Journal of Integer Sequences, Vol. 25 (2022),
Article 22.5.7

Alternating Variants of Multiple Poly-Bernoulli Numbers and Finite Multiple Zeta Values in Characteristic 0 and p

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Abstract

This paper has two parts: the characteristic 0 part and the characteristic p part. In the characteristic 0 part, we introduce an alternating extension of the multiple poly-Bernoulli numbers of M.-S. Kim and T. Kim. We obtain explicit representations of the alternating finite multiple zeta values, introduced by Zhao, in terms of the alternating extension of the multiple poly-Bernoulli numbers, which are alternating generalizations of the work of Imatomi, M. Kaneko, and Takeda. In the characteristic p part, we introduce positive characteristic analogs of alternating finite multiple zeta values, and express them as special values of finite Carlitz multiple polylogarithms defined by Chang and Mishiba. We introduce alternating variants of Harada's multiple poly-Bernoulli-Carlitz numbers, which are analogues of the multiple poly-Bernoulli numbers, to obtain explicit representations of the finite alternating multiple zeta values. We show that finite multiple zeta values with an integer index can be expressed as k-linear combination of FMZV's with all-positive indices.

1 Introduction

In this paper, we generalize the results of Imatomi, M. Kaneko, and Takeda [12] on multiple poly-Bernoulli numbers in characteristic 0 and their characteristic p analogues (p prime) established by Harada [10] to an alternating setting.

M.-S. Kim and T. Kim [17] introduced multiple poly-Bernoulli numbers, which are a generalization of Bernoulli numbers. Imatomi, M. Kaneko, and Takeda obtained connections of the multiple poly-Bernoulli numbers with Stirling numbers and finite multiple zeta values [12]. We generalize these results to an alternating setting (Theorems 7 and 9).

In the characteristic p case, Carlitz introduced analogues of Bernoulli numbers called Bernoulli-Carlitz numbers. Harada [10] generalized the notion to multiple poly-Bernoulli-Carlitz numbers and established their relationship with analogues of Stirling numbers introduced by H. Kaneko and Komatsu [13] and analogues of finite multiple zeta values introduced by Chang and Mishiba [5]. We further generalize these results to an alternating setting; alternating variants of the multiple poly-Bernoulli-Carlitz numbers are introduced in Definition 15, and we obtain explicit representations of the alternating finite multiple zeta values in terms of them (Theorems 16 and 20).

In appendix A, we show that FMZV with an integer index can be expressed as k-linear combination of FMZV's with all-positive indices.

2 Characteristic 0

This section disscusses the characteristic 0 part. In §2.1, we review the results of Imatomi, Kaneko, and Takeda: the connections of the multiple poly-Bernoulli numbers with Stirling numbers and finite multiple zeta values. In §2.2, we consider alternating extension of their results. We introduce alternating multiple poly-Bernoulli numbers (Definition 5) and obtain their relationships with Stirling numbers and alternating finite multiple zeta values (Theorems 7 and 9).

2.1 Review of the results in original (non-alternating) case

For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{Z}^r$ (where $r \in \mathbb{N}_{\geq 0}$), *the multiple polylogarithm* (*MPL* for short) $\mathrm{Li}_{\mathbf{s}}(z_1, \ldots, z_r)$ is the multivariable series defined by

$$\mathrm{Li}_{\mathbf{s}}(z_1, \ldots, z_r) = \sum_{n_1 > \cdots > n_r \geq 1} \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}}, \tag{1}$$

(cf. Zhao [20, Definition 2.3.1]). We define $\mathrm{Li}_{\mathbf{s}}(z) := 1$ if $r = 0$ by convention.

For $\mathbf{s} \in \mathbb{Z}^r$ and $n \in \mathbb{N}$, rational numbers $B_n^{\mathbf{s}}$ and $C_n^{\mathbf{s}}$ called the *multiple poly-Bernoulli numbers* (*MPBNs* for short) are defined by

$$\sum_{n \geq 0} B_n^{\mathbf{s}} \frac{x^n}{n!} = \frac{\mathrm{Li}_{\mathbf{s}}(1 - e^{-x}, 1, \ldots, 1)}{1 - e^{-x}}, \quad \sum_{n \geq 0} C_n^{\mathbf{s}} \frac{x^n}{n!} \qquad = e^{-x} \frac{\mathrm{Li}_{\mathbf{s}}(1 - e^{-x}, 1, \ldots, 1)}{1 - e^{-x}}.$$

M. Kim and T. Kim introduced rationals $C_n^{\mathbf{s}}$ [17, §2] (They call them generalized Bernoulli numbers). Imatomi, Kaneko, and Takeda introduced both of $B_n^{\mathbf{s}}$ and $C_n^{\mathbf{s}}$. If $r = 1$, then

$B_n^{\mathbf{s}}$ and $C_n^{\mathbf{s}}$ coincide with poly-Bernoulli numbers introduced by Kaneko [2, 14]; if $s_1 = 1$ in addition, these are equal to Bernoulli numbers.

Imatomi, Kaneko, and Takeda proved the following equations:

Proposition 1 ([12, Proposition 5])**.** *We have the equalities*

$$\begin{aligned}&\sum_{s_1,\,\dots,\,s_r\geq 1}\sum_{n\geq 0} B_n^{(-s_1,\,-s_2,\,\dots,\,-s_r)}\frac{x^n}{n!}\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!}\\ &= \frac{(1-e^{-x})^{r-1}}{(e^{-y_1}+e^{-x}-1)(e^{-y_1-y_2}+e^{-x}-1)\cdots(e^{-y_1-\cdots-y_r}+e^{-x}-1)}\end{aligned}$$

and

$$\begin{aligned}&\sum_{s_1,\,\dots,\,s_r\geq 1}\sum_{n\geq 0} C_n^{(-s_1,\,-s_2,\,\dots,\,-s_r)}\frac{x^n}{n!}\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!}\\ &= \frac{e^{-x}(1-e^{-x})^{r-1}}{(e^{-y_1}+e^{-x}-1)(e^{-y_1-y_2}+e^{-x}-1)\cdots(e^{-y_1-\cdots-y_r}+e^{-x}-1)}.\end{aligned}$$

2.1.1 Connection with the Stirling numbers

For m, $n \in \mathbb{N}$, the *Stirling numbers* $\left[{n \atop m}\right]$, $\left\{{n \atop m}\right\}$ of the *first* and the *second kind* are defined by the formulae [8, §6.1 and §7.4 (7.49)]

$$x(x+1)\cdots(x+n-1) = \sum_{m\geq 0}^{n}\left[{n \atop m}\right]x^m, \tag{2}$$

$$(e^x-1)^m = m!\sum_{n\geq m}\left\{{n \atop m}\right\}\frac{x^n}{n!}. \tag{3}$$

In this paper we use the formula

$$e^x(e^x-1)^{m-1} = (m-1)!\sum_{n\geq m-1}\left\{{n+1 \atop m}\right\}\frac{x^n}{n!}, \tag{4}$$

which is obtained by the differentiation, and the duality

$$\left[{n \atop m}\right] \equiv \left\{{l-n \atop l-m}\right\} \pmod{l} \tag{5}$$

between two kinds of Stirling numbers, which holds for prime l and $1 \leq m \leq n < l$ [11, §5]. Using these integers, we can write multiple poly-Bernoulli numbers down as finite sums:

Theorem 2 ([12, Theorem 3])**.** *For* $\mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{Z}^r$, *we have for* $n \in \mathbb{N}$ *the equalities*

$$B_n^{\mathbf{s}} = (-1)^n \sum_{n+1 \geq m_1 > \cdots > m_r > 0} \frac{(-1)^{m_1-1}(m_1-1)!\left\{ {n \atop m_1-1} \right\}}{m_1^{s_1} \cdots m_r^{s_r}},$$

$$C_n^{\mathbf{s}} = (-1)^n \sum_{n+1 \geq m_1 > \cdots > m_r > 0} \frac{(-1)^{m_1-1}(m_1-1)!\left\{ {n+1 \atop m_1} \right\}}{m_1^{s_1} \cdots m_r^{s_r}}. \tag{6}$$

2.1.2 Connection with finite multiple zeta values

The ring $\prod_l (\mathbb{Z}/l\mathbb{Z}) / \bigoplus_l (\mathbb{Z}/l\mathbb{Z})$, where the symbol l runs through the set of all prime numbers is denoted by $\mathcal{A}$. It should be noticed that the field $\mathbb{Q}$ can be canonically embedded into the ring $\mathcal{A}$, that is, the ring $\mathcal{A}$ is a $\mathbb{Q}$-algebra.

