Journal of Integer Sequences, Vol. 25 (2022),
Article 22.7.8

A Motzkin-Inspired Bijection

Peter Matsakis and Sam Vandervelde
Proof School
San Francisco, CA 94103
USA
pematsak@uwaterloo.ca
svandervelde@proofschool.org

Abstract

In this paper we briefly revisit Motzkin numbers before noting a novel manifestation of Motzkin numbers among certain fault-free tableaux. We next describe two other instances involving tableaux in which Motzkin numbers arise and present a bijection between them, yielding a new proof of the fact that standard Young tableaux with three or fewer rows are enumerated by the Motzkin numbers. We then extend the method of bijection to prove a new result that generalizes the original observation to larger tableaux. This approach motivates a natural way to generalize the Motzkin numbers, namely, as the sequences that result from fixing the height of the grid and letting the width vary. Finally, we note a few properties of these numbers and state two conjectures about their asymptotics.

1 Introduction

The Motzkin numbers (sequence A001006 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [9]) fall into that curious category of "sequences worthy of recognition that almost nobody recognizes." The sequence begins

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, \ldots,$$

and its properties (including parity, asymptotics, and a surprising connection to Riordan numbers) have been studied extensively by Aigner, Bernhart, Deutsch, and Sagan [1, 2, 3], among others. They are often overshadowed, however, by their role as a companion sequence to the more familiar Catalan numbers, in the sense that one expects Motzkin numbers to appear in settings that give rise to Catalan numbers, as Donaghey and Shapiro [4] amply illustrated. For instance, the nth Catalan number C_n counts the number of Dyck paths from $(0,0)$ to (n,n), consisting of unit steps either to the right or up and never entering the region above the line $y = x$. If we lengthen our horizontal and vertical steps to 2 units, and also introduce a diagonal step 1 unit up and to the right, then the number of paths from $(0,0)$ to (n,n) that avoid the region above $y = x$ is given by the nth Motzkin number, which we denote as M_n. Fig. 1 shows the four paths of lengthened steps for $n = 3$.

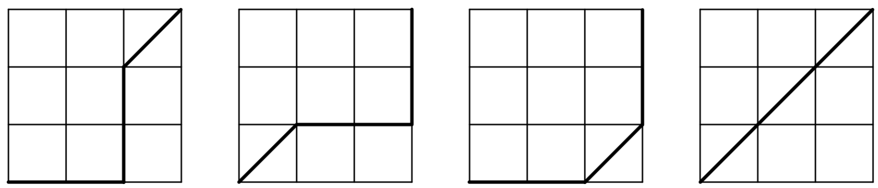


Figure 1: The four "Motzkin paths" corresponding to $M_3 = 4$.

The Motzkin numbers can be defined recursively by setting

$$M_0 = M_1 = 1, \qquad M_{n+1} = M_n + \sum_{j=0}^{n-1} M_j M_{n-1-j}, \quad n \geq 1.$$

Donaghey and Shapiro [4] note that they also relate to the Catalan numbers via the equalities

$$M_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} C_j, \qquad C_{n+1} = \sum_{j=0}^{n} \binom{n}{j} M_j,$$

and Donaghey [5] presents this fact visually as well.

Our purpose in this paper is to revisit the Motzkin numbers via a novel manifestation involving a variation on standard Young tableaux (SYT). We will next provide a new bijection between a known occurrence of Motzkin numbers and SYT with at most three rows, which differs from the one Eu [6] devised. Finally, we generalize this construction to obtain a new theorem involving the number of SYT whose shape has columns of heights 1, k, and $k+1$. The sequences of numbers that arise are extensions of Motzkin numbers, though ones unlike the generalization Sun [11] first introduced, which Wang and Zhang [12] later explored.

