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Scaled Fibonacci- and Lucas-Producing Rational Polynomials

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Abstract

We study families of interpolating rational polynomials that produce scaled Fibonacci or Lucas numbers on certain integer values. We use the expansions of these families in binomial polynomials and other formats to establish several identities involving harmonic numbers, binomial coefficients, and various recursively defined sequences.

1 Introduction

In a previous paper [3], we considered certain interpolating polynomials defined from points involving Fibonacci numbers which led to several identities involving Fibonacci numbers, harmonic numbers, and binomial coefficients. A key part of the analysis used the polynomial

$$\mathcal{F}_k(x) = F_1 x^{k-1} + F_2 x^{k-2} + \cdots + F_{k-1} x + F_k \tag{1}$$

and its derivatives evaluated at $x = 1$. Here we consider interpolating polynomials defined from points involving scaled Fibonacci and Lucas numbers whose study makes use of (1) at $x = -1, 2, -2, 3$. Our investigation of other integer values of x suggests that the corresponding polynomials use defining points whose expressions are more complicated than those here. Section 2 gives the statements of our main results along with a summary of the corresponding results of the previous work [3], followed by proofs in Section 3 where (1) is prominent. In Section 4, we express the various polynomial families in terms of binomial polynomials, which informs several of the many identities given in Section 5.

The Fibonacci numbers are given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. In addition, we use their extension to negative indices, $F_{-n} = (-1)^{n+1} F_n$ for $n > 0$. We will make frequent use of the related Lucas numbers $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, and $L_{-n} = (-1)^n L_n$ for $n > 0$. Also, we will need the following alternating sequences of Fibonacci and Lucas numbers. For $n \geq 0$, let

$$b_n = \begin{cases} L_n, & \text{if } n \text{ is even;} \\ F_n, & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

Similarly, for $n \geq 0$, let

$$c_n = \begin{cases} F_n, & \text{if } n \text{ is even;} \\ L_n, & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

These appear in the OEIS [4] as A005247 and A005013, respectively.

The harmonic numbers

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

for $n \geq 1$ with $H_0 = 0$ arise in several of the identities. The polynomial results use the binomial polynomials defined by $\binom{x}{0} = 1$, $\binom{x}{n} = 0$ for integers $n < 0$, and $\binom{x}{n} = (x)_n / n!$ for integers $n > 0$, where $(x)_n$ denotes the falling factorial $x(x-1)\cdots(x-n+1)$. Finally, $\lfloor \cdot \rfloor$ denotes the integer floor.

2 Main results

We begin by citing the primary result of our previous work [3]. Four related results follow, each with a table of examples of the polynomial families.

Theorem 1. *Given a nonnegative integer n, let $P_n(x)$ be the interpolating polynomial determined by the points (i, F_{n+2+i}) for $0 \leq i \leq n$. Then $P_n(x)$ has degree n with leading coefficient $1/n!$. For each integer $k > n$,*

$$P_n(k) = F_{k+n+2} - \sum_{i=1}^{k-n} F_i \binom{k-i}{n}$$

n	$P_n(x)$
0	1
1	$x+2$
2	$(x^2+3x+6)/2$
3	$(x^3+3x^2+14x+30)/6$
4	$(x^4+2x^3+23x^2+94x+192)/24$
5	$(x^5+35x^3+180x^2+744x+1560)/120$
6	$(x^6-3x^5+55x^4+255x^3+1744x^2+7308x+15120)/720$

Table 1: $P_n(x)$ for small values of n.

and, for each integer $k<0$,

$$P_n(k) = F_{k+n+2} + \sum_{i=1}^{-k-1} F_{-i}\binom{k+i}{n}.$$

Also, the polynomials $P_n(x)$ satisfy and, with initial values $P_0(x)=1$ and $P_1(x)=x+2$, are uniquely determined by each of the following recurrence relations. For all $n \geq 1$, we have

$$P_{n+1}(x) - P_n(x) - P_{n-1}(x) = \binom{x+1}{n+1}, \tag{4}$$

$$P_{n+1}(x) - 3P_n(x-1) + P_{n-1}(x-2) = \binom{x-1}{n+1}. \tag{5}$$

This combines parts of Theorem 2.1, its proof, and Propositions 2.3 and 2.4 of the earlier paper [3]. Examples of $P_n(x)$ are given in Table 1. Notice that the constant terms of these polynomials (before division by $n!$) are given by A078700 in the OEIS [4], which follows from the definition of $P_n(x)$.