Definition 3 ([15, §7])**.** For $\mathbf{s} \in \mathbb{Z}^r$, the element $\zeta_{\mathcal{A}}(\mathbf{s}) = (\zeta_{\mathcal{A}}(\mathbf{s})_l)_{l \text{ prime}}$ of $\mathcal{A}$ called the *finite multiple zeta value* (*FMZV* for short) is defined to be the image under the surjection $\prod_l (\mathbb{Z}/l\mathbb{Z}) \to \mathcal{A}$ of the elements of $\prod_l (\mathbb{Z}/l\mathbb{Z})$ whose component in the direct factor $\mathbb{Z}/l\mathbb{Z}$ is

$$\zeta_{\mathcal{A}}(\mathbf{s})_l := \sum_{l > n_1 > \cdots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{Z}/l\mathbb{Z}.$$

We call the natural number r the *depth* of the FMZV $\zeta_{\mathcal{A}}(\mathbf{s})$. The product of two FMZVs can be written by a $\mathbb{Q}$-linear combination of FMZVs [15, Section 7]. FMZVs are realized as special values of the finite version of the multiple polylogarithm $\mathfrak{L}_{\mathcal{A},\mathbf{s}}(z)$ introduced by Sakugawa, Seki [18, Definition 3.8], whose value $\mathfrak{L}_{\mathcal{A},\mathbf{s}}(\mathbf{a}_1, \ldots, \mathbf{a}_r) = (\mathfrak{L}_{\mathcal{A},\mathbf{s}}(\mathbf{a}_1, \ldots, \mathbf{a}_r))_l$ (each $\mathbf{a}_i = (a_{i,l})_{l\text{prime}}$ is an element of $\mathcal{A}$) is given by

$$(\mathfrak{L}_{\mathcal{A},\mathbf{s}}(\mathbf{a}_1, \ldots, \mathbf{a}_r))_l = \sum_{l > n_1 > \cdots > n_r > 0} \frac{a_{1,l}^{n_1} \cdots a_{r,l}^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{Z}/l\mathbb{Z} \tag{7}$$

for each prime l, in precise, we have the equality

$$\zeta_{\mathcal{A}}(\mathbf{s}) = \mathfrak{L}_{\mathcal{A},\mathbf{s}}(1, \ldots, 1) \tag{8}$$

for each $\mathbf{s} \in \mathbb{Z}^r$.

Imatomi, Kaneko, and Takeda obtained the following equalities:

Theorem 4 ([12, Theorem 8])**.**

1. *For a prime* l *and* $\mathbf{s} := (s_1, \ldots, s_r) \in \mathbb{Z}^r$, *we have the congruence*

$$\zeta_{\mathcal{A}}(\mathbf{s})_l \equiv -C_{l-2}^{s_1-1, s_2, \ldots, s_r} \pmod{l}.$$

2. *If we take* $r' \in \mathbb{N}$ *and put* $\bar{\mathbf{s}} := (1, \ldots, 1, s_1, \ldots, s_r) \in \mathbb{Z}^{r+r'}$, *then we have the congruence*

$$\zeta_{\mathcal{A}}(\bar{\mathbf{s}})_l \equiv -C_{l-r'-2}^{s_1-1, s_2, \ldots, s_r} \pmod{l}. \tag{9}$$

2.2 Alternating version

This subsection considers alternating extensions of notions and results in the previous subsection. We first give an alternating extension to the MPBNs by the following series:

Definition 5. For $\mathbf{s} \in \mathbb{Z}^r$ and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r) \in (\mathbb{Z}^\times)^r = \{\pm 1\}^r$, the sequences of rationals $B_n^{\mathbf{s};\boldsymbol{\epsilon}}$ and $C_n^{\mathbf{s};\boldsymbol{\epsilon}}$ are defined by the following series:

$$\sum_{n\geq 0} B_n^{\mathbf{s};\boldsymbol{\epsilon}} \frac{x^n}{n!} = \frac{\mathrm{Li}_{\mathbf{s}}((1-e^{-x})\epsilon_1, \epsilon_2, \ldots, \epsilon_r)}{1-e^{-x}}, \tag{10}$$

$$\sum_{n\geq 0} C_n^{\mathbf{s};\boldsymbol{\epsilon}} \frac{x^n}{n!} = e^{-x}\frac{\mathrm{Li}_{\mathbf{s}}((1-e^{-x})\epsilon_1, \epsilon_2, \ldots, \epsilon_r)}{1-e^{-x}}. \tag{11}$$

We call these rationals *alternating multiple poly-Bernoulli numbers* (*AMPBNs* for short).

The following proposition could be said as alternating extension of Proposition 1:

Proposition 6. *For $\boldsymbol{\epsilon} \in \{\pm 1\}^r$, the following equalities hold:*

$$\begin{aligned}&\sum_{s_1,\ldots,s_r\geq 0}\sum_{n\geq 0} B_n^{(-s_1,-s_2,\ldots,-s_r;\boldsymbol{\epsilon})}\frac{x^n}{n!}\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!}\\ &= \frac{(1-e^{-x})^{r-1}}{(\epsilon_1 e^{-y_1}+e^{-x}-1)(\epsilon_1\epsilon_2 e^{-y_1-y_2}+e^{-x}-1)\cdots(\epsilon_1\cdots\epsilon_r e^{-y_1-\cdots-y_r}+e^{-x}-1)}\end{aligned}$$

and

$$\begin{aligned}&\sum_{s_1,\ldots,s_r\geq 0}\sum_{n\geq 0} C_n^{(-s_1,-s_2,\ldots,-s_r;\boldsymbol{\epsilon})}\frac{x^n}{n!}\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!}\\ &= \frac{e^{-x}(1-e^{-x})^{r-1}}{(\epsilon_1 e^{-y_1}+e^{-x}-1)(\epsilon_1\epsilon_2 e^{-y_1-y_2}+e^{-x}-1)\cdots(\epsilon_1\cdots\epsilon_r e^{-y_1-\cdots-y_r}+e^{-x}-1)}.\end{aligned}$$

Proof. By the equality (10), we have

$$\begin{aligned}&\sum_{s_1,\ldots,s_r\geq 0}\sum_{n\geq 0} B_n^{(-s_1,-s_2,\ldots,-s_r;\boldsymbol{\epsilon})}\frac{x^n}{n!}\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!}\\ &= \sum_{s_1,\ldots,s_r\geq 0}\frac{\mathrm{Li}_{(-s_1,-s_2,\ldots,-s_r)}((1-e^{-x})\epsilon_1, \epsilon_2, \ldots, \epsilon_r)}{1-e^{-x}}\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!}\\ &= \sum_{s_1,\ldots,s_r\geq 0}(1-e^{-x})^{-1}\sum_{m_1>\cdots>m_r>0}\frac{\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(1-e^{-x})^{m_1}}{m_1^{-s_1}\cdots m_r^{-s_r}}\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!}\\ &= (1-e^{-x})^{-1}\sum_{m_1>\cdots>m_r>0}\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(1-e^{-x})^{m_1}\sum_{s_1,\ldots,s_r\geq 0}\frac{(m_1y_1)^{s_1}}{s_1!}\cdots\frac{(m_ry_r)^{s_r}}{s_r!}\\ &= (1-e^{-x})^{-1}\sum_{m_1>\cdots>m_r>0}\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(1-e^{-x})^{m_1}e^{m_1y_1}\cdots e^{m_ry_r}.\end{aligned}$$

In what follows we continue the previous calculation by putting $n_1 := m_1 - m_2$, $n_2 := m_2 - m_3$, …, $n_{r-1} := m_{r-1} - m_r$ and $n_r := m_r$.

$$\begin{aligned}
&= (1-e^{-x})^{-1} \sum_{n_1,\ldots,n_r\geq 1} \epsilon_1^{n_1+\cdots+n_r}\cdots\epsilon_r^{n_r}(1-e^{-x})^{n_1+\cdots+n_r} e^{(n_1+\cdots+n_r)y_1}\cdots e^{n_r y_r} \\
&= (1-e^{-x})^{-1} \sum_{n_1,\ldots,n_r\geq 1} \{\epsilon_1(1-e^{-x})e^{y_1}\}^{n_1}\{\epsilon_1\epsilon_2(1-e^{-x})e^{y_1+y_2}\}^{n_2}\cdots \\
&\quad \cdots\{\epsilon_1\cdots\epsilon_r(1-e^{-x})e^{y_1+\cdots+y_r}\}^{n_r} \\
&= (1-e^{-x})^{-1} \frac{\epsilon_1(1-e^{-x})e^{y_1}}{1-(\epsilon_1(1-e^{-x})e^{y_1})} \frac{\epsilon_1\epsilon_2(1-e^{-x})e^{y_1+y_2}}{1-(\epsilon_1\epsilon_2(1-e^{-x})e^{y_1+y_2})}\cdots \\
&\quad \cdots \frac{\epsilon_1\cdots\epsilon_r(1-e^{-x})e^{y_1+\cdots+y_r}}{1-(\epsilon_1\cdots\epsilon_r(1-e^{-x})e^{y_1+\cdots+y_r})} \\
&= (1-e^{-x})^{-1} \frac{1-e^{-x}}{\epsilon_1 e^{-y_1}+e^{-x}-1}\cdots\frac{1-e^{-x}}{\epsilon_1\ \cdots\epsilon_r e^{-y_1-\cdots-y_r}+e^{-x}-1} \\
&= \frac{(1-e^{-x})^{r-1}}{(\epsilon_1 e^{-y_1}+e^{-x}-1)(\epsilon_1\epsilon_2 e^{-y_1-y_2}+e^{-x}-1)\cdots(\epsilon_1\ \cdots\epsilon_r e^{-y_1-\cdots-y_r}+e^{-x}-1)},
\end{aligned}$$

(note $\epsilon_i^{-1} = \epsilon_i$ because $\epsilon_i = \pm 1$). Hence the first equality holds. By the equality (11), we have

$$\begin{aligned}
&\sum_{s_1,\ldots,s_r\geq 0}\sum_{n\geq 0} C_n^{(-s_1,-s_2,\ldots,-s_r;\,\boldsymbol{\epsilon})}\frac{x^n}{n!}\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!} \\
&= e^{-x}\sum_{s_1,\ldots,s_r\geq 0}\sum_{n\geq 0} B_n^{(-s_1,-s_2,\ldots,-s_r;\,\boldsymbol{\epsilon})}\frac{x^n}{n!}\frac{y_1^{s_1}}{s_1!}\cdots\frac{y_r^{s_r}}{s_r!} \\
&= \frac{e^{-x}(1-e^{-x})^{r-1}}{(\epsilon_1 e^{-y_1}+e^{-x}-1)(\epsilon_1\epsilon_2 e^{-y_1-y_2}+e^{-x}-1)\cdots(\epsilon_1\ \cdots\epsilon_r e^{-y_1-\cdots-y_r}+e^{-x}-1)}.
\end{aligned}$$

