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On a Ramanujan-Type Congruence for Partition Triples with 5-Cores

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Abstract

Let $B_5^{(3)}(n)$ denote the number of partition triples with 5-cores. In this short note, we prove a Ramanujan-type congruence modulo 5^α $(\alpha \geq 1)$ for $B_5^{(3)}(n)$ by using recurrence relations for the coefficients of $B_5^{(3)}(n)$.

1 Introduction

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers whose sum equals n. Let $p(n)$ be the number of partitions of n. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where $(a;q)_0 = 1$, $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, and $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$.

The most inspiring congruences of $p(n)$ discovered by Ramanujan for $n \geq 0$ are

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Ramanujan's discovery inspired researchers to study the arithmetic properties of the restricted partitions, such as the *t-core partition*. A *Ferrers-Young diagram* of a partition $n = \sum_{i=1}^{k} \lambda_i$, where $\lambda_i \geq \lambda_k$, $i \leq k$, is a pattern of dots, with λ_i dots in the i^{th} row. The hook number of a dot is defined by the number of dots directly below, together with the number of dots directly to the right, as well as the dot itself. For a positive integer $t \geq 2$, a partition is said to be *t-core*, if it has no hook numbers that are multiples of t.

Example 1. The Ferrers-Young diagram of the partition $\lambda = 4 + 3 + 2$ of 9 with the corresponding hook number is as follows:

$$\begin{array}{cccc} \bullet^{6} & \bullet^{5} & \bullet^{3} & \bullet^{1} \\ \bullet^{4} & \bullet^{3} & \bullet^{1} & \\ \bullet^{2} & \bullet^{1} & & \end{array}$$

Here λ is 7-core and λ is t-core for $t \geq 7$.

We let $B_t(n)$ denote the number of *t-core partitions* of n. The generating function for the $B_t(n)$ is given by [7, Eq. (2.1)]

$$\sum_{n=0}^{\infty} B_t(n) q^n = \frac{f_t^t}{f_1},$$

where $f_t = (q^t; q^t)_\infty$, for any integer $t \geq 2$. Arithmetic properties of t-core partitions have been studied by several mathematicians (see, for example [1, 3, 4, 5, 8, 9, 12, 13]).

A *partition triple* or a *tripartition* of n is a triple of partitions $(\lambda_1, \lambda_2, \lambda_3)$ such that the sum of all the parts of λ_1, λ_2, and λ_3 equals n. A tripartition with t-core of n is a partition triples $(\lambda_1, \lambda_2, \lambda_3)$ of n such that λ_1, λ_2, and λ_3 are t-cores. Let $B_t^{(3)}(n)$ denote the number of partition triples with t-cores of n. The generating function for $B_t^{(3)}(n)$ is given by

$$\sum_{n=0}^{\infty} B_t^{(3)}(n) q^n = \frac{f_t^{3t}}{f_1^3}. \tag{1}$$

Dasappa [6] proved a Ramanujan-type congruence modulo 5^α for bipartition with 5-cores, which motivated establishing a Ramanujan-type congruence for tripartitions with 5-core.

The ultimate aim of this note is to prove the following Ramanujan-type congruence for $B_5^{(3)}(n)$:

$$B_5^{(3)}(5^\alpha n + 5^\alpha - 3) \equiv 0 \pmod{5^\alpha}, \quad \alpha \geq 1.$$

The following are crucial lemmas that help prove our main congruence for $B_5^{(3)}(n)$:

Lemma 2. *Let* $\sum_{n=0}^{\infty} P_4(n)q^n = qf_5^9 f_1^3$. *Then*

$$\sum_{n=0}^{\infty} P_4(5n+4)q^n = 5f_5^3 f_1^9. \tag{2}$$

Proof. The following 5-dissection formula was first stated by Ramanujan [10, p. 212] without any proof:

$$f_1 = f_{25}\Big(\frac{1}{R(q^5)} - q - q^2 R(q^5)\Big), \tag{3}$$

where $R(q) = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$. Watson [11] provided a proof of (3).

