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On Remarkable Properties of Primes Near Factorials and Primorials

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Abstract

The distribution of primes is quite irregular. However, it is conjectured that if p is the smallest prime greater than $n!+1$, then $p-n!$ is also prime. We give a sufficient condition that guarantees when this conjecture is true. In particular, we prove that if a prime number p satisfies $n!+1 < p < n!+r^2$, where r is the smallest prime larger than a given natural number n, then $p-n!$ is also a prime. Similarly we treat another conjecture: If p is the largest prime smaller than $n!-1$, then $n!-p$ is also prime. Then

we establish further sufficient conditions also for the case when $n!$ is replaced by $q\#$, which is the product of all primes not exceeding the prime q.

1 Primes near factorials

Throughout this paper let $n \in \mathbb{N}$ be an arbitrary fixed natural number. Let $n! = 1 \cdot 2 \cdots n$ be its factorial.

Our main results are contained in Theorems 3,5,17, and 18. First, we present two well-known lemmas which illustrate that there are no primes in a close neighborhood above $n!+1$ and below $n! - 1$.

Lemma 1. *If a prime $p > n! + 1$, then $p > n! + n$.*

Proof. This lemma immediately follows from the fact that the consecutive numbers

$$n! + 2,\ n! + 3, \dots,\ n! + n$$

are all composite. □

Similarly we can prove the second lemma.

Lemma 2. *If a prime $p < n! - 1$ and $n > 3$, then $p < n! - n$.*

The assumption $n > 3$ excludes the undesirable initial case $n = p = 3$ for which the inequality $p < n! - n$ is obviously not valid.

Recall [9] that primes of the form $n! + 1$ are said to be *factorial primes.* For instance, if

$$n = 1,\ 2,\ 3,\ 11,\ 27,\ 37,\ 41,\ 73,\ 77,\ 116,\ 154,\ 320,\ 340,\ 399,\ 427,\ 872, \dots$$

then $n! + 1$ is prime. Primes of the form $n! - 1$ are also called *factorial primes.* We get them for

$$n = 3,\ 4,\ 6,\ 7,\ 12,\ 14,\ 30,\ 32,\ 33,\ 38,\ 94,\ 166,\ 324,\ 379,\ 469,\ 546,\ 974, \dots$$

Now we present the first of our main theorems.

Theorem 3. *Let r be the smallest prime such that $r > n$. If a prime p satisfies*

$$n! + 1 < p < n! + r^2, \tag{1}$$

then $p - n!$ is also prime.

Proof. The case $n = 1$ is obvious. So let $n > 1$ and let p be a prime satisfying (1). Assume to the contrary that $p - n!$ is composite. Then there exist a prime m and an integer $k \geq m$ such that

$$p - n! = mk.$$

From this and the inequality $p - n! < r^2$ from (1), we observe that $m < r$ and therefore, the prime m satisfies the inequality $m \leq n$. Since $m \mid n!$ and $m \mid (p - n!)$, we find that $m \mid p$, which contradicts the assumption that p is prime and the fact that $p > n! + 1 > n \geq m$. □

Example 4. Let $n = 5$. Then $r^2 = 49$ and for consecutive primes after 5! we have

$$\begin{aligned} 5! = 120 = 127 - 7 = 131 - 11 = 137 - 17 = 139 - 19 = 149 - 29 \\ = 151 - 31 = 157 - 37 = 163 - 43 = 167 - 47 \\ = 173 - 53 = 179 - 59 = 181 - 61 = 191 - 71 = 193 - 73 = \underline{197 - 7 \cdot 11}. \end{aligned} \tag{2}$$

So all these differences of primes yield the same number $5! = 120$. We observe that there are even more consecutive primes $p > n! + 1$ than those satisfying (1) for which $p - n!$ is also prime. Namely, the inequality (1) yields only the first two lines of (2), but we can continue in this manner until the underlined difference (cf. Table 1 below for $n = 5$).

Theorem 5. *Let $n > 2$ and let s be the largest prime such that $s < n$. If a prime p satisfies*

$$n! - s^2 < p < n! - 1, \tag{3}$$

then $n! - p$ is also prime.

The proof is done similarly as in Theorem 3. The additional assumption $n > 2$ only guarantees the existence of s.

