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Representation of Integers of the Form $x^2 + ky^2 - lz^2$

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Abstract

A positive integer l is k-special if for every integer n there exist non-zero integers x, y, and z such that $n = x^2 + ky^2 - lz^2$. For all odd integers k, we find infinite classes of k-special numbers and $2k$-special numbers.

1 Introduction

The representations of a natural number as a sum of squares have been widely studied by many mathematicians. Nowicki [3] and Lam [2] proved that all natural numbers can be written in the form $x^2 + y^2 - cz^2$ where $xyz \neq 0$ if and only if c is of the form q or $2q$, where either $q = 1$ or q is a product of primes of the form $4m + 1$. Later, Prugsapitak and Thongngam [4] defined a k-special number. A positive integer l is k-special if every integer n can be expressed as $n = x^2 + ky^2 - lz^2$ for some non-zero integers x, y, and z. They proved that 1 is k-special if and only if k is not divisible by 4.

In this article, we will be interested in representing natural numbers in the form $x^2 + ky^2 - lz^2$ where $xyz \neq 0$ for given positive integers k and l. For all odd integers k, we apply

Lam's method [2] to identify k-special numbers and $2k$-special numbers and additionally we show that there are infinitely many k-special numbers and $2k$-special numbers.

2 Main results

In this section, we present k-special and $2k$-special number for an odd integer k. We can show that $x^2 + 2ky^2$ where $\gcd(x, 2ky) = 1$ and $x^2 + ky^2$ where $\gcd(x, ky) = 1$ are $2k$-special and k-special respectively by modifying Lam's approach [2].

Theorem 1. *Let k and l be odd integers. If l can be written as $x^2 + 2ky^2$ for some positive integers x and y where $\gcd(x, 2ky) = 1$, then l is $2k$-special.*

Proof. Let l be an odd integer and $l = x^2 + 2ky^2$ for some positive integers x and y where $\gcd(x, 2ky) = 1$. We first find the representation of odd numbers of the form $a^2 + 2kb^2 - lc^2$ where $abc \neq 0$. Since $\gcd(x, 2ky) = 1$, there exist integers α_0 and β_0 such that

$$x\alpha_0 + 2ky\beta_0 = 1.$$

For a positive integer n, let $\alpha_n = \alpha_0 + 2nky$ and $\beta_n = \beta_0 - nx$. It is easy to see that

$$x\alpha_n + 2ky\beta_n = x(\alpha_0 + 2nky) + 2ky(\beta_0 - nx) = x\alpha_0 + 2ky\beta_0 = 1.$$

Next, let $a_n = xj + \alpha_n$, $b_n = yj + \beta_n$ and $c_n = j$, where j is an integer which will be chosen later. Thus

$$\begin{aligned} a_n^2 + 2kb_n^2 - lc_n^2 &= (xj + \alpha_n)^2 + 2k(yj + \beta_n)^2 - (x^2 + 2ky^2)j^2 \\ &= x^2j^2 + 2xj\alpha_n + \alpha_n^2 + 2ky^2j^2 + 4ykj\beta_n + 2k\beta_n^2 - x^2j^2 - 2ky^2j^2 \\ &= 2xj\alpha_n + 4ykj\beta_n + \alpha_n^2 + 2k\beta_n^2 \\ &= 2j(x\alpha_n + 2ky\beta_n) + \alpha_n^2 + 2k\beta_n^2. \end{aligned}$$

Since x is odd and $x\alpha_n + 2ky\beta_n = 1$, we can see that α_n is odd. Thus

$$a_i^2 + 2kb_i^2 - lc_i^2 = 2j + \alpha_i^2 + 2k\beta_i^2$$

for non-negative integers i. Let m be an odd integer. For a non-negative integer n, we choose a suitable value of an integer j_n such that

$$m = 2j_n + \alpha_n^2 + 2k\beta_n^2 = a_n^2 + 2kb_n^2 - lc_n^2$$

where $a_n = xj_n + \alpha_n$, $b_n = yj_n + \beta_n$ and $c_n = j_n$. We can see that $a_nb_nc_n = 0$ if and only if m is one of the following values; $\alpha_n^2 + 2k\beta_n^2$, $\alpha_n^2 + 2k\beta_n^2 - 2\alpha_n/x$ or $\alpha_n^2 + 2k\beta_n^2 - 2\beta_n/y$. Since

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} -\beta_n = \infty,$$

there exists a non-negative integer n such that $\alpha_n^2+2k\beta_n^2-2\alpha_n/x > m$ and $\beta_n < 0$. Therefore we obtain a representation for m, namely $m = a_n^2 + 2kb_n^2 - lc_n^2$ where $a_nb_nc_n \neq 0$.