Using (3), we obtain

$$\sum_{n=0}^{\infty} P_4(n)q^n = f_5^9 f_{25}^3\Big(\frac{q}{R(q^5)^3} - \frac{3q^2}{R(q^5)^2} + 5q^4 - 3q^6 R(q^5)^2 - q^7 R(q^5)^3\Big). \tag{4}$$

Extracting the terms involving q^{5n+4} from (4), dividing by q^4, and replacing q^5 by q, we obtain (2). □

Lemma 3. *Let* $\sum_{n=0}^{\infty} P_5(n)q^n = f_5^3 f_1^9$. *Then*

$$\sum_{n=0}^{\infty} P_5(5n+4)q^n = -90 f_5^3 f_1^9 - 625 q f_5^9 f_1^3.$$

Proof. Using (3), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} P_5(n)q^n = f_5^3 f_{25}^9\Big(&\frac{1}{R(q^5)^9} - \frac{9q}{R(q^5)^8} + \frac{27q^2}{R(q^5)^7} - \frac{12q^3}{R(q^5)^6} - \frac{90q^4}{R(q^5)^5} + \frac{126q^5}{R(q^5)^4} + \frac{126q^6}{R(q^5)^3} \\ &- \frac{288q^7}{R(q^5)^2} - \frac{117q^8}{R(q^5)} + 365q^6 + 117q^{10}R(q^5) - 288q^{11}R(q^5)^2 - 126q^{12}R(q^5)^3 \\ &+ 126q^{13}R(q^5)^4 + 90q^{14}R(q^5)^5 - 12q^{15}R(q^5)^6 - 27q^{16}R(q^5)^7 - 9q^{17}R(q^5)^8 \\ &- q^{18}R(q^5)^9\Big).\end{aligned} \tag{5}$$

Extracting the terms involving q^{5n+4} from (5), dividing by q^4, and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} P_5(5n+4)q^n = f_5^9 f_1^3\Big(90q^2 R(q)^5 + 365q - \frac{90}{R(q)^5}\Big). \tag{6}$$

Berndt [2, Thm. 7.4.4] gave the following identity:

$$\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}. \tag{7}$$

Employing (7) in (6), we obtain

$$\sum_{n=0}^{\infty} P_5(5n+4)q^n = f_5^9 f_1^3\Big(-90\frac{f_1^6}{f_5^6} - 625q\Big) = -90 f_5^3 f_1^9 - 625q f_5^9 f_1^3.$$

□

Lemma 4. *Let* $\sum_{n=0}^{\infty} P_3(n)q^n = \frac{1}{f_1^3}$. *Then*

$$\sum_{n=0}^{\infty} P_3(5n+2)q^n = 9\frac{f_5^3}{f_1^6} + 375q\frac{f_5^9}{f_1^{12}} + 3125q^2\frac{f_5^{15}}{f_1^{18}}.$$

Proof. [2, Eq. (7.4.14), p. 165] We have

$$\begin{aligned}\frac{1}{f_1} =& \frac{f_{25}^5}{f_5^6}\Big(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) \\ &+ 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4\Big). \end{aligned} \tag{8}$$

Using (8), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} P_3(n)q^n = \frac{f_{25}^{15}}{f_5^{18}}\Big(&\frac{1}{R(q^5)^{12}} + \frac{3q}{R(q^5)^{11}} + \frac{9q^2}{R(q^5)^{10}} + \frac{22q^3}{R(q^5)^9} + \frac{51q^4}{R(q^5)^8} + \frac{78q^5}{R(q^5)^7} + \frac{134q^6}{R(q^5)^6} \\ &+ \frac{177q^7}{R(q^5)^5} + \frac{216q^8}{R(q^5)^4} + \frac{153q^9}{R(q^5)^3} + \frac{219q^{10}}{R(q^5)^2} + \frac{57q^{11}}{R(q^5)} + 71q^{12} - 57q^{13}R(q^5) \\ &+ 219q^{14}R(q^5)^2 - 153q^{15}R(q^5)^3 + 216q^{16}R(q^5)^4 - 177q^{17}R(q^5)^5 + 134q^{18}R(q^5)^6 \\ &- 78q^{19}R(q^5)^7 + 51q^{20}R(q^5)^8 - 22q^{21}R(q^5)^9 + 9q^{22}R(q^5)^{10} - 3q^{23}R(q^5)^{11} \\ &+ q^{24}R(q^5)^{12}\Big). \end{aligned} \tag{9}$$

Extracting the terms involving q^{5n+2} from (9), dividing by q^2, and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} P_3(5n+2)q^n = \frac{f_5^{15}}{f_1^{18}}\Big(9q^4 R(q)^{10} - 177q^3 R(q)^5 + 71q^2 + 177\frac{q}{R(q)^5} + \frac{9}{R(q)^{10}}\Big). \tag{10}$$