□

2.2.1 Connection with Stirling numbers

The following theorem is an alternating extension of Theorem 2:

Theorem 7. *Take* $\mathbf{s} \in \mathbb{Z}^r$ *and* $\boldsymbol{\epsilon} \in \{\pm 1\}^r$ *as in Definition 5. Then we have the following equality for* $n \in \mathbb{N}$*:*

$$\begin{aligned}
B_n^{\mathbf{s};\,\boldsymbol{\epsilon}} &= (-1)^n \sum_{n+1\geq m_1>\cdots>m_r>0} \frac{\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(-1)^{m_1-1}(m_1-1)!\left\{ {n \atop m_1-1} \right\}}{m_1^{s_1}\cdots m_r^{s_r}} \\
C_n^{\mathbf{s};\,\boldsymbol{\epsilon}} &= (-1)^n \sum_{n+1\geq m_1>\cdots>m_r>0} \frac{\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(-1)^{m_1-1}(m_1-1)!\left\{ {n+1 \atop m_1} \right\}}{m_1^{s_1}\cdots m_r^{s_r}}. \qquad (12)
\end{aligned}$$

Proof. Using the formula (3) implies the formula

$$\begin{aligned}
&\sum_{n\geq 0} B_n^{\mathbf{s};\boldsymbol{\epsilon}}\frac{x^n}{n!} = \frac{\mathrm{Li}_{\mathbf{s}}((1-e^{-x})\epsilon_1,\,\epsilon_2,\,\ldots,\,\epsilon_r)}{1-e^{-x}}\\
&= \sum_{m_1>\cdots>m_r>0}\frac{(1-e^{-x})^{m_1-1}\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}}{m_1^{s_1}\cdots m_r^{s_r}}\\
&= \sum_{m_1>\cdots>m_r>0}\frac{(m_1-1)!\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(-1)^{m_1-1}}{m_1^{s_1}\cdots m_r^{s_r}}\frac{(e^{-x}-1)^{m_1-1}}{(m_1-1)!}\\
&= \sum_{m_1>\cdots>m_r>0}\frac{(m_1-1)!\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(-1)^{m_1-1}}{m_1^{s_1}\cdots m_r^{s_r}}\sum_{n\geq m_1-1}\left\{ {n \atop m_1-1} \right\}\frac{(-x)^n}{n!}\\
&= \sum_{n\geq 0}x^n(-1)^n\sum_{n+1\geq m_1>\cdots>m_r>0}\frac{\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(-1)^{m_1-1}(m_1-1)!\left\{ {n \atop m_1-1} \right\}}{n!m_1^{s_1}\cdots m_r^{s_r}}.
\end{aligned}$$

Then we can obtain the desired equality by comparing the coefficients x^n for each n.

By the formula (4), we have

$$\begin{aligned}
&\sum_{n\geq 0} C_n^{\mathbf{s};\boldsymbol{\epsilon}}\frac{x^n}{n!} = e^{-x}\frac{\mathrm{Li}_{\mathbf{s}}((1-e^{-x})\epsilon_1,\,\epsilon_2,\,\ldots,\,\epsilon_r)}{1-e^{-x}}\\
&= \sum_{m_1>\cdots>m_r>0}\frac{e^{-x}(1-e^{-x})^{m_1-1}\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}}{m_1^{s_1}\cdots m_r^{s_r}}\\
&= \sum_{m_1>\cdots>m_r>0}\frac{(m_1-1)!\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(-1)^{m_1-1}}{m_1^{s_1}\cdots m_r^{s_r}}\frac{e^{-x}(e^{-x}-1)^{m_1-1}}{(m_1-1)!}\\
&= \sum_{m_1>\cdots>m_r>0}\frac{(m_1-1)!\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(-1)^{m_1-1}}{m_1^{s_1}\cdots m_r^{s_r}}\sum_{n\geq m_1-1}\left\{ {n+1 \atop m_1} \right\}\frac{(-x)^n}{n!}\\
&= \sum_{n\geq 0}x^n(-1)^n\sum_{n+1\geq m_1>\cdots>m_r>0}\frac{\epsilon_1^{m_1}\cdots\epsilon_r^{m_r}(-1)^{m_1-1}(m_1-1)!\left\{ {n+1 \atop m_1} \right\}}{n!m_1^{s_1}\cdots m_r^{s_r}}.
\end{aligned}$$

Then comparing the coefficients x^n for each n results in the desired equality. □

Remark 8. D. Kim and T. Kim introduced degenerate versions of poly-Bernoulli numbers and Stirling numbers and obtain degenerate version of equality (6) in the case $r = 1$ [16, Theorem 4]. It might be interesting to consider multiple and alternating generalization of their result.

2.2.2 Connection with finite multiple zeta values

For each $\mathbf{s} \in \mathbb{Z}^r$ and $\boldsymbol{\epsilon} \in \{\pm 1\}^r$, the element $\zeta_{\mathcal{A}}(\mathbf{s};\, \boldsymbol{\epsilon}) = (\zeta_{\mathcal{A}}(\mathbf{s};\, \boldsymbol{\epsilon})_l)_{l \text{ prime}}$ of $\mathcal{A}$ is defined by

$$\zeta_{\mathcal{A}}(\mathbf{s};\, \boldsymbol{\epsilon})_l := \sum_{l > n_1 > \cdots > n_r > 0} \frac{\epsilon_1^{n_1} \cdots \epsilon_r^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{Z}/l\mathbb{Z}.$$

We call these elements of $\mathcal{A}$ *alternating finite multiple zeta values* (*AFMZVs* for short). When $\mathbf{s} \in \mathbb{N}_{>0}^r$, these are special examples with “superbity” 1 of *finite Euler sums* introduced by Zhao [19]. Sakugawa and Seki also treated these elements [18].

We have the following equality:

$$\zeta_{\mathcal{A}}(\mathbf{s};\, \boldsymbol{\epsilon}) = \mathfrak{L}_{\mathcal{A},\, \mathbf{s}}(\epsilon_1,\, \ldots,\, \epsilon_r) \tag{13}$$

for each $\mathbf{s}$ and $\boldsymbol{\epsilon}$. This is an alternating extension of the equality (8).

The following theorem is an alternating extension of Theorem 4.

Theorem 9.

1. *For $r \in \mathbb{N}_{>0}$, $\mathbf{s} = (s_1,\, \ldots,\, s_r) \in \mathbb{Z}^r$ and $\boldsymbol{\epsilon} = (\epsilon_1,\, \ldots,\, \epsilon_r) \in \{\pm 1\}^r$, we have the following congruence for each prime l:*

$$\zeta_{\mathcal{A}}(\mathbf{s};\, \boldsymbol{\epsilon})_l \equiv -C_{l-2}^{(s_1 - 1,\, s_2,\, \ldots,\, s_r);\, \boldsymbol{\epsilon}} \pmod{l}.$$

2. *For $r' \in \mathbb{N}$, $\bar{\mathbf{s}} = (1,\, \ldots, 1,\, s_1,\, \ldots,\, s_r) = \mathbb{Z}^{r+r'}$ and $\bar{\boldsymbol{\epsilon}} = (1,\, \ldots,\, 1,\, \epsilon_1,\, \ldots, \epsilon_r) \in \{\pm 1\}^{r+r'}$, the following congruence holds for each prime l:*

$$\zeta_{\mathcal{A}}(\bar{\mathbf{s}},\, \bar{\boldsymbol{\epsilon}})_l \equiv -C_{l-r'-2}^{(s_1 - 1,\, s_2,\, \ldots,\, s_r);\, \boldsymbol{\epsilon}} \pmod{l}. \tag{14}$$

Proof. First we note that the equality

$$m! \left\{ {l-1 \atop m} \right\} = \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} s^{l-1}$$

holds for each positive integer m [8, §6.1, (6.19)]. This implies the equation

$$(-1)^m m! \left\{ {l-1 \atop m} \right\} \equiv \sum_{s=1}^{m} \binom{m}{s} (-1)^s = (1-1)^m - 1 = -1 \pmod{l} \tag{15}$$

since we have $s^{l-1} \equiv 1$ for $1 \leq s \leq m < l$. By the equation (12) and the congruence (15), we have

$$\begin{aligned} C_{l-2}^{(s_1 - 1,\, s_2,\, \ldots,\, s_r);\, \boldsymbol{\epsilon}} &= (-1)^{l-2} \sum_{l-1 \geq m_1 > \cdots > m_r > 0} \frac{\epsilon_1^{m_1} \cdots \epsilon_r^{m_r} (-1)^{m_1 - 1} (m_1 - 1)! \left\{ {l-1 \atop m_1} \right\}}{m_1^{s_1 - 1} \cdots m_r^{s_r}} \\ &\equiv \sum_{l-1 \geq m_1 > \cdots > m_r > 0} \frac{\epsilon_1^{m_1} \cdots \epsilon_r^{m_r} (-1)^{m_1} (m_1)! \left\{ {l-1 \atop m_1} \right\}}{m_1^{s_1} \cdots m_r^{s_r}} \\ &\equiv \sum_{l-1 \geq m_1 > \cdots > m_r > 0} \frac{\epsilon_1^{m_1} \cdots \epsilon_r^{m_r} (-1)}{m_1^{s_1} \cdots m_r^{s_r}} = -\zeta_{\mathcal{A}}(\mathbf{s};\, \boldsymbol{\epsilon})_l \pmod{l}, \end{aligned}$$

hence we obtain the assertion (1).

