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A New Identity for Appell Polynomials

Farid Bencherif
LA3C, Faculty of Mathematics
USTHB
Algiers
Algeria
fbencherif@usthb.dz

Nouara Mokhtari[1] and Schehrazade Zerroukhat
LATN, Faculty of Mathematics
USTHB
Algiers
Algeria
nmokhtari1@usthb.dz
mokhtari.nouara@gmail.com
szerroukhat@usthb.dz

Abstract

In the present paper, we generalize an identity for some Appell polynomials, from which we deduce many explicit formulas for generalized Bernoulli and Euler numbers and polynomials.

1 Introduction and main result

The classical and generalized Bernoulli numbers and polynomials as well as the classical and generalized Euler numbers and polynomials have been extensively studied [4, 8]. In the present paper, we attempt to improve some results related to Bernoulli and Euler polynomials

[1]Corresponding author.

and extend a generalization of the Appell polynomials identity published in [2]. Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$ denote the set of integers, non-negative integers and complex numbers respectively. A polynomial sequence $(A_n(x))_{n\in\mathbb{N}}$ of $\mathbb{C}[x]$ is an *Appell polynomial sequence* [1] if $A_0(x)$ is a non-zero constant polynomial and $A_n'(x) = nA_{n-1}(x)$ for $n \geq 1$. The exponential generating series of the sequence $(A_n(x))_{n\in\mathbb{N}}$ can be formulated as follows:

$$\sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!} = S(z)e^{xz},$$

where $S(z)$ is a formal power series of $\mathbb{C}[[z]]$ with non-zero constant term. We focus now on the following theorem which generalizes [2, Theorem 1.1] and adds other properties.

Theorem 1. *Let* $S(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ *be a formal series of* $\mathbb{C}[[z]]$ *with* $a_0 = 1$. *For every* $\alpha \in \mathbb{C}$ *and* $(A_n^{(\alpha)}(x))_{n\in\mathbb{N}}$ *consider the Appell polynomial sequence defined by*

$$\sum_{n=0}^{\infty} A_n^{(\alpha)}(x)\frac{z^n}{n!} = S^{\alpha}(z)e^{xz}. \tag{1}$$

Then for all λ, $\alpha \in \mathbb{C}$, $\ell \in \mathbb{Z}$, *and non-negative integers* n, m, *with* $m \geq n$, *we have*

$$A_n^{(\alpha\ell)}(x) = \sum_{k=0}^{m} (-1)^k \binom{\alpha+m}{m-k}\binom{\alpha+k-1}{k} A_n^{(-k\ell)}(x+\lambda\ell(\alpha+k)) \tag{2}$$

and for $m \geq \lfloor \frac{n}{2} \rfloor$, *we have*

$$A_n^{(\alpha\ell)}(x) = \sum_{k=0}^{m} (-1)^k \binom{\alpha+m}{m-k}\binom{\alpha+k-1}{k} A_n^{(-k\ell)}(x+a_1\ell(\alpha+k)). \tag{3}$$

By using Relation (2) for $\lambda = 0$ and $\ell = 1$, we obtain [2, Theorem 1.1].

2 Applications

In this section, we give some applications of Theorem 1. Let us first recall that several sequences of remarkable polynomials in $\mathbb{C}[x]$ are Appell polynomial sequences. Indeed, it is clear that for $\alpha \in \mathbb{C}$, the sequences of generalized Bernoulli polynomials $(B_n^{(\alpha)}(x))_{n\in\mathbb{N}}$ and generalized Euler polynomials $(E_n^{(\alpha)}(x))_{n\in\mathbb{N}}$ defined by (see [9, 14])

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} = \left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} \tag{4}$$

and

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!} = \left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} \tag{5}$$

are Appell polynomial sequences, associated with the formal series $\left(\frac{z}{e^z-1}\right)^{\alpha}$, $\left(\frac{2}{e^z+1}\right)^{\alpha}$ respectively. The classical Bernoulli polynomials $B_n(x)$ and classical Euler polynomials $E_n(x)$ are defined by $B_n(x) = B_n^{(1)}(x)$ and $E_n(x) = E_n^{(1)}(x)$.