Example 6. Take $n = 7$ in Theorem 5. Then $s^2 = 25$ and

$$\begin{aligned} 7! = 5040 = 5039 + 1 = 5023 + 17 = 5021 + 19 \\ = 5011 + 29 = 5009 + 31 = 5003 + 37 = 4999 + 41 = 4997 + 43 \\ = 4987 + 53 = 4973 + 67 = 4969 + 71 = 4967 + 73 = 4957 + 83 \\ = 4951 + 89 = 4943 + 97 = 4937 + 103 = 4933 + 107 = 4931 + 109 = \underline{4919 + 11^2}. \end{aligned}$$

All these sums of primes yield the same number $7! = 5040$. We again get more consecutive primes p than those satisfying (3) for which $n! - p$ is prime until the underlined sum, see the last two columns of Table 1 and Remark 11. Theorem 5 thus reminds us of the well-known Goldbach conjecture [9, p. 79].

2 Further examples and open problems

In Figure 1, we observe a remarkable distribution of primes near $n!$.

$\vdots$

• $n! + r^2$

$\left\{\right.$ Here for each prime p we have that $p - n!$ is also prime by Theorem 3.

• $n! + n$

$\left\{\right.$ There are no primes by Lemma 1.

• $n! + 1$ A possible factorial prime.

• $n!$

• $n! - 1$ A possible factorial prime.

$\left\{\right.$ There are no primes by Lemma 2.

• $n! - n$

$\left\{\right.$ Here for each prime p we have that $n! - p$ is also prime by Theorem 5.

• $n! - s^2$

$\vdots$

• 3

• 2

• 1

Figure 1: Distribution of primes near $n!$ for $n > 2$.

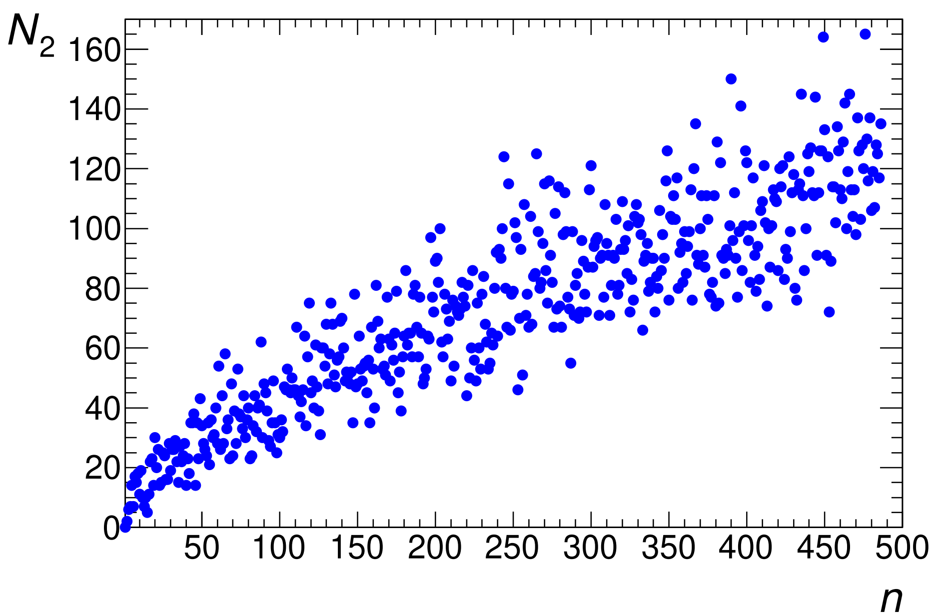

Figure 2: The number of consecutive primes p just above $n! + 1$ for which $p - n!$ is also prime for all $n \leq 486$.

It could happen, however, that the open intervals $(n!+1, n!+r^2)$ and $(n!-s^2, n!-1)$ appearing in (1) and (3) do not contain any prime number, although no such example is known. Therefore, Theorems 3 and 5 do not imply that the following conjectures are true.

Conjecture 7. If p is the smallest prime greater than $n!+1$, then $p-n!$ is also prime.

Conjecture 8. If p is the largest prime smaller than $n!-1$, then $n!-p$ is also prime.