We have

$$\begin{aligned}\zeta_{\mathcal{A}}(\bar{\mathbf{s}},\,\bar{\boldsymbol{\epsilon}})_l &= \sum_{l>i_1>\cdots>i_{r'}>m_1>\cdots>m_r>0} \frac{\epsilon_1^{m_1}\,\cdots\,\epsilon_r^{m_r}}{i_1\cdots i_{r'}m_1^{s_1}\,\cdots m_r^{s_r}} \\ &= \sum_{l-r'>m_1>\cdots>m_r>0} \frac{\epsilon_1^{m_1}\,\cdots\,\epsilon_r^{m_r}}{m_1^{s_1}\,\cdots m_r^{s_r}} \sum_{l>i_1>\cdots>i_{r'}>m_1} \frac{1}{i_1\cdots i_{r'}} \\ &\equiv \sum_{l-r'>m_1>\cdots>m_r>0} \frac{\epsilon_1^{m_1}\,\cdots\,\epsilon_r^{m_r}}{m_1^{s_1}\,\cdots m_r^{s_r}} \sum_{l-m_1>i_{r'}>\cdots>i_1\geq 1} \frac{(-1)^{r'}}{i_1\cdots i_{r'}}.\end{aligned}$$

The congruence of generating series

$$\begin{aligned}&\sum_{m=0}^{l-m_1-1}\left\{\sum_{l-m_1-1\geq i_m>\cdots>i_1\geq 1}\frac{(-1)^{r'}}{i_1\cdots i_m}\right\}x^{m+1} \\ &\equiv (-1)^{r'}\sum_{m=0}^{l-m_1-1}\left\{\frac{1}{(l-m_1-1)!}\sum_{l-m_1-1\geq j_1>\cdots>j_{N-m}\geq 1} j_1\cdots j_{N-m}\right\}x^{m+1} \pmod{l} \\ &= \frac{(-1)^{r'}}{(l-m_1-1)!}x(x+1)\cdots(x+l-m_1-1) = \frac{(-1)^{r'}}{(l-m_1-1)!}\sum_{m\geq 0}^{l-m_1-1}\begin{bmatrix} l-m_1 \\ m+1\end{bmatrix}x^{m+1} \\ &\equiv \frac{(-1)^{r'}}{(l-m_1-1)!}\sum_{m\geq 0}^{l-m_1-1}\begin{Bmatrix} l-m-1 \\ m_1\end{Bmatrix}x^{m+1} \pmod{l} \\ &\equiv (-1)^{r'+m_1+1}m_1!\sum_{m\geq 0}^{l-m_1-1}\begin{Bmatrix} l-m-1 \\ m_1\end{Bmatrix}x^{m+1} \pmod{l}\end{aligned}$$

(where $N = l - m_1 - 1$) follows from (5). Hence we obtain

$$\begin{aligned}\zeta_{\mathcal{A}}(\bar{\mathbf{s}},\,\bar{\boldsymbol{\epsilon}})_l &\equiv \sum_{l-r'>m_1>\cdots>m_r>0}\frac{\epsilon_1^{m_1}\,\cdots\,\epsilon_r^{m_r}(-1)^{m_1+r'+1}m_1!\left\{{l-r'-1 \atop m_1}\right\}}{m_1^{s_1}\,\cdots m_r^{s_r}} \\ &\equiv -C_{l-r'-2}^{(s_1-1,\,s_2,\,\ldots,\,s_r);\,\boldsymbol{\epsilon}}.\end{aligned}$$

□

3 Characteristic p

This subsection considers the characteristic p analogues of the notions and results in the previous section. After a review of results on positive characteristic analogues of the multiple poly-Bernoulli numbers and FMZVs of Harada [10], we generalize his results to an alternating setting. We introduce an alternating extension of the multiple poly-Bernoulli-Carlitz numbers (Definition 15) and establish their connection with Stirling-Carlitz numbers (Theorems 16). In Theorems 18, we write an alternating extension of finite multiple zeta values down in terms of special values of finite Carlitz multiple polylogarithm defined by Chang and Mishiba [5]. We obtain the relationship between alternating extensions of the multiple poly-Bernoulli-Carlitz numbers and finite multiple zeta values (Theorem 20).

3.1 Review of Harada's multiple poly-Bernoulli numbers

We fix a prime p and its power q. The symbol A denotes the polynomial ring $\mathbb{F}_q[\theta]$ in θ over the finite field $\mathbb{F}_q$ of q elements and k stands for the field $\mathbb{F}_q(\theta)$ of rational functions.

For each $n \in \mathbb{N}_{>0}$, the element $\theta^{q^n} - \theta$ of the set A_+ (of all monic polynomials) is denoted by $[n]$. Following Carlitz [3, 7], we put $D_n := [n]^{q^0}[n-1]^{q^1}\cdots[1]^{q^{n-1}} \in A_+$, $L_n := [n][n-1]\cdots[1](-1)^n \in A$ for $n \geq 1$ and $D_0 = L_0 := 1$.

For $n \in \mathbb{N}$ with the q-adic expansion $n = \sum_{j=0}^{d} \alpha_j q^j \quad (0 \leq \alpha_j < q)$, we put $\Gamma_{n+1} := \Pi(n) := \prod_{j=0}^{d} D_j^{\alpha_j} \in A_+$, which are called *the Carlitz gamma* and *the Carlitz factorial* respectively, following Carlitz [3, 7]. For each $d \in \mathbb{N}$ and $s \in \mathbb{Z}$, the sum $\sum_a \frac{1}{a^s} \in k$ (where a runs through all monic polynomials of degree d in A) is denoted by $S_d(s)$

Following Anderson and Thakur [1], we define polynomials $\mathfrak{H}_n(t,y) \in \mathbb{F}_q(t,y)$ $(n \geq 0)$ by

$$\sum_{n\geq 0} \frac{\mathfrak{H}_n(t,y)}{\Gamma_{n+1}|_{\theta=t}} x^n = \left(1 - \sum_{i\geq 0} \frac{G_i(t,y)}{D_i|_{\theta=t}} x^{q^i}\right)^{-1} \in \mathbb{F}_q(t,\, y)[[x]], \tag{16}$$

where $G_n(t,y) := \prod_{i=1}^{n}(t^{q^n} - y^{q^i})$, and we put $H_n(t) := \mathfrak{H}_n(t,\theta) \in A[t]$. These are called the *Anderson-Thakur polynomials.* Let us write

$$H_n(t) = \sum_{j=0}^{m_{n+1}} u_{n+1,\,j} t^j, \text{ with } u_{i,\,j} \in \mathrm{A} \text{ and } u_{m_{n+1}} \neq 0,$$

for $n \geq 0$. Anderson and Thakur [1] showed that we have

$$H_{n-1}^{(d)}(t)|_{t=\theta} := \left.\left(\sum_{j=0}^{m_n} u_{n,\,j}^{q^d} t^j\right)\right|_{t=\theta} = L_d^n \Gamma_n S_d(n)$$

for $d \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$. For each $\mathbf{s} = (s_1,\, \ldots,\, s_r) \in \mathbb{N}_{>0}^r$, we put

$$\mathfrak{I}_{\mathbf{s}} := \{\mathbf{j} = (j_1,\, \ldots,\, j_r) \in \mathbb{N}^r \mid 0 \leq j_i \leq \deg_t H_{s_i - 1} \text{ for } 1 \leq i \leq r.\}$$

and denote $\theta^{j_1+\cdots+j_r}$ by $\theta^{\mathbf{j}}$ for short.

We define the formal power series $e_C(z)$ called the *Carlitz exponential* as follows [3]:

$$e_C(z) := \sum_{i\geq 0} \frac{z^{q^i}}{D_i} \in k[[z]],$$

Following Chang [4], we define the power series $\mathrm{Li}_{\mathbf{s}}(z_1, \ldots, z_r)$ called *Carlitz multiple polylogarithm* for all $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ by

$$\mathrm{Li}_{\mathbf{s}}(z_1, \ldots, z_r) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in k[[z_1, \ldots, z_r]].$$

These are analogues of exponential and multiple polylogarithm functions.

Definition 10 ([10, Definition 21])**.** For each $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $\mathbf{j} = (j_1, \ldots, j_r) \in \mathfrak{J}_{\mathbf{s}}$, *multiple poly-Bernoulli-Carlitz numbers* (*MPBCNs* for short) $\mathrm{BC}_n^{\mathbf{s},\mathbf{j}}$ are elements of k defined by

$$\sum_{n\geq 0} \mathrm{BC}_n^{\mathbf{s},\mathbf{j}} \frac{z^n}{\Pi(n)} := \frac{\mathrm{Li}_{\mathbf{s}}(e_C(z)u_{s_1,j_1}, u_{s_2,j_2}, \ldots, u_{s_r,j_r})}{e_C(z)}.$$

The validity of the analogue of Proposition 1 is unclear, since Harada's multiple poly-Bernoulli-Carlitz numbers are defined only in the case when s_i are positive integers.