2 Multiple occurrence tableaux

Recall that a *standard Young tableau* is a labeling of a Young diagram for a given partition λ of n with the integers from 1 to n in which the numbers across each row and down each column are increasing. We can compute the number of such labelings via the hook length formula (which Greene, Nijenhuis, and Wilf [8] proved probabilistically); a routine

1	1	2	2
3	3	4	4

1	1	2	3
2	3	4	4

1	1	2	4
2	3	3	4

1	1	3	3
2	2	4	4

1	1	3	4
2	2	3	4

1	2	2	3
1	3	4	4

1	2	2	4
1	3	3	4

1	2	3	3
1	2	4	4

1	2	3	4
1	2	3	4

Figure 2: The nine 2-multiple occurrence tableaux for a 4×2 grid.

application reveals that the number of SYT for the partition $\lambda = \{n, n\}$ (labelings of an $n \times 2$ grid) is the Catalan number C_n.

Now consider a partition λ of N, where $N = kn$. One might naturally consider a related situation, in which we seek to label the cells of the Young diagram for λ with k copies of each of the positive integers from 1 to n, so that the numbers across each row and down each column are nondecreasing. We refer to such labelings as *k-multiple occurrence tableaux* or *k-MOTs*. To begin, we focus our attention on the partition $\lambda = \{n, n\}$. Fig. 2 depicts the nine 2-MOTs in the case $n = 4$. Not surprisingly, we claim that, in general, these tableaux are enumerated by Motzkin numbers.

Proposition 1. *The number of 2-multiple occurrence tableaux for the partition $\lambda = \{n, n\}$ is equal to M_n.*

Proof. We construct a Motzkin path associated with a given 2-MOT for an $n \times 2$ grid as follows. For a given integer j in the range $1 \leq j \leq n$, the two js appear in either the top row, the bottom row, or there is a single j in each row. Let these options correspond to the jth step in our path moving 2 units right, 2 units up, or diagonally up and right.

There are n squares per row, which ensures that the path ends at (n, n). Furthermore, the numbers increase down each column, implying that as we place the numbers from 1 to n into the 2-MOT, the occupied squares on the lower row never surpass those on the upper row, which means that the associated path does not enter the region above $y = x$. Finally, by reversing this process, a Motzkin path clearly gives rise to a unique 2-MOT, so we have a bijection. Fig. 3 illustrates this association. $\square$

Motzkin numbers surface in a less obvious fashion once we introduce the notion of fault-free labelings.

Definition 2. A *fault line* of a 2-MOT on an $n \times 2$ grid is a vertical division along an internal grid line such that all numbers to the left of the fault line are strictly less than all numbers to its right.

1	1	2	4	4
2	3	3	5	5

$\longleftrightarrow$

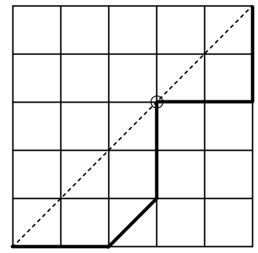


Figure 3: The Motzkin path associated with a particular 2-MOT, with fault line indicated.

The fault line in the 2-MOT shown on the left in Fig. 3 is indicated by double vertical lines. A fault-free labeling has no fault lines, of course; we list the four such 2-MOTs for a 5×2 grid below.

1	1	2	2	3
3	4	4	5	5

1	1	2	2	4
3	3	4	5	5

1	1	2	3	3
2	4	4	5	5

1	1	2	3	4
2	3	4	5	5

Proposition 3. *The number of fault-free 2-multiple occurrence tableaux for the partition $\lambda = \{n, n\}$ is equal to M_{n-2}.*

Proof. The bijection described in the proof of Proposition 1 indicates that a fault line corresponds to a point along the associated Motzkin path where the point touches the line $y = x$, highlighted in Fig. 3 by a circle. Therefore a fault-free 2-MOT must correspond to a path that begins with a step right, concludes with a step up, and stays on or below the line $y = x - 2$ in between. Such paths are clearly in one-to-one correspondence with the M_{n-2} Motzkin paths from $(2, 0)$ to $(n, n-2)$. $\square$

3 A bijection with standard Young tableaux

According to the OEIS [9], Motzkin numbers also count the number of standard Young tableaux (SYT) of height 3 or less, meaning that the Young diagrams involved all have columns of height 1, 2, or 3. This observation prompted a search for a bijection between such SYT and 2-MOTs on an $n \times 2$ grid. We now present this bijection, which complements Gouyou-Beauchamps's [7] work.