While the $P_n(x)$ of Theorem 1 use a particular increasing sequence of Fibonacci numbers, the polynomials of the next theorem are determined by a particular decreasing sequence of Fibonacci numbers.

Theorem 2. *Given a nonnegative integer n, let $Q_n(x)$ be the interpolating polynomial determined by the points (i, F_{2n+1-i}) for $0 \leq i \leq n$. Then $Q_n(x)$ has degree n and leading coefficient $(-1)^n/n!$. For each integer $k>n$,*

$$Q_n(k) = F_{2n-k+1} + (-1)^n \sum_{i=1}^{k-n} F_{-i}\binom{k-i}{n}$$

and, for each integer $k<0$,

$$Q_n(k) = F_{2n-k+1} + (-1)^{n+1} \sum_{i=1}^{-k-1} F_{i}\binom{k+i}{n}.$$

n	$Q_n(x)$
0	1
1	$-x+2$
2	$(x^2-5x+10)/2$
3	$(-x^3+9x^2-38x+78)/6$
4	$(x^4-14x^3+95x^2-394x+816)/24$
5	$(-x^5+20x^4-195x^3+1240x^2-5144x+10680)/120$
6	$(x^6-27x^5+355x^4-3105x^3+19444x^2-80748x+167760)/720$

Table 2: $Q_n(x)$ for small values of n.

Also, the polynomials $Q_n(x)$ satisfy and, with initial values $Q_0(x)=1$ and $Q_1(x)=-x+2$, are uniquely determined by each of the following recurrence relations. For all $n \geq 1$,

$$Q_{n+1}(x) - 3Q_n(x) + Q_{n-1}(x) = (-1)^{n+1}\binom{x+1}{n+1},$$
$$Q_{n+1}(x) - Q_n(x-1) - Q_{n-1}(x-2) = (-1)^{n+1}\binom{x-1}{n+1}. \tag{6}$$

See the next section for notes on the proof. Examples of $Q_n(x)$ are given in Table 2. Notice that the constant terms of these polynomials (before division by $n!$) are given by A052568 in the OEIS [4], which follows from the definition of $Q_n(x)$.

The polynomials of the next theorem are determined by a particular increasing sequence of Fibonacci numbers scaled by certain decreasing powers of two.

Theorem 3. *Given a nonnegative integer n, let $R_n(x)$ be the interpolating polynomial determined by the points $(i, 2^{n-1-i}F_{2n+3+i})$ for $0 \leq i \leq n$. Then $R_n(x)$ has degree n and leading coefficient $(-1)^n/n!$. For each integer $k > n$,*

$$R_n(k) = 2^{n-k-1}F_{k+2n+3} + (-1)^n \sum_{i=1}^{k-n} \frac{F_i}{2^{i+1}}\binom{k-i}{n}$$

and, for each integer $k < 0$,

$$R_n(k) = 2^{n-k-1}F_{k+2n+3} + (-1)^{n+1} \sum_{i=1}^{-k-1} 2^{i-1}F_{-i}\binom{k+i}{n}. \tag{7}$$

Also, the polynomials $R_n(x)$ satisfy and, with initial values $R_0(x)=1$ and $R_1(x)=-x+5$,

n	$R_n(x)$
0	1
1	$-x+5$
2	$(x^2-11x+52)/2$
3	$(-x^3+18x^2-173x+816)/6$
4	$(x^4-26x^3+383x^2-3622x+17088)/24$
5	$(-x^5+35x^4-705x^3+10045x^2-94814x+447360)/120$
6	$(x^6-45x^5+1165x^4-22275x^3+315634x^2-2978640x+14054400)/720$

Table 3: $R_n(x)$ for small values of n.

are uniquely determined by each of the following recurrence relations. For all $n \geq 1$,

$$R_{n+1}(x) - 6R_n(x) + 4R_{n-1}(x) = (-1)^{n+1}\binom{x+1}{n+1}, \tag{8}$$

$$R_{n+1}(x) - 4R_n(x-1) - R_{n-1}(x-2) = (-1)^{n+1}\binom{x-1}{n+1}. \tag{9}$$

See the next section for the proof. Examples of $R_n(x)$ are given in Table 3.

The even and odd degree polynomials of the next theorem are determined separately, by Lucas and Fibonacci numbers scaled by powers of negative two, respectively. Nonetheless, they enjoy some recurrence relations independent of parity.