Employing (7) in (10), we deduce that

$$\sum_{n=0}^{\infty} P_3(5n+2)q^n = \frac{f_5^{15}}{f_1^{18}}\left(9\frac{f_1^{12}}{f_5^{12}} + 375q\frac{f_1^6}{f_5^6} + 3125q^2\right)$$
$$= 9\frac{f_5^3}{f_1^6} + 375q\frac{f_5^9}{f_1^{12}} + 3125q^2\frac{f_5^{15}}{f_1^{18}}.$$

□

Theorem 5. *For all integers $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} B_5^{(3)}(5^{\alpha+1}n + 5^{\alpha+1} - 3)q^n = A_\alpha f_5^3 f_1^9 + B_\alpha q f_5^9 f_1^3 + C_\alpha q^2 \sum_{n=0}^{\infty} B_5^{(3)}(n)q^n, \tag{11}$$

where $A_0 = 9$, $B_0 = 375$, $C_0 = 3125$, and for any integer $n \geq 1$

$$A_n = -90A_{n-1} + 5B_{n-1} + 9C_{n-1}, \tag{12}$$
$$B_n = -625A_{n-1} + 5B_{n-1} + 375C_{n-1}, \tag{13}$$
$$C_n = C_0^{n+1}. \tag{14}$$

Proof. From (1), we have

$$\sum_{n=0}^{\infty} B_5^{(3)}(n) = \frac{f_5^{15}}{f_1^3}.$$

Employing Lemma 4, we obtain

$$\sum_{n=0}^{\infty} B_5^{(3)}(5n+2)q^n = 9f_5^3 f_1^9 + 375q f_5^9 f_1^3 + 3125q^2 \sum_{n=0}^{\infty} B_5^{(3)}(n)q^n. \tag{15}$$

Eq. (15) is the $\alpha = 0$ case of Eq. (11). Now assume for $\alpha \geq 0$. Replacing n by $5n+4$ in (11), using Lemmas (2) and (3), and (15), we obtain

$$\sum_{n=0}^{\infty} B_5^{(3)}(5^{\alpha+2}n + 5^{\alpha+2} - 3)q^n$$
$$= A_\alpha\left(-90f_5^3 f_1^9 - 625q f_5^9 f_1^3\right) + B_\alpha 5 f_5^3 f_1^9 + C_\alpha\left(9f_5^3 f_1^9 + 375q f_5^9 f_1^3 + 3125q^2 \sum_{n=0}^{\infty} B_5^{(3)}(n)q^n\right)$$
$$= \left(-90A_\alpha + 5B_\alpha + 9C_\alpha\right) f_5^3 f_1^9 + \left(-625A_\alpha + 375C_\alpha\right) f_5^9 f_1^3 + 3125C_\alpha q^2 \sum_{n=0}^{\infty} B_5^{(3)}(n)q^n$$
$$= A_{\alpha+1} f_5^3 f_1^9 + B_{\alpha+1} q f_5^9 f_1^3 + C_{\alpha+1} q^2 \sum_{n=0}^{\infty} B_5^{(3)}(n)q^n.$$

That is, (11) holds for $\alpha + 1$. This completes the proof by induction of (11). □

Theorem 6. *For all integers $\alpha \geq 1$ and $n \geq 0$, we have*

$$B_5^{(3)}(5^\alpha n + 5^\alpha - 3) \equiv 0 \pmod{5^\alpha}. \tag{16}$$

Proof. From (12), (13), and (14), we see that

$$\begin{array}{lll} A_1 \equiv 0 \pmod{5}, & B_1 \equiv 0 \pmod{5^3}, & C_1 \equiv 0 \pmod{5^5}, \\ A_2 \equiv 0 \pmod{5^2}, & B_2 \equiv 0 \pmod{5^4}, & C_2 \equiv 0 \pmod{5^{10}}, \\ \vdots & \vdots & \vdots \\ A_\alpha \equiv 0 \pmod{5^\alpha}, & B_\alpha \equiv 0 \pmod{5^{\alpha+2}}, & C_\alpha \equiv 0 \pmod{5^{5\alpha}}. \end{array}$$

That is, A_α, B_α, and C_α are multiples of 5^α. Then (11) implies (16). □

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