Remark 9. From the well-known Stirling formula [10, p. 343]

$$\lim_{n\to\infty} \frac{n!}{n^n \mathrm{e}^{-n}\sqrt{n}} = \sqrt{2\pi},$$

we find an asymptotic expression for the factorial $n! \approx \sqrt{2\pi}\, n^n \mathrm{e}^{-n}\sqrt{n}$ and thus

$$\ln(n!) \approx n\ln(n) - n + 0.5\ln(n) + 0.5\ln(2\pi).$$

According to the celebrated Gauss prime number theorem [9], the probability that n is a prime number is about $1/\ln(n)$. Hence, consecutive primes on the order of $n!$ should differ by about $\ln(n!)$ which is approximately

$$\ln(n!) = n\ln(n) - n + O(\ln(n)) \qquad \text{as } n \to \infty, \tag{4}$$

where $O(\cdot)$ stands for the usual Landau symbol. However, this is by relation (4) much less than $r^2 \quad (> n^2)$ appearing on the right-hand side of (1). This provides support that Conjecture 7 (and similarly Conjecture 8) might be true.

Example 10. Another argument for the validity of Conjectures 7 and 8 are numerical tests. The statements of Theorems 3 and 5 for $n = 2, 3, \ldots, 10$ are given in Table 1.

Numerical tests calculated by Maple for all $n \le 500$ indicate that if $\overline{p}$ is the smallest prime greater than $n!+1$ for which $\overline{p}-n!$ is composite, then $\overline{p}-n!$ is always the product of two not necessarily different primes, cf. Table 2. In Figure 2, we see an increasing trend of the number of consecutive primes p above $n!+1$ for which $p-n!$ is also prime.

Another open problem is whether the difference $\overline{p}-n!$ from the last column of Table 2 is always the product of two (not necessarily different) primes that are greater than n.

Remark 11. If the upper bound $n!+r^2$ appearing in (1) is a prime $\tilde{p}$, then $N_1 = N_2$ in Table 1, since the difference $\tilde{p}-n! = (n!+r^2)-n! = r^2$ is composite. Hence, the sequence of consecutive primes p just above $n!+1$, for which $p-n!$ is also prime, finishes before $n!+r^2$. For instance, $2!+3^2 = 11$, $3!+5^2 = 31$, and $6!+7^2 = 769$ are primes, cf. Table 1 for $n \in \{2,3,6\}$. Also $100!+101^2$ and $350!+353^2$ are primes, cf. Table 2 for $n \in \{100, 350\}$.

On the other hand, the lower bound $n!-s^2$ appearing in (3) is never prime except for the trivial case $n = 3$ when $N_3 = N_4 = 1$. The reason is that $s \mid n!$, and thus $s \mid (n!-s^2)$.

n	N_1	N_2	N_3	N_4
2	2	2	–	–
3	6	6	1	1
4	6	7	2	6
5	9	14	1	10
6	7	7	2	10
7	12	17	2	17
8	8	15	3	10
9	11	18	3	4
10	7	11	5	8

Table 1: Here N_1 denotes the number of primes satisfying (1), N_2 is the number of consecutive primes just above $n!+1$ for which $p-n!$ is prime, N_3 is the number of primes satisfying (3), N_4 is the number of consecutive primes just below $n!-1$ for which $n!-p$ is prime, $N_1 \leq N_2$, and $N_3 \leq N_4$.

n	N_1	N_2	$\overline{p}-n!$
10	7	11	$169 = 13^2$
50	27	34	$3481 = 59^2$
100	30	30	$10201 = 101^2$
150	37	48	$31133 = 163 \cdot 191$
200	54	89	$76729 = 277^2$
250	55	79	$88579 = 283 \cdot 313$
300	77	121	$176959 = 311 \cdot 569$
350	76	76	$124609 = 353^2$
400	85	122	$242321 = 443 \cdot 547$
450	95	133	$307297 = 487 \cdot 631$
500	95	105	$294319 = 521 \cdot 569$

Table 2: Here N_1 denotes the number of primes satisfying (1), N_2 is the number of consecutive primes just above $n!+1$ for which $p-n!$ is also prime, and $\overline{p}$ is the smallest prime greater than $n!+1$ for which $\overline{p}-n!$ is composite.