Proposition 4. *The number of standard Young tableaux of size n with three or fewer rows is equal to M_n.*

Proof. We construct a bijection between a 2-MOT on an $n \times 2$ grid and an SYT with three or fewer rows as follows. Every j in the range $1 \leq j \leq n$ either appears in only the top row of the 2-MOT, only the bottom row, or both. For each j from 1 to n, in order, if both js appear in the top row, put j in the top of a grid with two rows, working from left to right. Likewise, if both js appear in the bottom row, put j in the bottom row of the tableau. If j

appears in both the top and bottom rows of the 2-MOT, keep j in reserve. The occupied squares on the lower row never surpass those on the upper row so this will be a valid SYT with some numbers missing. After this, append the reserve (in order) to the first row of the SYT. If the resulting tableau is valid, we are done.

$$\begin{array}{cccccc} 1 & 1 & 2 & 2 & 3 & 5 \\ 3 & 4 & 4 & 5 & 6 & 6 \end{array} \longrightarrow \begin{array}{cc} 1 & 2 \\ 4 & 6 \end{array} + \begin{array}{cc} 3 & 5 \\ & \end{array} \longrightarrow \begin{array}{cccc} 1 & 2 & 3 & 5 \\ 4 & 6 & & \end{array}$$

However, some elements from the reserve may be less than elements in the first row that are not from the reserve, so that the top row is not increasing. If so, we find the first element in the top row that is less than its neighbor to the left and move it to the left until it reaches the leftmost element greater than it. Then we push all numbers in that column down one space, to make room for the moved element. We repeat this process until the top row is ascending. Note that the reserve numbers (the only numbers that were moved) stay on the top row, since they were in order to begin with. Because of this, the tableau will only have three rows at this point. Furthermore, every column will be increasing, since a reserve element only moves into a column when it is less than the element it displaces, and the columns began in increasing order. We are now done constructing the top row of the new SYT.

$$\begin{array}{cccccc} 1 & 1 & 2 & 3 & 3 & 5 \\ 2 & 4 & 4 & 5 & 6 & 6 \end{array} \longrightarrow \begin{array}{cccc} 1 & 3 & 2 & 5 \\ 4 & 6 & & \end{array} \longrightarrow \begin{array}{ccc} 1 & 2 & 5 \\ 4 & 3 & \\ & 6 & \end{array}$$

For the middle row, we use a similar process. Starting on the left and moving right, for each out-of-place element in the second row use the same process described above with one modification. Now, when moving an element, pull the elements below it back up. We continue until the entire middle row is ascending. Once again the columns are increasing downwards since they started out ascending. We also know that the only potentially out-of-place elements are from the original top row, since the middle row started out ascending. Because of this, any column of size 3 must have an element from the top row in the middle of it.

$$\begin{array}{ccc} 1 & 2 & 5 \\ 4 & 3 & \\ & 6 & \end{array} \longrightarrow \begin{array}{ccc} 1 & 2 & 5 \\ 3 & & \\ 4 & 6 & \end{array} \longrightarrow \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 6 & \\ 4 & & \end{array}$$

Observe that all the out-of-place elements are from the top row, which was ascending at the start, and we move from left to right. This means that, as before, no two elements from the original top row will end up in the same column and our tableau will not have any columns of size four. Observe that all the elements in the bottom row came from the middle row and are therefore ascending.

Finally, we move all the elements in the bottom row as far left as possible without changing the order. Since the current middle row is increasing, moving an element in the bottom row to the left only puts it below smaller elements than before, so the columns remain ascending. Recall that the bottom row was ascending before we moved the elements, thus the bottom row is ascending as well, while all the columns remain ascending. We have now finished with the process of turning a 2-MOT on an $n \times 2$ grid into a standard Young tableau. Fig. 4 illustrates the entire process.