We next find the representation of even integers. Since $\gcd(x, 2ky) = 1$, we have $\gcd(x, ky) = 1$. Then there exist integers α_0 and β_0 such that

$$x\alpha_0 + ky\beta_0 = 1.$$

For a positive integer n, let $\alpha_n = \alpha_0 + kny$ and $\beta_n = \beta_0 - nx$. Then

$$x\alpha_n + ky\beta_n = x(\alpha_0 + kny) + ky(\beta_0 - nx) = x\alpha_0 + kxny + ky\beta_0 - kynx = x\alpha_0 + ky\beta_0 = 1.$$

Now let $a_n = xj+2\alpha_n$, $b_n = yj+\beta_n$ and $c_n = j$ where j is an integer which will be determined later. Then

$$\begin{aligned} a_n^2 + 2kb_n^2 - lc_n^2 &= (xj + 2\alpha_n)^2 + 2k(yj + \beta_n)^2 - (x^2 + 2ky^2)j^2 \\ &= x^2j^2 + 4xj\alpha_n + 4\alpha_n^2 + 2ky^2j^2 + 4ykj\beta_n + 2k\beta_n^2 - x^2j^2 - 2ky^2j^2 \\ &= 4j(x\alpha_n + ky\beta_n) + 4\alpha_n^2 + 2k\beta_n^2. \end{aligned}$$

Since x is odd, we have

$$\begin{aligned} 4\alpha_n^2 + 2k\beta_n^2 &= 4(\alpha_0 + kny)^2 + 2k(\beta_0 - nx)^2 \\ &= 4\alpha_0^2 + 8kny\alpha_0 + 4k^2n^2y^2 + 2k\beta_0^2 - 4knx\beta_0 + 2kn^2x^2 \\ &\equiv 2k\beta_0^2 + 2kn^2x^2 \pmod 4 \\ &\equiv 2k\beta_0^2 + 2kn^2 \pmod 4. \end{aligned}$$

Thus,

$$4\alpha_n^2 + 2k\beta_n^2 \equiv \begin{cases} 2k\beta_0^2 \pmod 4, & \text{if } n \text{ is even;} \\ 2k(\beta_0^2 + 1) \pmod 4, & \text{if } n \text{ is odd.} \end{cases}$$

For every non-negative integer r, we obtain two identities for representing even integers given by

$$a_{2r}^2 + 2kb_{2r}^2 - lc_{2r}^2 = (xj + 2\alpha_{2r})^2 + 2k(yj + \beta_{2r})^2 - (x^2 + 2y^2)j^2 = 4j + 4\alpha_{2r}^2 + 2k\beta_{2r}^2$$

and

$$\begin{aligned} a_{2r+1}^2 + 2kb_{2r+1}^2 - lc_{2r+1}^2 &= (xj + 2\alpha_{2r+1})^2 + 2(yj + \beta_{2r+1})^2 - (x^2 + 2y^2)j^2 \\ &= 4j + 4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2. \end{aligned}$$

Let m be an even integer. There are four cases to consider; namely,

- *Case 1:* $m \equiv 0 \pmod 4$ and $\beta_0^2 \equiv 0 \pmod 4$,
- *Case 2*: $m \equiv 2 \pmod 4$ and $\beta_0^2 \equiv 0 \pmod 4$,

- *Case 3*: $m \equiv 0 \pmod 4$ and $\beta_0^2 \equiv 1 \pmod 4$,
- *Case 4*: $m \equiv 2 \pmod 4$ and $\beta_0^2 \equiv 1 \pmod 4$.