3.1.1 Connection with Stirling-Carlitz numbers

Let us recall the definition and properties of the positive characteristic analogues of Stirling numbers (of the second kind) introduced by H. Kaneko and Komatsu.

The *Stirling-Carlitz numbers (of the second kind)* $\left\{ {n \atop m} \right\}_C$ (n, $m \in \mathbb{N}$) are defined by

$$\frac{(e_C(z))^m}{\Pi(m)} = \sum_{n\geq 0} \left\{ {n \atop m} \right\}_C \frac{z^n}{\Pi(n)}.$$

The definition is due to Kaneko and Komatsu [13]. We note that they also introduced analogues of the first kind Stirling numbers. Kaneko and Komatsu [13, (17)] showed that the equation $\left\{ {n \atop m} \right\}_C = 0$ holds if $n < m$. Harada [10, (10)] obtained

$$\left\{ {q^n - 1 \atop q^m - 1} \right\}_C = \begin{cases} 0, & \text{if } n \neq m; \\ 1, & \text{if } n = m. \end{cases} \tag{17}$$

The following theorem is an analogue of Theorem 2 obtained by Harada [10], which describes MPBCNs as finite sums in terms of Stirling-Carlitz numbers.

Theorem 11 ([10, Theorem 27])**.** *If* $\mathbf{s}$ *and* $\mathbf{j}$ *are as in Definition 10, then the following equality in* k *holds:*

$$\mathrm{BC}_n^{\mathbf{s},\mathbf{j}} = \sum_{\log_q(n+1) \geq d_1 > \cdots > d_r \geq 0} \Gamma_{q^{d_1}} \left\{ {n \atop q^{d_1} - 1} \right\}_C \frac{u_{s_1,j_1}^{q^{d_1}} \cdots u_{s_r,j_r}^{q^{d_r}}}{L_{d_1}^{s_1} \cdots L_{d_r}^{s_r}}.$$

3.1.2 Connection with finite multiple zeta values

Characteristic p analogues of FMZVs are introduced by Chang and Mishiba [5]. Let $\mathcal{A}_k$ be the quotient ring $\prod_P(A/(P))/\bigoplus_P(A/(P))$. Here, the symbol P runs through the set $\operatorname{Spm} A$ of all monic irreducible polynomials in A. The ring $\mathcal{A}_k$ is naturally equipped with k-algebra structure.

Definition 12 ([5, §2]). For each $\mathbf{s} \in \mathbb{Z}^r$, the element $\zeta_{\mathcal{A}_k}(\mathbf{s}) = (\zeta_{\mathcal{A}_k}(\mathbf{s})_P)_{P\in\operatorname{Spm} A}$ of $\mathcal{A}_k$ is defined by

$$\zeta_{\mathcal{A}_k}(\mathbf{s})_P :\equiv \sum_{\substack{\deg P>\deg a_1>\cdots>\deg a_r\geq 0\\ a_i \text{ monic}}} \frac{1}{a_1^{s_1}\cdots a_r^{s_r}} \in A/(P);$$

these elements of $\mathcal{A}_k$ are called *finite multiple zeta values* (*FMZV* for short). We call the natural number r the *depth* of the FMZV $\zeta_{\mathcal{A}_k}(\mathbf{s})$.

Chang and Mishiba introduced the *finite Carlitz multiple polylogarithm* $\operatorname{Li}_{\mathcal{A}_k,\mathbf{s}}(z)$ (*FCMPL* for short) as a finite variant of CMPL. For $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{N}^r$ and tuple $\mathbf{a} = ((a_{P,1})_P, \ldots, (a_{P,r})_P) \in \mathcal{A}_k^r$ with $a_{P,i} \in A/P$, the value $\operatorname{Li}_{\mathcal{A}_k,\mathbf{s}}(\mathbf{a}) = (\operatorname{Li}_{\mathcal{A}_k,\mathbf{s}}(\mathbf{a})_P)_{P\in\operatorname{Spm} A}$ at $\mathbf{a} \in \mathcal{A}_k$ is given by

$$\operatorname{Li}_{\mathcal{A}_k,\mathbf{s}}(\mathbf{a})_P :\equiv \sum_{\deg P>i_1>\cdots>i_r\geq 0} \frac{a_{P,1}^{q^{i_1}}\cdots a_{P,r}^{q^{i_r}}}{L_{i_1}^{s_1}\cdots L_{i_r}^{s_r}} \in A/(P).$$

It is clear that the value $\operatorname{Li}_{\mathcal{A}_k,\mathbf{s}}(\mathbf{a})$ is independent on the choices of representatives of $(a_{P,1}), \ldots, (a_{P,r-1})$ and $(a_{P,r})$.

Chang and Mishiba obtained the following analogue of the equality (8):

Theorem 13 ([5, Theorem 3.7]). *For all $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r_{>0}$, the equations*

$$\zeta_{\mathcal{A}_k}(\mathbf{s}) = \frac{1}{\Gamma_{s_1}\cdots\Gamma_{s_r}} \sum_{\mathbf{j}\in\mathfrak{J}_{\mathbf{s}}} \theta^{\mathbf{j}} \operatorname{Li}_{\mathcal{A}_k,\mathbf{s}}(u_{s_1,j_1}, \ldots, u_{s_r,j_r})$$

hold in $\mathcal{A}_k$.

Using elements $\mathrm{BC}_n^{\mathbf{s},\mathbf{j}}$ of k, we can write down FMZVs as follows:

Theorem 14 ([10, Theorem 32]).

1. *For $\mathbf{s} \in \mathbb{N}^r_{>0}$, the congruence*

$$\zeta_{\mathcal{A}_k}(\mathbf{s})_P \equiv \frac{1}{\Gamma_{s_1}\cdots\Gamma_{s_r}} \sum_{\mathbf{j}\in\mathfrak{J}_{\mathbf{s}}} \theta^{\mathbf{j}} \sum_{d=r-1}^{\deg P-1} \frac{\mathrm{BC}_{q^d-1}^{\mathbf{s},\mathbf{j}}}{L_d\,\mathrm{BC}_{q^d-1}^{(1),(0)}} \tag{18}$$

in the residue field $A/(P)$ holds for $P \in \operatorname{Spm} A$ such that $P \nmid \Gamma_{s_i}$ for $1 \leq i \leq r$.

2. Moreover, if $r' \in \mathbb{N}$ and $\bar{\mathbf{s}} = (1, \ldots, 1, s_1, \ldots, s_r) \in \mathbb{N}_{>0}^{r+r'}$, the congruence

$$\zeta_{\mathcal{A}_k}(\bar{\mathbf{s}})_P \equiv \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\mathbf{j} \in \mathfrak{J}_{\mathbf{s}}} \theta^{\mathbf{j}} \sum_{\deg P > d_0 > \cdots > d_{r'} \geq r-1} \frac{\mathrm{BC}^{\mathbf{s},\mathbf{j}}_{q^{d_{r'}}-1}}{L_{d_0} \cdots L_{d_{r'}} \, \mathrm{BC}^{(1),(0)}_{q^{d_{r'}}-1}} \tag{19}$$

in $A/(P)$ holds for $P \in \operatorname{Spm} A$ such that $P \nmid \Gamma_{s_i}$ for $1 \leq i \leq r$.

This is an analogue of Theorem 4.

3.2 Alternating multiple poly-Bernoulli-Carlitz numbers

This section extends the results of Harada [10] explained in §3.1 to the alternating case.

Definition 15. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, tuples $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_r) \in (\overline{\mathbb{F}}_q^{\times})^r$ of invertible elements of the algebraic closure of $\mathbb{F}_q$ and $\mathbf{j} = (j_1, \ldots, j_r) \in \mathfrak{J}_{\mathbf{s}}$, the *alternating multiple poly-Bernoulli-Carlitz numbers* (*AMPBCNs* for short) $\mathrm{BC}_n^{\mathbf{s},\boldsymbol{\gamma},\mathbf{j}} \in \bar{k}$ are defined by

$$\sum_{n \geq 0} \mathrm{BC}_n^{\mathbf{s},\boldsymbol{\gamma},\mathbf{j}} \frac{z^n}{\Pi(n)} = \frac{\mathrm{Li}_{\mathbf{s}}(e_C(z)\gamma_1 u_{s_1,j_1}, \gamma_2 u_{s_2,j_2}, \ldots, \gamma_r u_{s_r,j_r})}{e_C(z)}.$$

This is an alternating extension of Definition 10.

3.2.1 Connection with Stirling-Carlitz numbers

We describe the above numbers as finite sums in terms of Stirling-Carlitz numbers, which could be regarded as an alternating extension of Theorem 11 and as an analogue of Theorem 7.