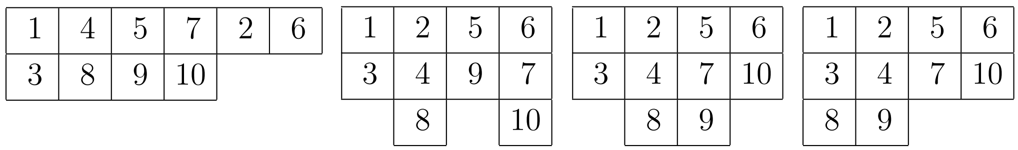
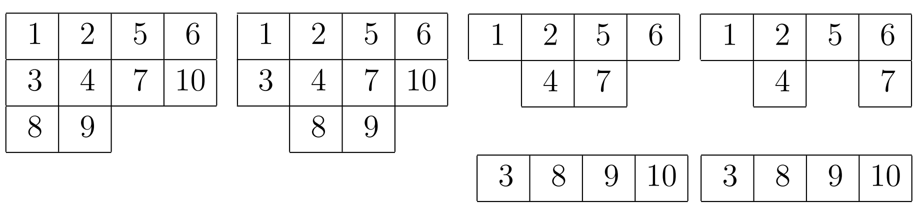


Figure 4: Turning a 2-MOT into a standard Young tableau.

We will show that the above process is uniquely reversible, starting with the final step. In order, from right to left, move each element j in the bottom row right so that j lands below the rightmost element of the middle row satisfying two conditions: it does not already have an element below it and is less than j. Recall that all the elements in the bottom row were from the original middle row. Therefore, taking the lowest element of each column should yield an ascending sequence. Moving every j below the rightmost element less than j is the only way to do this. Moving j further right would make a non-ascending column, which did not exist at the beginning of the final step. Not moving j right far enough would contradict the fact that the second row started out ascending.



Now, we have to undo the second-to-last step. Since the elements at the bottom of each column are already in their proper columns, we remove them for now. The remaining elements are all from the original first row and the reserve. Note that all the elements in the middle row were from the original top row. We use exactly the same process from before to return all the elements to their original columns. From here, we know that any element at the bottom of a column that had an element we temporarily removed last time is from the original top row. After removing these, only the reserve remains.

Now all we need to do is turn these three sets of numbers (the original top row, the original bottom row, and the reserve) into a 2-MOT. Start with an empty $n \times 2$ grid and iterate over j in the range $1 \leq j \leq n$. If j is in the top row, put two copies of j into the grid

2		6		Reserve
1	4	5	7	Top row
3	8	9	10	Bottom row

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 & 5 & 5 & 6 & 7 & 7 \\ \hline 2 & 3 & 3 & 6 & 8 & 8 & 9 & 9 & 10 & 10 \\ \hline \end{array}$$

on the top row (filling it in from left to right). If j is in the reserve, put one copy of j on the top and one on the bottom. Otherwise, put two copies of j on the bottom row of the grid. This completes the process. $\square$

Below are all nine 2-MOTs with four columns beside their corresponding standard Young tableaux.

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&3&4&4\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&4&4\\\hline\end{array} \longrightarrow \begin{array}{lll} 1&2&3\\ 4&& \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&3&3&4\\\hline\end{array} \longrightarrow \begin{array}{lll} 1&2&4\\ 3&& \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&2&4&4\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 2&2&3&4\\\hline\end{array} \longrightarrow \begin{array}{lll} 1&3&4\\ 2&& \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 1&3&4&4\\\hline\end{array} \longrightarrow \begin{array}{ll} 1&3\\ 2&\\ 3& \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&4\\\hline 1&3&3&4\\\hline\end{array} \longrightarrow \begin{array}{ll} 1&4\\ 2&\\ 3& \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&2&3&3\\\hline 1&2&4&4\\\hline\end{array} \longrightarrow \begin{array}{ll} 1&2\\ 3&\\ 4& \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 1&2&3&4\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$$

Now we will generalize this result.

Theorem 5. *The number of m-MOTs on an $n \times m$ grid, where all the occurrences of a number appear in the same row or all different rows, is equal to the number of standard Young tableaux with columns of size 1, m, or $m+1$.*

Proof. We will prove this by demonstrating a process for turning m-MOTs on an $n \times m$ grid where all the occurrences of a number appear in the same row or all different rows into SYTs where each column has 1, m, or $m+1$ elements, and then showing that it is uniquely reversible.