For $k \in \mathbb{N}$, from (4) and (5) we deduce the explicit expressions for $B_n^{(-k)}(x)$ and $E_n^{(-k)}(x)$:

$$B_n^{(-k)}(x) = \frac{1}{k!}\binom{n+k}{k}^{-1} \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}(x+j)^{n+k}$$

and

$$E_n^{(-k)}(x) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(x+j)^n.$$

The Stirling numbers of second kind $S(n,k)$ (see sequence A008277 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [13]) are defined by

$$\sum_{n=0}^{\infty} S(n,k)\frac{z^n}{n!} = \frac{1}{k!}(e^z-1)^k.$$

The generalized Stirling numbers of second kind $S(n,k,x)$ [3, Eq. (3.9), p. 152] are defined by

$$\sum_{n=0}^{\infty} S(n,k,x)\frac{z^n}{n!} = \frac{1}{k!}e^{xz}(e^z-1)^k.$$

We have

$$\sum_{n=0}^{\infty} S(n,k,x) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n.$$

and

$$S(n,k) = S(n,k,0) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

by using the generalized Stirling numbers of second kind, we can express $B_n^{(-k)}(x)$ as follows:

$$B_n^{(-k)}(x) = \binom{n+k}{k}^{-1} S(n+k,k,x). \tag{6}$$

Corollary 2. *Let $\alpha, \lambda \in \mathbb{C}$, $\ell \in \mathbb{Z}$ and $n, m \in \mathbb{N}$. For $m \geq n$, we have*

$$B_n^{(\alpha\ell)}(x) = \sum_{k=0}^{m}(-1)^k\binom{\alpha+m}{m-k}\binom{\alpha+k-1}{k}B_n^{(-k\ell)}(x+\lambda\ell(\alpha+k)), \tag{7}$$

and for $m \geq \lfloor \frac{n}{2} \rfloor$*, we have*

$$B_n^{(\alpha\ell)}(x) = \sum_{k=0}^{m} (-1)^k \binom{\alpha+m}{m-k} \binom{\alpha+k-1}{k} B_n^{(-k\ell)}(x - \frac{\ell}{2}(\alpha+k)). \tag{8}$$

Corollary 3. *Let* $\alpha, \lambda \in \mathbb{C}$*,* $\ell \in \mathbb{Z}$*,* $n, m \in \mathbb{N}$*. For* $m \geq n$*, we have*

$$E_n^{(\alpha\ell)}(x) = \sum_{k=0}^{m} (-1)^k \binom{\alpha+m}{m-k} \binom{\alpha+k-1}{k} E_n^{(-k\ell)}(x + \lambda\ell(\alpha+k)),$$

and for $m \geq \lfloor \frac{n}{2} \rfloor$*, we have*

$$E_n^{(\alpha\ell)}(x) = \sum_{k=0}^{m} (-1)^k \binom{\alpha+m}{m-k} \binom{\alpha+k-1}{k} E_n^{(-k\ell)}(x - \frac{\ell}{2}(\alpha+k)). \tag{9}$$

Corollaries 2 and 3 are deduced from Theorem 1 by choosing successively $S(z) = \frac{z}{e^z-1}$ then $S(z) = \frac{2}{e^z+1}$ and using Definitions (4) and (5). The generalized Bernoulli numbers $B_n^{(\alpha)}$ and Euler numbers $E_n^{(\alpha)}$ are defined by $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ and $E_n^{(\alpha)} = 2^n E_n^{(\alpha)}(\frac{\alpha}{2})$. The classical Bernoulli numbers B_n and classical Euler numbers E_n A122045 are defined by $B_n = B_n^{(1)} = B_n^{(1)}(0)$ and $E_n = E_n^{(1)} = 2^n E_n(\frac{1}{2})$. We also define the polynomials $\widehat{B}_n^{(\alpha)}$ by $\widehat{B}_n^{(\alpha)} = B_n^{(\alpha)}(\alpha/2)$ [15, p. 259].

Corollary 4. *Let* $\ell, n, m \in \mathbb{N}$*. For* $m \geq n$*, we have*

$$B_n^{(\ell)}(x) = \sum_{k=0}^{m} (-1)^k \frac{\binom{m+1}{k+1}}{\binom{n+k\ell}{n}} S(n+k\ell, k\ell, x). \tag{10}$$

This follows by using Relations (6) and (7) for $\alpha = 1$.