Remark 12. The verification of Conjecture 7 for any $n \leq 4003$ follows from sequence A037153 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [11]. Also see sequences A033932, A037151, and A087421. The verification of Conjecture 8 for $n \leq 1000$ follows from A037155. Conjectures 7 and 8 are also related to a paper by Flórez and James [1]. Nevertheless, one has to keep in mind the so-called strong law of small numbers [2, 3, 5, 6, 7], when the validity of some apparent regular pattern is violated for $n \gg 1$.

Now we will modify our previous results to another class of numbers.

3 Primes near primorials

From now on, let q be an arbitrary fixed prime. Denote by $q\#$ the product of all primes not exceeding q, i.e., $q\# = 2 \cdot 3 \cdot 5 \cdots q$. It is called the *primorial* of q.

Conjecture 13. If p is the smallest prime greater than $q\# + 1$, then $p - q\#$ is also prime.

Conjecture 14. If p is the largest prime smaller than $q\# - 1$, then $q\# - p$ is also prime.

The classical proof of Euclid's theorem on the infinity of primes is done by contradiction [9]. It is assumed that there exist only a finite number of primes and that the largest prime is q. Then one investigates the number $q\#+1$ which leads to a contradiction, since $q\#+1$ is a new prime or $q\#+1$ is composite and divisible by a prime greater than q. For this reason, prime numbers of the form $q\# + 1$ are called *Euclidean primes,* see e.g.,[8]. For example,

$$2\# + 1 = 3, \quad 3\# + 1 = 7, \quad 5\# + 1 = 31, \quad 7\# + 1 = 211, \quad 11\# + 1 = 2311$$

are Euclidean primes. However, not every number of this form is prime, since

$$13\# + 1 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 59 \cdot 509.$$

Note that $q\# + 1$ is a Euclidean prime only for

$$q = 2,\ 3,\ 5,\ 7,\ 11,\ 31,\ 379, \ldots$$

Similarly we can investigate numbers of the form $q\# - 1$, which are primes for

$$q = 3,\ 5,\ 11,\ 13,\ 41,\ 89,\ 317,\ 337,\ 991, \ldots$$

In this case they are called *primorial primes.*

Lemma 15. *If a prime* $p > q\# + 1$*, then* $p > q\# + q$.

Proof. Since the consecutive numbers

$$q\# + 2,\ q\# + 3, \ldots,\ q\# + q$$

are all composite, the lemma follows. □

Similarly we can prove the next lemma.

Lemma 16. *If a prime $p < q\# - 1$ and $q > 3$, then $p < q\# - q$.*

Theorem 17. *Let $q < r$ be consecutive primes. If a prime p satisfies*

$$q\# + 1 < p < q\# + r^2, \tag{5}$$

then $p - q\#$ is also prime.

Proof. We shall proceed similarly as in the proof of Theorem 3. Let p be a prime satisfying (5). Suppose to the contrary that $p - q\#$ is composite. Then there exist a prime m and an integer $k \geq m$ such that

$$p - q\# = mk. \tag{6}$$

From this and the inequality $p - q\# < r^2$ arising from (5), we see that $mk < r^2$ and thus $m \leq q < r$. Since $m \mid q\#$ and $m \mid (p - q\#)$ by (6), we find that $m \mid p$ which is a contradiction with the assumption that p is prime and the fact that $p > q\# + 1 > q \geq m$. □

In a similar way we can prove the following statement.

Theorem 18. *Let $s < q$ be consecutive primes. If a prime p satisfies*

$$q\# - s^2 < p < q\# - 1, \tag{7}$$

then $q\# - p$ is also prime.

Example 19. To illustrate the meaning of Lemmas 15 and 16 and also Theorems 17 and 18, we set $q = 13$. We observe similar remarkable properties of consecutive primes near $q\# \pm 1$ as in Examples 4 and 6, namely,

$$\begin{aligned} 13\# = 30030 &= 30047 - 17 = 30059 - 29 = 30071 - 41 = 30089 - 59 = 30091 - 61 \\ &= 30097 - 67 = 30103 - 73 = 30109 - 79 = 30113 - 83 = 30119 - 89 \\ &= 30133 - 103 = 30137 - 107 = 30139 - 109 = 30161 - 131 = 30169 - 139 \\ &= 30181 - 151 = 30187 - 157 = 30197 - 167 = 30203 - 173 = 30211 - 181 \\ &= 30223 - 193 = 30241 - 211 = 30253 - 223 = 30259 - 229 = 30269 - 239 \\ &= 30271 - 241 = 30293 - 263 = 30307 - 277 = 30313 - 283 = \underline{30319 - 17^2}, \end{aligned}$$