The forward process (i.e., MOTs to SYTs) is very similar to what we used before. Every j in the range $1 \leq j \leq n$ either appears in only the kth row of the m-MOT or in all rows. For each j from 1 to n, in order, if all js appear in the kth row, put j in the kth row of a grid with m rows, working from left to right. If j appears in all of the rows in the m-MOT, keep it in reserve. The occupied squares on lower rows never surpass those on the higher rows, so we have a valid SYT with some numbers missing. After this, append the reserve (in order) to the first row of the SYT. Note that all columns have height 1 if they are from the reserve or height m if they are not (since a column with height less than m would mean the rows in the original m-MOT had different lengths). If this is a valid tableau, we are done.

The top row, however, may not be increasing. If so, we find the first element in the top row that is less than its neighbor to the left and move it to the left until it reaches the leftmost element greater than it. Then we push all numbers in that column down one space, to make room for the moved element. We repeat this process until the top row is ascending. Note that this process will yield an increasing row and maintain increasing columns for the same reasons it did before. Furthermore, the columns will all have height 1, m, or $m+1$. We are now done with constructing the top row of the new SYT.

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&4&4&4&5\\\hline 2&3&3&3&5&7&7&7\\\hline 2&5&6&6&6&8&8&8\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1&4&2&5\\\hline 3&7\\\cline{1-2} 6&8\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6&7\\\cline{1-2} \multicolumn{1}{c|}{}&8\\\cline{2-2}\end{array}$$

Next, for every row, in order, starting from the second-highest row and ending with the mth row, we use the same process as we used for the middle row of the 2-MOT to SYT transformation. While making the rows increasing, we will also show property $(*)$ holds at the end of the ith iteration of this procedure for all i.

> $(*)$ At the end of the ith iteration, a column is of size $m+1$ if and only if it has an element which began in row $i+1$ in row $i+2$.

Observe that property $(*)$ holds when we begin modifying the second-highest row by construction (i.e., the beginning of iteration $i=1$). For each row, note that the only potentially out-of-place elements in that row are from row i, since all rows started out ascending. Therefore, by $(*)$, the only columns that lose an element have $m+1$ elements and those that gain an element have m so the sizes of the columns are all 1, m, or $m+1$ during each step of the process.

Furthermore, property $(*)$ still holds after we make row $i+1$ ascending. If a column has size $m+1$, we know it either starts with or receives an element that began in the row i. Observe that all elements that started in row i but are now in row $i+1$ will end directly above an element that began in row $i+1$ (but is now in row $i+2$). We know each column with size $m+1$ has an element from row i in row $i+1$, thus proving the forward direction of $(*)$. Since the elements from row i will not end in the same column as one another and property $(*)$ holds at the beginning, only the columns with one of these elements at the end will have size $m+1$. Observe that all columns with an element that began in row $i+1$ in row $i+2$ must have an element that began in row i directly above it, thus showing that the reverse direction of $(*)$ also holds.

After these steps are complete, we have created a configuration with increasing rows and increasing columns of height 1, m, or $m+1$. Then, move all the elements in the bottom row,

