_{k=1}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{3/2} - \frac{1}{4\pi} \zeta\left(\frac{3}{2}\right) + x\sqrt{x} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l-5)!!}{2^{l-1} l!} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}. \tag{48}$$

4.) Generalized Faulhaber formula for the partial sums of $\zeta(-3/2)$:

For every natural number $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{k=1}^{n} k\sqrt{k} \\
&= \frac{2}{5} n^{5/2} + \frac{1}{2} n^{3/2} + \frac{1}{8}\sqrt{n} - \frac{3}{16\pi^2} \zeta\left(\frac{5}{2}\right) + 3\sqrt{n} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(2l-3)!!}{2^{l+1}(l+2)!} B_{l+2} S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)} \\
&= \frac{2}{5} n^{5/2} + \frac{1}{2} n^{3/2} + \frac{1}{8}\sqrt{n} - \frac{3}{16\pi^2} \zeta\left(\frac{5}{2}\right) + \frac{\sqrt{n}}{1920(n+1)(n+2)} \\
&\quad + \frac{\sqrt{n}}{640(n+1)(n+2)(n+3)} + \frac{611\sqrt{n}}{107520(n+1)(n+2)(n+3)(n+4)} \\
&\quad + \frac{275\sqrt{n}}{10752(n+1)(n+2)(n+3)(n+4)(n+5)} \\
&\quad + \frac{159157\sqrt{n}}{1146880(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} + \cdots .
\end{aligned} \tag{49}$$

5.) Generalized Faulhaber formula for the partial sums of $\zeta(-5/2)$:

For every natural number $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&\sum_{k=1}^{n} k^2\sqrt{k} \\
&= \frac{2}{7}n^{7/2} + \frac{1}{2}n^{5/2} + \frac{5}{24}n^{3/2} + \frac{15}{64\pi^3}\zeta\left(\frac{7}{2}\right) + 15\sqrt{n}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(2l-3)!!}{2^{l+2}(l+3)!}B_{l+3}S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)} \\
&= \frac{2}{7}n^{7/2} + \frac{1}{2}n^{5/2} + \frac{5}{24}n^{3/2} + \frac{15}{64\pi^3}\zeta\left(\frac{7}{2}\right) - \frac{\sqrt{n}}{384(n+1)} - \frac{\sqrt{n}}{384(n+1)(n+2)} \\
&\quad - \frac{37\sqrt{n}}{7168(n+1)(n+2)(n+3)} - \frac{55\sqrt{n}}{3584(n+1)(n+2)(n+3)(n+4)} \\
&\quad - \frac{1995\sqrt{n}}{32768(n+1)(n+2)(n+3)(n+4)(n+5)} - \cdots .
\end{aligned} \tag{50}$$

6.) Generalized Faulhaber formula for the sum of the inverses of the square roots:

For every natural number $n \in \mathbb{N}$, we get that

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{\sqrt{k}} &= 2\sqrt{n} + \zeta\left(\frac{1}{2}\right) + \frac{1}{2\sqrt{n}} + \frac{1}{\sqrt{n}}\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}\frac{(2l-1)!!}{2^l(l+1)!}B_{l+1}S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)} \\
&= 2\sqrt{n} + \zeta\left(\frac{1}{2}\right) + \frac{1}{2\sqrt{n}} - \frac{1}{24\sqrt{n}(n+1)} - \frac{1}{24\sqrt{n}(n+1)(n+2)} \\
&\quad - \frac{31}{384\sqrt{n}(n+1)(n+2)(n+3)} - \frac{15}{64\sqrt{n}(n+1)(n+2)(n+3)(n+4)} - \cdots .
\end{aligned} \tag{51}$$

7.) Extended generalized Faulhaber formula for the partial sums of $\zeta(3/2)$:

For every real number $x \in \mathbb{R}^+$, we have

$$\sum_{k=1}^{\lfloor x\rfloor}\frac{1}{k\sqrt{k}} = \zeta\left(\frac{3}{2}\right) - \frac{2}{\sqrt{x}} - \frac{B_1(\{x\})}{x\sqrt{x}} + \frac{2}{\sqrt{x}}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l(2l+1)!!}{2^{l+1}(l+1)!}S_k^{(1)}(l)B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}. \tag{52}$$

8.) Extended generalized Faulhaber formula for the partial sums of $\zeta(5/2)$:

For every real number $x \in \mathbb{R}^+$, we obtain

$$\sum_{k=1}^{\lfloor x\rfloor}\frac{1}{k^2\sqrt{k}} = \zeta\left(\frac{5}{2}\right) - \frac{2}{3x^{3/2}} + \frac{4}{3\sqrt{x}}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l(2l+1)!!}{2^{l+1}l!}S_k^{(1)}(l)B_l(\{x\})}{x(x+1)(x+2)\cdots(x+k)}. \tag{53}$$

In equations 3.), 4.) and 5.), we have used that $\zeta\left(-\frac{1}{2}\right) = -\frac{1}{4\pi}\zeta\left(\frac{3}{2}\right)$, $\zeta\left(-\frac{3}{2}\right) = -\frac{3}{16\pi^2}\zeta\left(\frac{5}{2}\right)$ and that $\zeta\left(-\frac{5}{2}\right) = \frac{15}{64\pi^3}\zeta\left(\frac{7}{2}\right)$, which follow from the functional equation of the Riemann zeta function [14, (11.05), p. 63] and are also given in [23, (2), (5) and (9)].

6 Conclusion

We have proved a rapidly convergent generalization of Faulhaber's formula to sums of arbitrary complex powers $m \in \mathbb{C}$. In our eyes, these formulas are useful because of their rapid convergence. We believe that they may also have applications in physics [18], such as the extended version of Faulhaber's formula [19, 20]. With the universal technique, explained in this paper, one can obtain other summation formulas of this type [21, 22]. A generalization of Faulhaber's formula for alternating sums can be found in [22].

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