Since all cases can be handled similarly, we provide the proof for Case 1 as follows. We choose a suitable value of an integer j_{2r} such that

$$m = 4j_{2r} + 4\alpha_{2r}^2 + 2k\beta_{2r}^2 = a_{2r}^2 + 2kb_{2r}^2 - lc_{2r}^2$$

where $a_{2r} = xj_{2r} + 2\alpha_{2r}$, $b_{2r} = yj_{2r} + \beta_{2r}$ and $c_{2r} = j_{2r}$. We can see that $a_{2r}b_{2r}c_{2r} = 0$ if and only if m is one of the following values: $4\alpha_{2r}^2 + 2k\beta_{2r}^2, 4\alpha_{2r}^2 + 2k\beta_{2r}^2 - 8\alpha_{2r}/x$ or $4\alpha_{2r}^2 + 2k\beta_{2r}^2 - 4\beta_{2r}/y$. Since

$$\lim_{r\to\infty} \alpha_{2r} = \lim_{r\to\infty} -\beta_{2r} = \infty,$$

there exists a non-negative integer r such that $4\alpha_{2r}^2 + 2k\beta_{2r}^2 - 8\alpha_{2r}/x > m$ and $\beta_{2r} < 0$. Therefore we obtain a representation for m, namely $m = a_{2r}^2 + 2kb_{2r}^2 - lc_{2r}^2$ where $a_{2r}b_{2r}c_{2r} \neq 0$. □

For an odd integer k, we obtain a sufficient condition for an odd integer to be $2k$-special. The necessary condition is unknown to us. However we can provide a necessary condition for the 2-special number. That is, if d is 2-special, then $d = m^2 + 2n^2$ for some integers m and n. We first recall a well-known lemma (see, for example, [1]) before proving our mentioned conclusion.

Lemma 2. *Let p be a prime. Then $p = x^2 + 2y^2$ for some integers x and y if and only if $p = 2$ or $p \equiv 1, 3 \pmod 8$.*

Using the above lemma, we have the following.

Lemma 3. *A positive integer n can be written as $x^2 + 2y^2$ for some integers x and y if and only if all primes of the form $8k + 5$ or $8k + 7$ have an even exponent in the prime factorization of n.*

Proof. If $n = x^2 + 2y^2$ for some integers x and y, then

$$n = x^2 + 2y^2 = (x + y\sqrt{-2})(x - y\sqrt{-2}).$$

Let p be a prime of the form $8k + 5$ or $8k + 7$ and p divides n. Since -2 is a quadratic non-residue modulo p, we see that p is a prime in $\mathbb{Z}[\sqrt{-2}]$. Thus p divides $x + y\sqrt{-2}$ or p divides $x - y\sqrt{-2}$. If p divides $x + y\sqrt{-2}$, then p divides $x - y\sqrt{-2}$. Hence p divides $2x$ and p divides $2y$. Since p is odd, we have that p divides both x and y. Similarly, we can show that if p divides $x - y\sqrt{-2}$, then p divides both x and y. Write $x = px_1$ and $y = py_1$ for some integers x_1 and y_1. Therefore $n = p^2x_1^2 + 2p^2y_1^2 = p^2(x_1^2 + 2y_1^2)$. If $p \nmid x_1^2 + 2y_1^2$, then $p^2 || n$. If p divides $x_1^2 + 2y_1^2$, then p^2 divides $x_1^2 + 2y_1^2$. We can repeat the process until we arrive at the conclusion that p has even multiplicity in the prime factorization of n. Conversely, we know that 2 and all primes p where $p \equiv 1, 3 \pmod 8$ can be written as $a^2 + 2b^2$ for some integers a and b and a product of integers of the form $a^2 + 2b^2$ is still an integer of the form $a^2 + 2b^2$. Therefore the result follows. □

Theorem 4. *If d is 2-special, then $d = m^2 + 2n^2$ for some integers m and n.*

Proof. Let d be 2-special. For all integers c there exist non-zero integers x, y, and z such that $x^2 + 2y^2 - dz^2 = 2dc^2$. So $x^2 + 2y^2 = d(z^2 + 2c^2)$. By Lemma 3, all primes of the form $8k+5$ or $8k+7$ have even exponent in the prime factorization of x^2+2y^2 and $2c^2+z^2$. Thus all primes of the form $8k + 5$ or $8k + 7$ have even exponent in the prime factorization of d. Hence again by Lemma 3, we have that d is of the form $m^2 + 2n^2$ for some integers m and n. □

The converse of the above theorem is not true, as we will discover in the next theorem that 8 is not 2-special.