Theorem 16. *If $\mathbf{s}$, $\boldsymbol{\gamma}$ and $\mathbf{j}$ are as in Definition 15, the following equality holds:*

$$\mathrm{BC}_n^{\mathbf{s},\boldsymbol{\gamma},\mathbf{j}} = \sum_{\log_q(n+1) \geq d_1 > \cdots > d_r \geq 0} \Gamma_{q^{d_1}} \begin{Bmatrix} n \\ q^{d_1}-1 \end{Bmatrix}_C \frac{(\gamma_1 u_{s_1,j_1})^{q^{d_1}} \cdots (\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_{d_1}^{s_1} \cdots L_{d_r}^{s_r}}.$$

Proof. We have

$$\begin{aligned} &\frac{\mathrm{Li}_{\mathbf{s}}(e_C(z)\gamma_1 u_{s_1,j_1}, \gamma_2 u_{s_2,j_2}, \ldots, \gamma_r u_{s_r,j_r})}{e_C(z)} \\ &= \sum_{d_1 > \cdots > d_r \geq 0} e_C(z)^{q^{d_1}-1} \frac{(\gamma_1 u_{s_1,j_1})^{q^{d_1}} \cdots (\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_{d_1}^{s_1} \cdots L_{d_r}^{s_r}} \\ &= \sum_{d_1 > \cdots > d_r \geq 0} \left(\sum_{n \geq 0} \Gamma_{q^{d_1}} \begin{Bmatrix} n \\ q^{d_1}-1 \end{Bmatrix}_C \frac{z^n}{\Pi(n)} \frac{(\gamma_1 u_{s_1,j_1})^{q^{d_1}} \cdots (\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_{d_1}^{s_1} \cdots L_{d_r}^{s_r}} \right) \end{aligned}$$

$$= \sum_{n \geq 0} \left(\sum_{d_1 > \cdots > d_r \geq 0} \Gamma_{q^{d_1}} \left\{ {n \atop q^{d_1} - 1} \right\}_C \frac{(\gamma_1 u_{s_1, j_1})^{q^{d_1}} \cdots (\gamma_r u_{s_r, j_r})^{q^{d_r}}}{L_{d_1}^{s_1} \cdots L_{d_r}^{s_r}} \right) \frac{z^n}{\Pi(n)}$$

$$= \sum_{n \geq 0} \left(\sum_{\log_q (n+1) \geq d_1 > \cdots > d_r \geq 0} \Gamma_{q^{d_1}} \left\{ {n \atop q^{d_1} - 1} \right\}_C \frac{(\gamma_1 u_{s_1, j_1})^{q^{d_1}} \cdots (\gamma_r u_{s_r, j_r})^{q^{d_r}}}{L_{d_1}^{s_1} \cdots L_{d_r}^{s_r}} \right) \frac{z^n}{\Pi(n)};$$

the second equality follows from the definition of Stirling-Carlitz numbers and the fourth holds by the equality (17). Then the comparing coefficients of z^n for each n results in the desired equalities. $\square$

Using Theorem 16 and the equality (17), we obtain:

$$\mathrm{BC}_{q^m - 1}^{\mathbf{s}, \boldsymbol{\gamma}, \mathbf{j}} = \Gamma_{q^m} \sum_{m > d_2 > \cdots > d_r \geq 0} \frac{(\gamma_1 u_{s_1, j_1})^{q^m} \cdots (\gamma_r u_{s_r, j_r})^{q^{d_r}}}{L_m^{s_1} \cdots L_{d_r}^{s_r}}. \tag{20}$$

where $m \in \mathbb{N}$, which is a generalization of [10, Corollary 28].

3.2.2 Connection with finite alternating multiple zeta values

Definition 17. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{Z}^r$ and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r) \in (A^{\times})^r$, the *alternating finite multiple zeta value* (*AFMZV* for short) $\zeta_{\mathcal{A}_k}(\mathbf{s}\,;\boldsymbol{\epsilon}) = (\zeta_{\mathcal{A}_k}(\mathbf{s}\,;\boldsymbol{\epsilon})_P)_{P \in \operatorname{Spm} A} \in \mathcal{A}_k$ is defined by

$$\zeta_{\mathcal{A}_k}(\mathbf{s}\,;\boldsymbol{\epsilon})_P := \sum_{\substack{\deg P > \deg a_1 > \cdots > \deg a_r \geq 0 \\ a_i \text{ monic}}} \frac{{\epsilon_1}^{\deg a_1} {\epsilon_2}^{\deg a_2} \cdots {\epsilon_r}^{\deg a_r}}{a_1^{s_1} \cdots a_r^{s_r}} \in A/(P).$$

It is a characteristic p analogue of AFMZV. It immediately follows from Theorem 2.6 in Harada's paper [9] that the product of two AFMZVs are $\mathbb{F}_q$-linear combination of AFMZVs.

To obtain an alternating extension of Theorem 13, we extend the domain of FCMPLs from $\mathcal{A}_k$ to the ring $\mathcal{A}_{k'}$ defined as follows: Let q' be a power of q. We define A', k' and $\mathcal{A}_{k'}$ by the same ways as those of A, k and $\mathcal{A}_k$ but substituting q by q', and regard A, k as subrings of A', k' by canonical ways, respectively. For each element (a_P) of $\prod_P (A/(P))$ and each irreducible monic polynomial Q_1 in A' above $P_1 \in \operatorname{Spm} A$, define b_{Q_1} to be the image of a_{P_1} under the canonical embedding $A/(P_1) \hookrightarrow A'/(Q_1)$ induced by the inclusion $A \to A'$. Then the ring homomorphism from $\prod_P (A/(P))$ to $\prod_{Q \in \operatorname{Spm} A'} (A'/(Q))$ which maps (a_P) to (b_Q) induces an embedding of $\mathcal{A}_k$ into $\mathcal{A}_{k'}$.

The FCMPLs can be extended to the multivariable functions on $\mathcal{A}_{k'}$; for tuples $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{N}^r$ and $\mathbf{b} = ((b_{Q,1}), \ldots, (b_{Q,r})) \in \mathcal{A}_{k'}^r$ with $(b_{Q,i}) \in A'/Q$, we define the value $\mathrm{Li}_{\mathcal{A}_{k'}, \mathbf{s}}(\mathbf{b}) = (\mathrm{Li}_{\mathcal{A}_k, \mathbf{s}}(\mathbf{b})_Q)_{Q \in \operatorname{Spm} A'}$ by

$$\mathrm{Li}_{\mathcal{A}_k, \mathbf{s}}(\mathbf{b})_Q := \sum_{\deg P > i_1 > \cdots > i_r \geq 0} \frac{b_{Q,1}^{q^{i_1}} \cdots b_{Q,r}^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in A'/(Q),$$

where the symbol P in the right hand side stands for the monic irreducible polynomial in A which is divided by Q in A'.

So far we put $q' := q^{q-1}$. We note that, for $\epsilon \in A^\times = \mathbb{F}_q^\times$, the set $A'^\times$ contains all $(q-1)$-th roots of ϵ.

Theorem 18. *Let $q' := q^{q-1}$ and let $\boldsymbol{s}$ and $\boldsymbol{\epsilon}$ be as in the Definition 17 and $\gamma_1, \ldots, \gamma_r \in A'^\times$ be $(q-1)$-th roots of $\epsilon_1, \ldots, \epsilon_r$, respectively. Then the equality*

$$\zeta_{\mathcal{A}_k}(\mathbf{s}\,;\boldsymbol{\epsilon}) = \frac{1}{\gamma_1\Gamma_{s_1}\cdots\gamma_r\Gamma_{s_r}}\sum_{\mathbf{j}\in\mathfrak{I}_{\mathbf{s}}}\theta^{\mathbf{j}}\mathrm{Li}_{\mathcal{A}_k,\mathbf{s}}(\gamma_1 u_{s_1,j_1},\ldots,\gamma_r u_{s_r,j_r}) \tag{21}$$

in $\mathcal{A}_k$ holds.

Though the elements $\gamma_1, \ldots, \gamma_r$ are not in $\mathcal{A}_k$ but in $\mathcal{A}_{k'}$, it is clear that the right hand side of the equality (21) is in $\mathcal{A}_k$ as the left hand side is.

Proof. It is sufficient to show that the congruences

$$\zeta_{\mathcal{A}_k}(\mathbf{s}\,;\boldsymbol{\epsilon})_P \equiv \frac{1}{\gamma_1\Gamma_{s_1}\cdots\gamma_r\Gamma_{s_r}}\sum_{\mathbf{j}\in\mathfrak{I}_{\mathbf{s}}}\theta^{\mathbf{j}}\mathrm{Li}_{\mathcal{A}_k,\mathbf{s}}(\gamma_1 u_{s_1,j_1},\ldots,\gamma_r u_{s_r,j_r})_P \pmod{P}$$

in $A'/(P) \simeq \prod_{Q|P} A'/(Q)$ hold for all but finite irreducible polynomial P in A. Let P be an element of $\operatorname{Spm} A$ such that $P \nmid \Gamma_{s_i}$ for all i. We have the equalities and congruences:

$$\begin{aligned}
&\zeta_{\mathcal{A}_k}(\mathbf{s}\,;\boldsymbol{\epsilon})_P \\
&= \sum_{\deg P > d_1 > \cdots > d_r \geq 0} \epsilon_1^{d_1} S_{d_1}(s_1)\cdots\epsilon_r^{d_r} S_{d_r}(s_r) \\
&\equiv \frac{1}{\Gamma_{s_1}\cdots\Gamma_{s_r}} \sum_{\deg P > d_1 > \cdots > d_r \geq 0} \frac{\epsilon_1^{d_1} H_{s_1-1}^{(d_1)}(\theta)\cdots\epsilon_r^{d_r} H_{s_r-1}^{(d_r)}(\theta)}{L_{d_1}^{s_1}\cdots L_{d_r}^{s_r}} \pmod{P} \\
&= \frac{1}{\Gamma_{s_1}\cdots\Gamma_{s_r}} \sum_{\deg P > d_1 > \cdots > d_r \geq 0}\sum_{\mathbf{j}\in\mathfrak{I}_{\mathbf{s}}} \theta^{\mathbf{j}} \frac{\epsilon_1^{d_1} u_{s_1,j_1}^{q^{d_1}}\cdots\epsilon_r^{d_r} u_{s_r,j_r}^{q^{d_r}}}{L_{d_1}^{s_1}\cdots L_{d_r}^{s_r}} \\
&\equiv \frac{1}{\gamma_1\Gamma_{s_1}\cdots\gamma_r\Gamma_{s_r}} \sum_{\mathbf{j}\in\mathfrak{I}_{\mathbf{s}}} \theta^{\mathbf{j}} \sum_{\deg P > d_1 > \cdots > d_r \geq 0} \frac{(\gamma_1 u_{s_1,j_1})^{q^{d_1}}\cdots(\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_{d_1}^{s_1}\cdots L_{d_r}^{s_r}} \pmod{P} \\
&= \frac{1}{\gamma_1\Gamma_{s_1}\cdots\gamma_r\Gamma_{s_r}} \sum_{\mathbf{j}\in\mathfrak{I}_{\mathbf{s}}} \theta^{\mathbf{j}} \mathrm{Li}_{\mathcal{A}_k,\mathbf{s}}(\gamma_1 u_{s_1,j_1},\ldots,\gamma_r u_{s_r,j_r})_P,
\end{aligned}$$

where the second congruence is by equations $\gamma_i^{q^d} = \epsilon_i^d \gamma_i$ which holds for $1 \leq i \leq r$ and $d \geq 0$. □

The following lemma is an alternating extension of Lemma 31 in Harada's paper [10].