$$\begin{aligned} 13\# = 30030 &= 30029 + 1 = 30013 + 17 = 30011 + 19 = 29989 + 41 = 29983 + 47 \\ &= 29959 + 71 = 29947 + 83 = 29927 + 103 = 29921 + 109 = 29917 + 113 \\ &= 29881 + 149 = 29879 + 151 = 29873 + 157 = 29867 + 163 = 29863 + 167 \\ &= 29851 + 179 = 29837 + 193 = 29833 + 197 = 29819 + 211 = 29803 + 227 \\ &= 29789 + 241 = 29761 + 269 = 29759 + 271 = 29753 + 277 = \underline{29741 + 17^2}. \end{aligned}$$

n	Q_1	Q_2	Q_3	Q_4
2	2	2	–	–
3	6	6	1	1
5	10	10	1	6
7	19	19	4	22
11	23	25	7	19
13	29	29	9	23
17	25	36	10	33
19	38	42	20	32

Table 3: Here Q_1 denotes the number of primes satisfying (5), Q_2 is the number of consecutive primes just above $q\# + 1$ for which $p - q\#$ is prime, Q_3 is the number of primes satisfying (7), Q_4 is the number of consecutive primes just below $q\# - 1$ for which $q\# - p$ is prime, where $q > 2$, $Q_1 \leq Q_2$, and $Q_3 \leq Q_4$.

Figure 1 can be easily modified to primes near $q\#$. Also Table 3 corresponding to these primes is similar to Table 1.

Taking into account that

$$4! < 5\# < 5! < 7\# < 6! < 11\# < 7! < 13\# < 8! < 17\# < 9! < 10! < 19\#,$$

we find that numbers Q_i in particular columns are generally greater than N_i from Table 1. when $(n-1)! < q\# < n!$.

Remark 20. If the upper bound $q\#+r^2$ appearing in (5) is a prime $\tilde{p}$, then $Q_1 = Q_2$ in Table 3, since the difference $\tilde{p} - q\# = (q\# + r^2) - q\# = r^2$ is composite. Hence, the sequence of consecutive primes p just above $q\# + 1$, for which $p - q\#$ is also prime, finishes before $q\# + r^2$. For instance, $2\# + 3^2 = 11$, $3\# + 5^2 = 31$, $5\# + 7^2 = 79$, $7\# + 11^2 = 331$, and $13\# + 17^2 = 30319$ are primes, cf. Table 3 for $q \in \{2, 3, 5, 7, 13\}$.

On the other hand, the lower bound $q\# - s^2$ appearing in (7) is never prime except for the trivial case $q = 3$ when $Q_3 = Q_4 = 1$. The reason is that $s \mid q\#$, and thus $s \mid (q\# - s^2)$.

Remark 21. We note that a Fortunate number, named after Reo Franklin Fortune, is the smallest integer $m > 1$ such that for a given prime q, $q\# + m$ is a prime number (see [2, 3, 4, 5, 7] for a discussion of Fortunate numbers). The sequence of Fortunate numbers begins: 3, 5, 7, 13, 23, 17, 19, 23, 37, ... Conjecture 13 which was introduced by Fortune, states that all Fortunate numbers are primes. The verification of Conjecture 13 for the first 3000 primes q follows from A005235 (also see A046066, A035346, A098168). The verification of Conjecture 14 for the first 2000 primes q follows from A055211 (also see A098166).

Remark 22. The distribution of primes is quite irregular. However, Theorems 3–18 imply that there are some regular patterns. Moreover, Theorems 17 and 18 can be easily extended to the case when $q\#$ is everywhere substituted by the product $i(q\#)$ for any fixed integer

$i \in \mathbb{N}$. For example, for $i = 31$ and $q = 3$ we have $31 \cdot 6 = 186 = 191 - 5 = 193 - 7 = 197 - 11 = 199 - 13 = \underline{211 - 5^2}$.

This extension covers the case investigated in Section 1, since we may set $i = n!/q\#$ for some $n \geq q$. See, for example, identities (2) for $n = q = 5$ yielding $i = 120/30 = 4$.

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