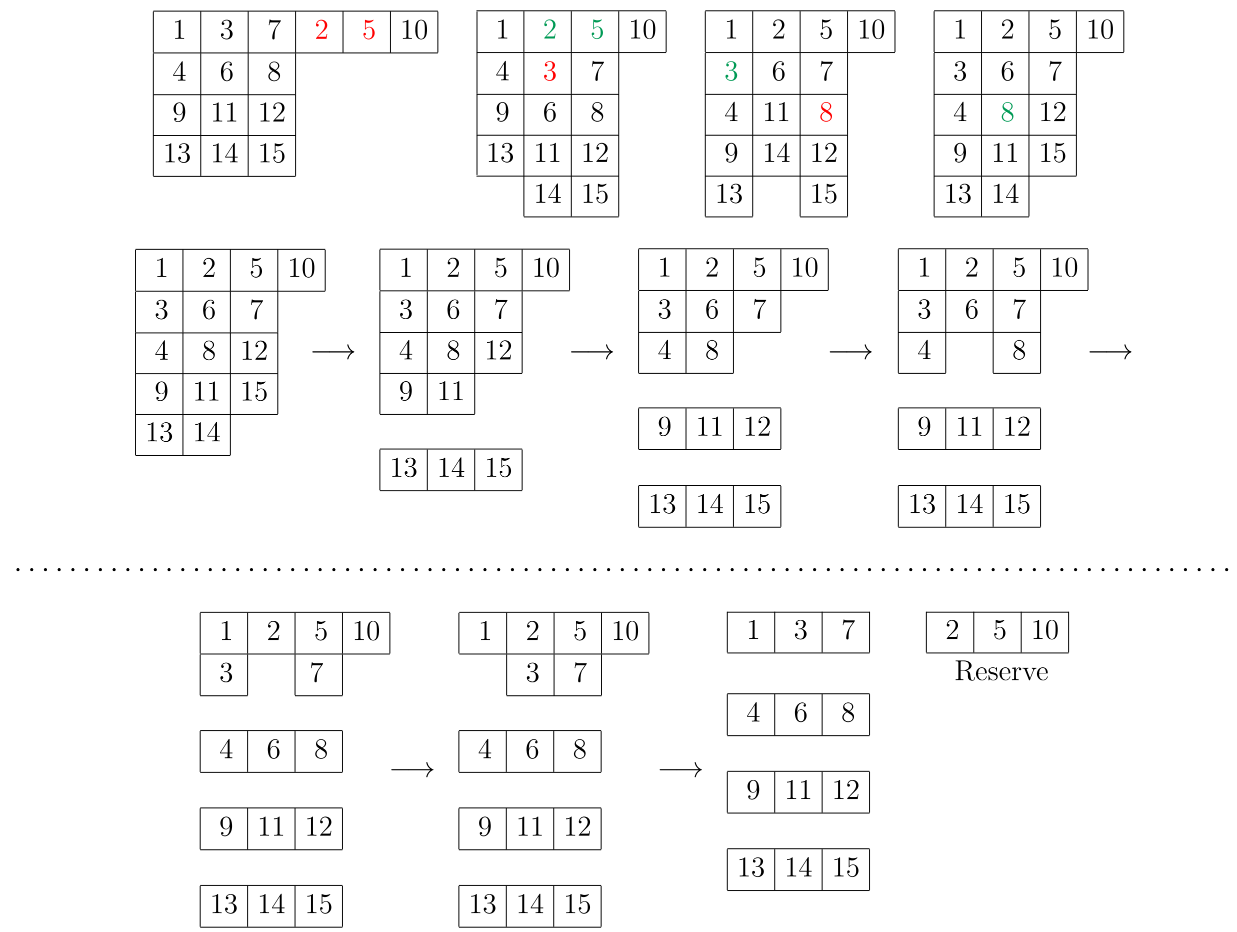


Figure 5: Reversing the process to obtain a 4-MOT from a standard Young tableau with columns of height 1, 4, and 5.

without changing the order, as far left as possible. We now have a valid SYT with columns of height 1, m, or $m+1$, as shown above.

As before, we now need to show that this process is uniquely reversible. We will do this by reversing each of the steps, starting from the last step. Like before, we know there is only one possible place each element in the final row could have come from so we can put each of the elements in the final row back in the correct column. Now remove the bottom-most element from each column of height m or $m+1$. Similarly, we can identify the remaining elements in the second-lowest row that came from the third-lowest row, so we put these back in their original columns. Then we remove the lowest element in each column that had size m or $m+1$ at the beginning. This condition ensures that reserve elements are not mistakenly put into one of the main rows. We continue this process until we reach the first row. At this point, we have separated out all the original rows and the reserve elements. Now we only need to turn these rows back into an m-MOT. For every number j in the range $1 \leq j \leq n$, fill in an empty $n \times m$ grid, working from left to right, as follows. If j appears in the kth row, put m instances of j in the kth row. If j appears in the reserve, put one copy of j in each row. We have now returned the SYT into the original m-MOT on an $n \times m$ grid, as Fig. 5 illustrates. □