Replacing $x = 0$ and $m = n$ in Relation (10) we obtain the following relation which can be found in [6, p. 60]:

$$B_n^{(\ell)} = \sum_{k=0}^{n} (-1)^k \frac{\binom{n+1}{k+1}}{\binom{n+k\ell}{n}} S(n+k\ell, k\ell).$$

Replacing $x = 0$, $m = n$, and $\ell = 1$ in Relation (10) it follows that

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{\binom{n+1}{k+1}}{\binom{n+k}{n}} S(n+k, k).$$

Note that this formula has been proven by numerous authors: see, for example, [4, Eq. (11), p. 48], [5, Eq. (6), p. 27], [6, p. 59], [7, p. 219], [10, Eq. (1.3), p. 91], [12, p. 140].

Corollary 5. *Let $\alpha \in \mathbb{C}$, $\ell \in \mathbb{Z}$ and $n, m \in \mathbb{N}$. For $m \geq \lfloor \frac{n}{2} \rfloor$, the following hold:*

$$\widehat{B}_n^{(\ell\alpha)} = \sum_{k=0}^{m} (-1)^k \binom{\alpha+m}{m-k} \binom{\alpha+k-1}{k} \widehat{B}_n^{(-k\ell)}, \tag{11}$$

and

$$\widehat{B}_{2n}^{(\ell\alpha)} = \sum_{k=0}^{n} (-1)^k \binom{\alpha+n}{n-k} \binom{\alpha+k-1}{k} \widehat{B}_{2n}^{(-k\ell)}.$$

This follows immediately from Relation (8) for $x = \frac{1}{2}\alpha\ell$. For $\ell = 1$ and $m = n$, Relation (11) can be written as

$$\widehat{B}_n^{(\alpha)} = \sum_{k=0}^{n} (-1)^k \binom{\alpha+n}{n-k} \binom{\alpha+k-1}{k} \widehat{B}_n^{(-k)}. \tag{12}$$

Note that (12) is exactly Relation (6.12) of [15, Theorem 6.3].

Corollary 6. *For $\alpha \in \mathbb{C}$, $\ell \in \mathbb{N}$, n, $m \in \mathbb{N}$, with $m \geq \lfloor \frac{n}{2} \rfloor$, we have*

$$E_n^{(\alpha\ell)} = \sum_{k=0}^{m} \frac{(-1)^k}{2^{k\ell}} \binom{\alpha+m}{m-k} \binom{\alpha+k-1}{k} \sum_{j=0}^{k\ell} \binom{k\ell}{j} (k\ell-2j)^n, \tag{13}$$

$$E_{2n}^{(\alpha\ell)} = \sum_{k=0}^{n} \frac{(-1)^k}{2^{k\ell}} \binom{\alpha+n}{n-k} \binom{\alpha+k-1}{k} \sum_{j=0}^{k\ell} \binom{k\ell}{j} (k\ell-2j)^{2n},$$

$$E_{2n}^{(\ell)} = \sum_{k=0}^{n} \frac{(-1)^k}{2^{k\ell}} \binom{n+1}{k+1} \sum_{j=0}^{k\ell} \binom{k\ell}{j} (k\ell-2j)^{2n}.$$

This follows immediately from Relation (9) for $x = \frac{1}{2}\alpha\ell$, and by noticing that $E_n^{(\alpha\ell)} = 0$ for n odd. For $m = n$ and $\ell = 1$ the relation (13) can be written as follows:

$$E_n^{(\alpha)} = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \binom{\alpha+n}{n-k} \binom{\alpha+k-1}{k} \sum_{j=0}^{k} \binom{k}{j} (k-2j)^n. \tag{14}$$

We notice that (14) is nothing else than the relation obtained by Luo [8].

3 Proof of Theorem 1

The following lemma, proved in [2], will be useful for the proof of the main result.