Lemma 19. *If we take* $\mathbf{s}$, $\boldsymbol{\gamma}$ *and* $\mathbf{j}$ *as in Definition 15, then the recursive formula*

$$\mathrm{BC}^{\mathbf{s},\boldsymbol{\gamma},\mathbf{j}}_{q^m-1} = \mathrm{BC}^{s_1,\gamma_1,\, j_1}_{q^m-1} \sum_{d=r-2}^{m-1} \frac{1}{\Gamma_{q^d}} \mathrm{BC}^{\mathbf{s}^*,\boldsymbol{\gamma}^*,\mathbf{j}^*}_{q^d-1}$$

holds for $m \in \mathbb{N}_{>0}$ *where* $\mathbf{s}^* := (s_2, \ldots, s_r)$, $\boldsymbol{\gamma}^* := (\gamma_2, \ldots \gamma_r)$ *and* $\mathbf{j}^* := (j_2, \ldots, j_r)$.

Proof. The assertion is obtained as follows:

$$\begin{aligned}
\mathrm{BC}^{\mathbf{s},\boldsymbol{\gamma},\mathbf{j}}_{q^m-1} &= \sum_{m>d_2>\cdots>d_r\geq 0} \Gamma_{q^m} \frac{(\gamma_1 u_{s_1,j_1})^{q^m} \cdots (\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_m^{s_1} \cdots L_{d_r}^{s_r}} \\
&= \Pi(q^m-1) \frac{(\gamma_1 u_{s_1,\, j_1})^{q^m}}{L_m^{s_1}} \sum_{m>d_2>\cdots>d_r\geq 0} \frac{(\gamma_2 u_{s_2,\, j_2})^{q^{d_2}} \cdots (\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_{d_2}^{s_2} \cdots L_{d_r}^{s_r}} \\
&= \mathrm{BC}^{s_1,\gamma_1,\, j_1}_{q^m-1} \sum_{m>d_2>\cdots>d_r\geq 0} \frac{(\gamma_2 u_{s_2,\, j_2})^{q^{d_2}} \cdots (\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_{d_2}^{s_2} \cdots L_{d_r}^{s_r}} \\
&= \mathrm{BC}^{s_1,\gamma_1,\, j_1}_{q^m-1} \sum_{d_2=r-2}^{m-1} \frac{1}{\Gamma_{q^{d_2}}} \sum_{d_2>d_3>\cdots>d_r\geq 0} \Gamma_{q^{d_2}} \frac{(\gamma_2 u_{s_2,\, j_2})^{q^{d_2}} \cdots (\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_{d_2}^{s_2} L_{d_3}^{s_3} \cdots L_{d_r}^{s_r}} \\
&= \mathrm{BC}^{s_1,\gamma_1,\, j_1}_{q^m-1} \sum_{d=r-2}^{m-1} \frac{1}{\Gamma_{q^d}} \mathrm{BC}^{\mathbf{s}^*,\boldsymbol{\gamma}^*,\mathbf{j}^*}_{q^d-1},
\end{aligned}$$

where the first, third and the fifth equalities are because of the equality (20). ☐

The following theorem could be seen as an alternating extension of Theorem 14 and also as an analogue of Theorem 9.

Theorem 20. *We put* $q' := q^{q-1}$. *The following formulas hold.*

1. *Taking* $\boldsymbol{s}$ *and* $\boldsymbol{\epsilon}$ *be as in the Definition 17 and* $(q-1)$*-th roots* $\gamma_1, \ldots, \gamma_r \in \mathbb{F}_{q'}$ *of* $\epsilon_1, \ldots, \epsilon_r$*, respectively, then the congruences*

$$\zeta_{\mathcal{A}_k}(\mathbf{s}\,;\,\boldsymbol{\epsilon})_P \equiv \frac{1}{\gamma_1\Gamma_{s_1}\cdots\gamma_r\Gamma_{s_r}} \sum_{\mathbf{j}\in\mathfrak{J}_{\mathbf{s}}} \theta^{\mathbf{j}} \sum_{d=r-1}^{\deg P-1} \frac{\mathrm{BC}^{\mathbf{s},\boldsymbol{\gamma},\mathbf{j}}_{q^d-1}}{L_d\,\mathrm{BC}_{q^d-1}} \tag{22}$$

   in the residue ring $A'/(P)$ *hold for all* $P \in \operatorname{Spm} A$ *such that* $P \nmid \Gamma_{s_i}$ *for* $1 \leq i \leq r$.

2. *For* $r' \in \mathbb{N}$*, we put* $\bar{\mathbf{s}} = (1, \ldots, 1, s_1, \ldots, s_r) \in \mathbb{N}^{r+r'}$ *and* $\bar{\boldsymbol{\epsilon}} = (1, \ldots, 1, \epsilon_1, \ldots, \epsilon_r) \in (\mathbb{F}_q^{\times})^{r+r'}$. *Then the congruences*

$$\zeta_{\mathcal{A}_k}(\bar{\mathbf{s}}\,;\,\bar{\boldsymbol{\epsilon}})_P \equiv \frac{1}{\gamma_1\Gamma_{s_1}\cdots\gamma_r\Gamma_{s_r}} \sum_{\mathbf{j}\in\mathfrak{J}_{\mathbf{s}}} \theta^{\mathbf{j}} \sum_{\deg P>d_0>\cdots>d_{r'}\geq r-1} \frac{\mathrm{BC}^{\mathbf{s},\boldsymbol{\gamma},\mathbf{j}}_{q^{d_{r'}}-1}}{L_{d_0}\cdots L_{d_{r'}}\,\mathrm{BC}_{q^{d_{r'}}-1}}$$

   in $A'/(P)$ *hold for all* $P \in \operatorname{Spm} A$ *such that* $P \nmid \Gamma_{s_i}$ *for* $1 \leq i \leq r$.

Proof. We have

$$\begin{aligned}
\gamma_1\Gamma_{s_1}\cdots\gamma_r\Gamma_{s_r}\zeta_{\mathcal{A}_k}(\mathbf{s}\,;\,\boldsymbol{\epsilon})_P &= \sum_{\mathbf{j}\in\mathfrak{J}_{\mathbf{s}}}\theta^{\mathbf{j}}\mathrm{Li}_{\mathcal{A}_k,\,\mathbf{s}}(\gamma_1 u_{s_1,\,j_1},\ldots,\,\gamma_r u_{s_r,\,j_r})_P \\
&= \sum_{\mathbf{j}\in\mathfrak{J}_{\mathbf{s}}}\theta^{\mathbf{j}}\sum_{\deg P>d_1>\cdots>d_r\geq 0}\frac{(\gamma_1 u_{s_1,j_1})^{q^{d_1}}\cdots(\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_{d_1}^{s_1}\cdots L_{d_r}^{s_r}} \\
&= \sum_{\mathbf{j}\in\mathfrak{J}_{\mathbf{s}}}\theta^{\mathbf{j}}\sum_{d=r-1}^{\deg P-1}\frac{1}{\Gamma_{q^d}}\sum_{d>d_2>\cdots>d_r\geq 0}\Gamma_{q^d}\frac{(\gamma_1 u_{s_1,j_1})^{q^{d}}\cdots(\gamma_r u_{s_r,j_r})^{q^{d_r}}}{L_{d}^{s_1}\cdots L_{d_r}^{s_r}} \\
&= \sum_{\mathbf{j}\in\mathfrak{J}_{\mathbf{s}}}\theta^{\mathbf{j}}\sum_{d=r-1}^{\deg P-1}\frac{1}{\Gamma_{q^d}}\mathrm{BC}_{q^d-1}^{\mathbf{s},\,\boldsymbol{\gamma},\,\mathbf{j}} = \sum_{\mathbf{j}\in\mathfrak{J}_{\mathbf{s}}}\theta^{\mathbf{j}}\sum_{d=r-1}^{\deg P-1}\frac{\mathrm{BC}_{q^d-1}^{\mathbf{s},\,\boldsymbol{\gamma},\,\mathbf{j}}}{L_d\,\mathrm{BC}_{q^d-1}},
\end{aligned}$$

for such a P. Hence we obtain the first assertion. The last equality is from the following equation

$$\frac{\mathrm{BC}_{q^d-1}}{\Gamma_{q^d}} = \frac{1}{L_d}$$

which holds for each $d\geq 0$; this is from the equality (20).