4 Enumeration of the generalized Motzkin numbers

As we have demonstrated, by counting the number of ways to place $m \geq 2$ copies of each of the numbers from 1 to n into an $n \times m$ grid so that all rows and columns are nondecreasing and each of the m copies of a given number either occupy the same row or all different rows, we can generalize the Motzkin numbers, which are realized by $m = 2$. We refer to this value as $M_{m,n}$, the number of m-MOTs on an $n \times m$ grid. (We also set $M_{m,0} = 1$.) The table below enumerates the first few generalized Motzkin numbers.

m/n	0	1	2	3	4	5	6	7	8	9
2	1	1	2	4	9	21	51	127	323	835
3	1	1	1	2	5	11	26	71	197	547
4	1	1	1	1	2	6	16	36	85	253
5	1	1	1	1	1	2	7	22	57	127
6	1	1	1	1	1	1	2	8	29	85

Table 1: Number of m-MOTs with n columns for small m and n

There are a couple of noteworthy patterns to observe. One can easily show that

- $M_{m,n} = 1$ for all m and n such that $m > n$,
- $M_{m,n} = 2$ when $m = n$,
- $M_{m,n} = n + 1$ when $m = n - 1$.

Also, for $m = n-2$ when $m > 2$ we have $M_{m,n} = \frac{1}{2}(n^2 - n + 2)$. This is because there are $M_{m,n-1} = n$ m-MOTs with the 1s all in the first column, and an additional $\binom{n-1}{2}$ m-MOTs with the 1s all in the first row. (Choosing which two elements appear in every row uniquely determines an m-MOT of this size.) Observe that the pattern breaks for $m = 2$ since it is possible to have fewer than two elements which appear in every row.

Note that neither of the apparent patterns $M_{m,n} > M_{m+1,n}$ and $M_{m,n} < M_{m+1,n+1}$ hold for all m and n. The smallest counterexamples are

$$M_{2,13} = 41835 < 44903 = M_{3,13} \quad \text{and} \quad M_{5,12} = 3565 > 3433 = M_{6,13},$$

respectively. The following conjectures concern the properties of such counterexamples.

Conjecture 6. Given distinct m_1 and m_2, there exists n such that $M_{m_1,n} < M_{m_2,n}$.

Conjecture 7. If $M_{m_1,n} < M_{m_2,n}$ for any m_1, m_2, and n where $m_1 < m_2$, then $M_{m_1,n'} < M_{m_2,n'}$ for all $n' > n$.

References

[1] M. Aigner, Motzkin numbers, *European J. Combin.* **19** (1998), 663–675.

[2] F. R. Bernhart, Catalan, Motzkin, and Riordan numbers, *Discrete Math.* **204** (1999), 73–112.

[3] E. Deutsch and B. E. Sagan, Congruences for Catalan and Motzkin numbers and related sequences, *J. Number Theory* **117** (2006), 191–215.

[4] R. Donaghey and L. W. Shapiro, Motzkin numbers, *J. Combin. Theory Ser. A* **23** (1977), 291–301.

[5] R. Donaghey, Restricted plane tree representations of four Motzkin–Catalan equations, *J. Combin. Theory Ser. B* **22** (1977), 114–121.

[6] S. Eu, Skew-standard tableaux with three rows, *Adv. in Appl. Math.* **45** (2010), 463–469.

[7] D. Gouyou-Beauchamps, Standard Young tableaux of height 4 and 5, *European J. Combin.* **10** (1989), 69–82.

[8] C. Greene, A. Nijenhuis, and H. Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape, *Adv. Math.* **31** (1979), 104–109.

[9] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org, October 2022.

[10] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, *Adv. Math.* **41** (1981), 115–136.

[11] Z. Sun, Congruences involving generalized central trinomial coefficients, *Sci. China Math.* **57** (2014), 1375–1400.

[12] Y. Wang and Z. Zhang, Combinatorics of generalized Motzkin numbers, *J. Integer Sequences* **18** (2015), Article 15.2.4.

2020 *Mathematics Subject Classification*: Primary 11P81; Secondary 05A17.
Keywords: Motzkin number, Young tableau, generalized Motzkin sequence, fault-free.

(Concerned with sequence A001006.)

Received February 3 2022; revised version received August 1 2022. Published in *Journal of Integer Sequences*, October 6 2022.

Return to Journal of Integer Sequences home page.