Theorem 4. *Let* n *be a nonnegative integer. If* n *is even, let* $S_n(x)$ *be the interpolating polynomial determined by the points* $(i, (-2)^{n-1-i}L_{n-i})$ *for* $0 \leq i \leq n$. *Then* $S_n(x)$ *has degree* n *and leading coefficient* $-5^{n/2}/n!$. *For each integer* $k > n$,

$$S_n(k) = (-2)^{n-k-1}L_{k-n} - 5^{\frac{n+2}{2}}\sum_{i=1}^{k-n}\frac{F_i}{(-2)^{i+1}}\binom{k-i}{n} \tag{10}$$

and, for each integer $k < 0$,

$$S_n(k) = (-2)^{n-k-1}L_{k-n} + 5^{\frac{n+2}{2}}\sum_{i=1}^{-k-1}(-2)^{i-1}F_{-i}\binom{k+i}{n}.$$

If n *is odd, let the points* $(i, (-2)^{n-1-i}F_{i-n})$ *for* $0 \leq i \leq n$ *determine the interpolating polynomial* $S_n(x)$. *Then* $S_n(x)$ *has degree* n *and leading coefficient* $-5^{(n-1)/2}/n!$. *For each integer* $k > n$,

$$S_n(k) = (-2)^{n-k-1}F_{k-n} - 5^{\frac{n+1}{2}}\sum_{i=1}^{k-n}\frac{F_i}{(-2)^{i+1}}\binom{k-i}{n}$$

n	$S_n(x)$
0	-1
1	$-x+1$
2	$(-5x^2+15x-12)/2$
3	$(-5x^3+30x^2-61x+48)/6$
4	$(-25x^4+250x^3-935x^2+1670x-1344)/24$
5	$(-25x^5+375x^4-2225x^3+6825x^2-11670x+9600)/120$
6	$(-125x^6+2625x^5-22625x^4+105375x^3-294050x^2+496800x-414720)/720$

Table 4: $S_n(x)$ for small values of n.

and, for each integer $k<0$,

$$S_n(k) = (-2)^{n-k-1}F_{k-n} + 5^{\frac{n+1}{2}} \sum_{i=1}^{-k-1} (-2)^{i-1} F_{-i} \binom{k+i}{n}.$$

Also, the polynomials $S_n(x)$ satisfy and, with initial values $S_0(x) = -1$ and $S_1(x) = -x+1$, are uniquely determined by each of the following recurrence relations. For all $n \geq 1$,

$$S_{n+1}(x) + 2\cdot 5^{\frac{1-(-1)^n}{2}} S_n(x) + 4S_{n-1}(x) = -5^{\lfloor\frac{n+1}{2}\rfloor}\binom{x+1}{n+1}, \tag{11}$$

$$S_{n+1}(x) - S_{n-1}(x-2) = -5^{\lfloor\frac{n+1}{2}\rfloor}\binom{x-1}{n+1}. \tag{12}$$

See the next section for the proof. Examples of $S_n(x)$ are given in Table 4.

The polynomials of the last theorem have a structure similar to those of Theorem 4 with the scaling here by powers of three.

Theorem 5. *Let n be a nonnegative integer. If n is even, let $T_n(x)$ be the interpolating polynomial determined by the points $(i, 3^{n-1-i}L_{n+2+i})$ for $0 \leq i \leq n$. Then $T_n(x)$ has degree n and leading coefficient $5^{n/2}/n!$. For each integer $k > n$,*

$$T_n(k) = 3^{n-k-1}L_{k+n+2} + 5^{\frac{n+2}{2}} \sum_{i=1}^{k-n} \frac{F_i}{3^{i+1}} \binom{k-i}{n}$$

and, for each integer $k<0$,

$$T_n(k) = 3^{n-k-1}L_{k+n+2} - 5^{\frac{n+2}{2}} \sum_{i=1}^{-k-1} 3^{i-1} F_{-i} \binom{k+i}{n}.$$

n	$T_n(x)$
0	1
1	$-x+2$
2	$(5x^2-25x+42)/2$
3	$(-5x^3+45x^2-166x+270)/6$
4	$(25x^4-350x^3+2135x^2-7210x+11664)/24$
5	$(-25x^5+500x^4-4475x^3+23800x^2-78120x+126360)/120$
6	$(125x^6-3375x^5+41375x^4-310125x^3+1562300x^2-5081100x+8223120)/720$

Table 5: $T_n(x)$ for small values of n.