To show the second assertion, take P such that $P\nmid\Gamma_{s_i}$ for all i. If $\bar{\boldsymbol{\gamma}}$ stands for the tuple $(1,\,\ldots,1,\,\gamma_1,\ldots,\gamma_r)\in(\mathbb{F}_{q'}^{\times})^{r+r'}$, the assertion (1) of Theorem 20 yields the equality

$$\begin{aligned}
\zeta_{\mathcal{A}_k}(\bar{\mathbf{s}}\,;\,\bar{\boldsymbol{\epsilon}})_P &= \frac{1}{\Gamma_1^{r'}\gamma_1\Gamma_{s_1}\cdots\gamma_r\Gamma_{s_r}}\sum_{\mathbf{j}\in\mathfrak{J}_{\bar{\mathbf{s}}}}\theta^{\mathbf{j}}\sum_{d_0=r-1}^{\deg P-1}\frac{\mathrm{BC}_{q^{d_0}-1}^{\bar{\mathbf{s}},\,\bar{\boldsymbol{\gamma}},\,\bar{\mathbf{j}}}}{L_{d_0}\,\mathrm{BC}_{q^{d_0}-1}} \\
&= \frac{1}{\gamma_1\Gamma_{s_1}\cdots\gamma_r\Gamma_{s_r}}\sum_{\mathbf{j}\in\mathfrak{J}_{\bar{\mathbf{s}}}}\theta^{\mathbf{j}}\sum_{d_0=r-1}^{\deg P-1}\frac{\mathrm{BC}_{q^{d_0}-1}^{\bar{\mathbf{s}},\,\bar{\boldsymbol{\gamma}},\,\bar{\mathbf{j}}}}{\Pi(q^{d_0}-1)}.
\end{aligned}$$

By applying the Lemma 19, we can calculate as follows for $d_0\geq r'+r-1$:

$$\begin{aligned}
\frac{\mathrm{BC}_{q^{d_0}-1}^{\bar{\mathbf{s}},\,\bar{\boldsymbol{\gamma}},\,\bar{\mathbf{j}}}}{\Pi(q^{d_0}-1)} &= \frac{\mathrm{BC}_{q^{d_0}-1}^{(1),(1),(0)}}{\Gamma_{q^{d_0}}}\sum_{d_1=r+r'-2}^{d_0-1}\frac{\mathrm{BC}_{q^{d_1}-1}^{\bar{\mathbf{s}}^*,\,\bar{\boldsymbol{\gamma}}^*,\,\bar{\mathbf{j}}^*}}{\Gamma_{q^{d_1}}} \\
&= \frac{\mathrm{BC}_{q^{d_0}-1}^{(1),(1),(0)}}{\Gamma_{q^{d_0}}}\sum_{d_1=r+r'-2}^{d_0-1}\frac{\mathrm{BC}_{q^{d_1}-1}^{(1),(1),(0)}}{\Gamma_{q^{d_1}}}\sum_{d_2=r+r'-2}^{d_1-1}\frac{\mathrm{BC}_{q^{d_2}-1}^{\bar{\mathbf{s}}^{**},\,\bar{\boldsymbol{\gamma}}^{**},\,\bar{\mathbf{j}}^{**}}}{\Gamma_{q^{d_2}}}.
\end{aligned}$$

Repeating this procedure, we obtain

$$\frac{\mathrm{BC}^{\bar{\mathbf{s}},\bar{\boldsymbol{\gamma}},\bar{\mathbf{j}}}_{q^{d_0}-1}}{\Pi(q^{d_0}-1)} = \sum_{d_0>\cdots>d_{r'}>r-1}\left(\prod_{i=0}^{r'}\frac{\mathrm{BC}^{(1),(1),(0)}_{q^{d_i}-1}}{\Gamma_{q^{d_i}}}\right)\frac{\mathrm{BC}^{\mathbf{s},\boldsymbol{\gamma},\mathbf{j}}_{q^{d_{r'}}-1}}{\mathrm{BC}^{(1),(0)}_{q^{d_{r'}}-1}}$$

$$= \sum_{d_0>\cdots>d_{r'}>r-1}\left(\prod_{i=0}^{r'}\frac{1}{L_{d_i}}\right)\frac{\mathrm{BC}^{\mathbf{s},\boldsymbol{\gamma},\mathbf{j}}_{q^{d_{r'}}-1}}{\mathrm{BC}^{(1),(0)}_{q^{d_{r'}}-1}}.$$

Hence we obtain the desired equality. □

4 Acknowledgment

The author is deeply grateful to Professor H. Furusho; without his profound instruction and continuous encouragement the present paper would never be accomplished. He is also grateful to R. Harada, who guided him in the research on positive characteristic arithmetic, and to Y. Mishiba for his useful comments on the manuscript. He thanks the reviewers for offering valuable suggestions.

A Finite multiple zeta values with non-all-positive indices

In the characteristic 0 case, it is known that a FMZV with an integer index can be expressed as $\mathbb{Q}$-linear combination of 1 and FMZV's with all-positive indices (cf. Kaneko [15]). Here, we show that the same is true in the case of characteristic p (Theorem 23).

We recall that the sum $\sum_a \frac{1}{a^s}$ (where a runs through all monic polynomials of degree d in A) is denoted by $S_d(s)$ (cf. §3.1). The following proposition is a special case of the result of Goss [6, Proposition 4.1]:

Proposition 21. *For $s \in \mathbb{N}_{\geq 0}$, there is $N(s) \in \mathbb{N}$ such that $S_d(-s) = 0$ for $d \geq N(s)$.*

Proof. If $s = 0$, it is sufficient to put $N(s) = 1$ since a number of elements of the set of all monic polynomials of degree d is q^d for $d \in \mathbb{N}$.

For general s, it is enough to out $N(s) := \max\{N(t) + 1 \mid t < s\}$. Indeed, for $d \geq N(s)$ we have

$$S_d(-s) = \sum_{\substack{\deg a = d-1 \\ a \text{ monic} \\ b \in \mathbb{F}_q}} (\theta a + b)^s = \sum_{\substack{\deg a = d-1 \\ a \text{ monic} \\ b \in \mathbb{F}_q}} \sum_{t=0}^{s} (\theta a)^t b^{s-t} \binom{s}{t}$$

$$= \theta^s \sum_{\substack{\deg a = d-1 \\ a \text{ monic}}} a^s \sum_{b \in \mathbb{F}_q} b^0 = 0.$$

□

We need the following lemma:

Lemma 22. *For a tuple* $(s_1, \ldots, s_r) \in \mathbb{Z}^r$, $0 \leq M \leq r$ *and* $N \in \mathbb{N}$, *the element*

$$\left(\sum_{\deg P > d_1 > \cdots > d_M \geq N > d_{M+1} > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) \right)_P$$

of $\mathcal{A}_k$ *is a* k*-linear combination of FMZVs with depth equal to or less than* r*.*

Proof. Induction proves this for depth r. If $M < r$, we have

$$\begin{aligned} &\sum_{\deg P > d_1 > \cdots > d_M \geq N > d_{M+1} > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) \\ = &\sum_{N > d_{M+1} > \cdots > d_r \geq 0} S_{d_{M+1}}(s_{M+1}) \cdots S_{d_r}(s_r) \times \sum_{\deg P > d_1 > \cdots > d_M \geq N} S_{d_1}(s_1) \cdots S_{d_M}(s_M), \end{aligned}$$

hence the induction hypothesis implies the desired result. In the case $M = r$, the equation

$$\begin{aligned} &\sum_{\deg P > d_1 > \cdots > d_r \geq N} S_{d_1}(s_1) \cdots S_{d_r}(s_r) \\ = &\zeta_{\mathcal{A}}(s_1, \ldots, s_r)_P - \sum_{M'=0}^{r-1} \left(\sum_{\deg P > d_1 > \cdots > d_{M'} \geq N > d_{M'+1} > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) \right) \end{aligned}$$

holds. Therefore we have the result.

□

Theorem 23. *An FMZV with an integer index can be expressed as a* k*-linear combination of* 1 *and FMZVs with all positive indices.*

Proof. We use induction on depth. We consider a FMZV $\zeta_{\mathcal{A}}(s_1, \ldots, s_r)$. If $s_1 \leq 0$, then Proposition 21 implies that $\zeta_{\mathcal{A}}(s_1, \ldots, s_r) \in k$. Assume that $s_{M+1} \leq 0$ for some M with $1 \leq M \leq r-1$. Then we can take $N \in \mathbb{N}$ such that $S_d(s_{M+1}) = 0$ for $d \geq M$. Then we have

$$\zeta_{\mathcal{A}}(s_1, \ldots, s_r) = \left(\sum_{\deg P > d_1 > \cdots > d_M \geq N > d_{M+1} > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) \right)_P .$$

Hence, the desired result follows from the induction hypothesis and Lemma 22. □

Remark 24.

1. similarly, we can show that an AFMZV with an integer index can be expressed as a k-linear combination of AFMZVs with all positive indices.

2. If $s_i \leq 0$ for all i, a FMZV $\zeta_{\mathcal{A}}(s_1, \ldots, s_r)$ is in A.

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2020 *Mathematics Subject Classification*: Primary 11B68; Secondary 11M32, 11M38.
Keywords: alternating multiple poly-Bernoulli number, Stirling number, alternating finite multiple zeta value, function field.

Received May 4 2021; revised versions received May 12 2022; June 21 2022. Published in *Journal of Integer Sequences*, June 22 2022.

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