Lemma 7. *For $m,\, q \in \mathbb{N}$, we have*

$$1 = \sum_{k=0}^{m} (-1)^k \binom{q+m}{m-k} \binom{q+k-1}{k} x^{q+k} - (-1)^m \binom{q+m}{q} \sum_{k=1}^{q} \frac{k}{m+k} \binom{q}{k} (x-1)^{m+k}.$$

Let us consider the formal series $S(z) = 1 + \sum_{n=1}^{\infty} a_n \frac{z^n}{n!}$. For every $\alpha \in \mathbb{C}$, let $(A_n^{(\alpha)}(x))_{n\in\mathbb{N}}$ be the Appell polynomial sequence defined by (1). For non-negative integers n, m, a given integer ℓ and for every $\lambda \in \mathbb{C}$, we consider the polynomial

$$P_\lambda(x, \alpha) = A_n^{(\alpha\ell)}(x) - \sum_{k=0}^{m} (-1)^k \binom{\alpha+m}{m-k} \binom{\alpha+k-1}{k} A_n^{(-k\ell)}(x + \lambda\ell(\alpha+k)).$$

Let $n_\lambda = n$ if $\lambda \neq a_1$ and $n_\lambda = \lfloor \frac{n}{2} \rfloor$ if $\lambda = a_1$. To prove our theorem it is equivalent to show that $P_\lambda(x, \alpha) = 0$ for $m \geq n_\lambda$. For this, since $P_\lambda(x, \alpha)$ is a polynomial in x and α, this amounts to proving that $P_\lambda(x, q) = 0$ for every non-negative integer q, provided that $m \geq n_\lambda$.

Let D denote the derivation operator of $\mathbb{C}[x]$. In the commutative $\mathbb{C}$-algebra $\mathbb{C}[[D]]$ of operators of composition of $\mathbb{C}[x]$, consider the operator of translation $T_\beta = \exp(\beta D) = \sum_{n=0}^{\infty} \beta^n \frac{D^n}{n!}$ for $\beta \in \mathbb{C}$ and the automorphism $\Omega = S(D) = 1 + \sum_{n=1}^{\infty} a_n \frac{D^n}{n!}$ [11, p. 200], we can write $T_\beta(x^n) = (x+\beta)^n$ and $\Omega^q(x^n) = A_n^{(q)}(x)$. We remark that for all non-negative integer n, $P_\lambda(x, q) = \Lambda_\lambda(x^n)$ with

$$\Lambda_\lambda = \Omega^{q\ell} - \sum_{k=0}^{m} (-1)^k \binom{q+m}{m-k} \binom{q+k-1}{k} \Omega^{-k\ell} \circ T_{\lambda\ell(q+k)}.$$

Then, to get the desired result we must show that $\Lambda_\lambda(x^n) = 0$ for $m \geq n_\lambda$. We can write

$$\Lambda_\lambda = \Omega^{q\ell} \circ \Psi_\lambda \tag{15}$$

with

$$\Psi_\lambda = 1 - \sum_{k=0}^{m} (-1)^k \binom{q+m}{m-k} \binom{q+k-1}{k} (\Omega^{-\ell} \circ T_\lambda^\ell)^{q+k}.$$

Lemma 7 allows us to express Ψ_λ as follows:

$$\Psi_\lambda = (-1)^{m+1} \binom{q+m}{q} \sum_{k=1}^{q} \frac{k}{m+k} \binom{q}{k} (\Omega^{-\ell} \circ T_\lambda^\ell - 1)^{m+k}.$$

Let T be a nonzero composition operator. We can write $T = \sum_{j\geq i} a_j D^j$ with a first nonzero coefficient $a_i \neq 0$. In this case, we say that the composition operator T has order i. Noting then that for every λ, $\mathrm{ord}(\Omega^{-\ell} \circ T_\lambda^\ell - 1) \geq 1$, and for the particular case $\lambda = a_1$, we have $\mathrm{ord}(\Omega^{-\ell} \circ T_\lambda^\ell - 1) \geq 2$, we deduce (for $k \geq 1$) that $\mathrm{ord}(\Omega^{-\ell} \circ T_\lambda^\ell - 1)^{m+k} > m$ and $\mathrm{ord}(\Omega^{-\ell} \circ T_\lambda^\ell - 1)^{m+k} > 2m$ in the special case where $\lambda = a_1$. Thus for every integer $m \geq n_\lambda$, we have $\Psi_\lambda(x^n) = 0$. Using (15), we obtain the equality $\Lambda_\lambda(x^n) = 0$ for $m \geq n_\lambda$, which completes the proof.

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