Theorem 5. *Let k be an odd integer. If d is divisible by 8, then d is not $2k$-special.*

Proof. Let d be divisible by 8. Suppose on the contrary that d is $2k$-special. Then

$$x^2 + 2ky^2 - dz^2 = 5$$

for some non-zero integers x, y, and z. So $x^2 + 2ky^2 \equiv 5 \pmod 8$. Since $x^2 \equiv 0, 1, 4 \pmod 8$ and $2ky^2 \equiv 0, 2k \pmod 8$, it is easy to see that $x^2 + 2ky^2 \equiv 0, 1, 4, 2k, 2k + 1, 2k + 4 \pmod 8$. Since k is odd, we can see that $x^2 + 2ky^2 \not\equiv 5 \pmod 8$. This is a contradiction. □

Next, we consider k-special numbers where k is odd.

Theorem 6. *Let l and k be odd integers. If $l = x^2 + ky^2$ for some positive integers x and y and $\gcd(x, ky) = 1$, then l is k-special.*

Proof. Suppose $l = x^2 + ky^2$ for some positive integers x and y where $\gcd(x, ky) = 1$. Since $\gcd(x, ky) = 1$, there exist integers α_0 and β_0 such that $x\alpha_0 + ky\beta_0 = 1$.

For a positive integer n, we define $\alpha_n = \alpha_0 + nky$ and $\beta_n = \beta_0 - nx$. Then

$$\begin{aligned} x\alpha_n + ky\beta_n &= x(\alpha_0 + nky) + ky(\beta_0 - nx) \\ &= x\alpha_0 + xnky + ky\beta_0 - nkyx \\ &= x\alpha_0 + ky\beta_0 = 1. \end{aligned}$$

Let $a_n = xj + \alpha_n$, $b_n = yj + \beta_n$ and $c_n = j$, where j is an integer which will be selected later. Thus

$$\begin{aligned} a_n^2 + kb_n^2 - lc_n^2 &= (xj + \alpha_n)^2 + k(yj + \beta_n)^2 - (x^2 + ky^2)j^2 \\ &= x^2j^2 + 2xj\alpha_n + \alpha_n^2 + ky^2j^2 + 2kyj\beta_n + k\beta_n^2 - x^2j^2 - ky^2j^2 \\ &= 2xj\alpha_n + \alpha_n^2 + 2kyj\beta_n + k\beta_n^2 \\ &= 2j(x\alpha_n + ky\beta_n) + \alpha_n^2 + k\beta_n^2. \end{aligned}$$

We have

$$\begin{aligned}\alpha_n^2 + k\beta_n^2 &= (\alpha_0 + nky)^2 + k(\beta_0 - nx)^2 \\ &= \alpha_0^2 + 2kn\alpha_0 y + n^2k^2y^2 + k\beta_0^2 - 2nk\beta_0 x + kn^2x^2 \\ &\equiv \alpha_0^2 + k\beta_0^2 + n^2k^2y^2 + kn^2x^2 \pmod 2.\end{aligned}$$

Therefore,

$$\alpha_n^2 + k\beta_n^2 \equiv \begin{cases} \alpha_0^2 + k\beta_0^2 \pmod 2, & \text{if } n \text{ is even;} \\ \alpha_0^2 + k\beta_0^2 + y^2 + x^2 \pmod 2, & \text{if } n \text{ is odd.} \end{cases}$$

Since l and k are odd, we can see that x and y have different parities. Thus

$$\alpha_n^2 + k\beta_n^2 \equiv \begin{cases} \alpha_0^2 + k\beta_0^2 \pmod 2, & \text{if } n \text{ is even;} \\ \alpha_0^2 + k\beta_0^2 + 1 \pmod 2, & \text{if } n \text{ is odd.} \end{cases}$$

For all non-negative integers r, we obtain the following identities

$$\begin{aligned} a_{2r}^2 + kb_{2r}^2 - lc_{2r}^2 &= 2j + \alpha_{2r}^2 + k\beta_{2r}^2 \equiv 2j + \alpha_0^2 + k\beta_0^2 \pmod 2, \\ a_{2r+1}^2 + kb_{2r+1}^2 - lc_{2r+1}^2 &= 2j + \alpha_{2r+1}^2 + k\beta_{2r+1}^2 \equiv 2j + \alpha_0^2 + k\beta_0^2 + 1 \pmod 2. \end{aligned}$$

By using both identities, we can demonstrate that all integers can be expressed in the form $a^2 + kb^2 - lc^2$. Next, we will illustrate how to choose a, b, and c so that $abc \neq 0$.

Case 1: $\alpha_0^2 + k\beta_0^2 \equiv 0 \pmod 2$.

We first consider how to represent an even integer m. We select an appropriate value for j_{2r} such that

$$m = 2j_{2r} + \alpha_{2r}^2 + k\beta_{2r}^2 = a_{2r}^2 + kb_{2r}^2 - lc_{2r}^2$$

where $a_{2r} = xj_{2r} + \alpha_{2r}$, $b_{2r} = yj_{2r} + \beta_{2r}$ and $c_{2r} = j_{2r}$. We can see that $a_{2r}b_{2r}c_{2r} = 0$ if and only if m is one of the following values: $\alpha_{2r}^2 + k\beta_{2r}^2, \alpha_{2r}^2 + k\beta_{2r}^2 - 2\alpha_{2r}/x$ or $\alpha_{2r}^2 + k\beta_{2r}^2 - 2\beta_{2r}/y$. Since

$$\lim_{r\to\infty} \alpha_{2r} = \lim_{r\to\infty} -\beta_{2r} = \infty,$$

there exists a non-negative integer r for which $\alpha_{2r}^2 + k\beta_{2r}^2 - 2\alpha_{2r}/x > m$ and $\beta_{2r} < 0$. As a result, we have a representation for m, namely $m = a_{2r}^2 + kb_{2r}^2 - lc_{2r}^2$ where $a_{2r}b_{2r}c_{2r} \neq 0$.

Next, we consider a representation for an odd integer m. Let m be an odd integer. We select an appropriate value for j_{2r+1} such that

$$m = 2j_{2r+1} + \alpha_{2r+1}^2 + k\beta_{2r+1}^2 = a_{2r+1}^2 + kb_{2r+1}^2 - lc_{2r+1}^2$$

where $a_{2r+1} = xj_{2r+1} + \alpha_{2r+1}$, $b_{2r+1} = yj_{2r+1} + \beta_{2r+1}$ and $c_{2r+1} = j_{2r+1}$. It is easy to see that $a_{2r+1}b_{2r+1}c_{2r+1} = 0$ if and only if m is one of the values listed below: $\alpha_{2r+1}^2 + k\beta_{2r+1}^2, \alpha_{2r+1}^2 + k\beta_{2r+1}^2 - 2\alpha_{2r+1}/x$ or $\alpha_{2r+1}^2 + k\beta_{2r+1}^2 - 2\beta_{2r+1}/y$. Since

$$\lim_{r\to\infty} \alpha_{2r+1} = \lim_{r\to\infty} -\beta_{2r+1} = \infty,$$

there exists a non-negative integer r such that $\alpha_{2r+1}^2 + k\beta_{2r+1}^2 - 2\alpha_{2r+1}/x > m$ and $\beta_{2r+1} < 0$. As a consequence, a representation for m is obtained as follows: $m = a_{2r+1}^2 + kb_{2r+1}^2 - lc_{2r+1}^2$ where $a_{2r+1}b_{2r+1}c_{2r+1} \neq 0$.

Case 2: $\alpha_0^2 + k\beta_0^2 \equiv 1 \pmod 2$. This case can be handled similarly as in Case 1. Thus l is k-special as desired. □

Finally, we obtain the following consequence of Theorem 1 and Theorem 6.

Corollary 7. *Let k be an odd integer. There are infinitely many k-special numbers and $2k$-special numbers.*

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