If n is odd, let the points $(i, 3^{n-1-i}F_{n+2+i})$ for $0 \le i \le n$ determine the interpolating polynomial $T_n(x)$. Then $T_n(x)$ has degree n and leading coefficient $-5^{(n-1)/2}/n!$. For each integer $k > n$,

$$T_n(k) = 3^{n-k-1}F_{k+n+2} - 5^{\frac{n+1}{2}} \sum_{i=1}^{k-n} \frac{F_i}{3^{i+1}} \binom{k-i}{n}$$

and, for each integer $k < 0$,

$$T_n(k) = 3^{n-k-1}F_{k+n+2} + 5^{\frac{n+1}{2}} \sum_{i=1}^{-k-1} 3^{i-1}F_{-i} \binom{k+i}{n}.$$

Also, the polynomials $T_n(x)$ satisfy and, with initial values $T_0(x) = 1$ and $T_1(x) = -x+2$, are uniquely determined by each of the following recurrence relations.

For all $n \ge 1$,

$$T_{n+1}(x) - 3 \cdot 5^{\frac{1-(-1)^n}{2}} T_n(x) + 9T_{n-1}(x) = (-1)^{n+1} 5^{\lfloor \frac{n+1}{2} \rfloor} \binom{x+1}{n+1}, \tag{13}$$

$$T_{n+1}(x) - 5^{\frac{1-(-1)^n}{2}} T_n(x-1) - T_{n-1}(x-2) = (-1)^{n+1} 5^{\lfloor \frac{n+1}{2} \rfloor} \binom{x-1}{n+1}. \tag{14}$$

See the next section for notes on the proof. Examples of $T_n(x)$ are given in Table 5.

3 Proofs of main results

Of the four new theorems above, we present two full proofs. The other two theorems have analogous proofs.

First, we collect various Fibonacci and Lucas number results that we will use. Proofs are available in many standard sources such as Vajda [5].

Lemma 6. *Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. For all integers m and n,*

$$F_m F_{n+1} - F_n F_{m+1} = (-1)^n F_{m-n}, \tag{15}$$
$$F_n = 3F_{n-2} + F_{n-4}, \tag{16}$$
$$F_n = 4F_{n-3} + F_{n-6}, \tag{17}$$
$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \tag{18}$$
$$5F_n = L_{n+1} + L_{n-1}, \tag{19}$$
$$L_n = \alpha^n + \beta^n. \tag{20}$$

Also, we will need Lemma 2.2 from the previous work [3] concerning (1).

Lemma 7. *Let $\mathcal{F}_k(x) = F_1 x^{k-1} + F_2 x^{k-2} + \cdots + F_{k-1}x + F_k$. The nth derivative of $\mathcal{F}_k(x)$ is*

$$\mathcal{F}_k^{(n)}(x) = \begin{cases} \displaystyle\sum_{i=1}^{k-n} x^{k-n-i} F_i (k-i)_n, & \text{if } n \le k-1; \\ 0, & \text{if } n \ge k. \end{cases} \tag{21}$$

The product of $\mathcal{F}_k(x)$ and $x^2 - x - 1$, the characteristic polynomial of the Fibonacci sequence, is

$$(x^2 - x - 1)\mathcal{F}_k(x) = x^{k+1} - F_{k+1}x - F_k. \tag{22}$$

The derivatives of the product of $\mathcal{F}_k(x)$ and $x^2 - x - 1$ are

$$(x^2 - x - 1)\mathcal{F}_k'(x) + (2x-1)\mathcal{F}_k(x) = (k+1)x^k - F_{k+1} \tag{23}$$

and

$$\begin{aligned} (x^2 - x - 1)\mathcal{F}_k^{(n)}(x) + n(2x-1)\mathcal{F}_k^{(n-1)}(x) + n(n-1)\mathcal{F}_k^{(n-2)}(x) \\ = \begin{cases} x^{k-n+1}(k+1)_n, & \text{if } 2 \le n \le k+1; \\ 0, & \text{if } n \ge k+2. \end{cases} \end{aligned} \tag{24}$$

The proofs of Theorems 1, 2, and 3 follow the same format. We present the proof of Theorem 3 which includes scalar factors (powers of 2).

Proof of Theorem 2. The proof is very similar to those of Theorem 2.1, Proposition 2.3, and Proposition 2.4 of our previous article [3]. □

Proof of Theorem 3. Lagrange's interpolation formula yields

$$\begin{aligned} R_n(x) &= \sum_{i=0}^{n} 2^{n-1-i} F_{2n+i+3} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-j}{i-j} \\ &= (-1)^n 2^{n-1}(n+1)\binom{x}{n+1} \sum_{i=0}^{n} \frac{(-1)^i}{2^i}\binom{n}{i}\frac{F_{2n+i+3}}{x-i}, \end{aligned} \tag{25}$$

where we set the empty product to 1.

We proceed by induction on n. As we will see, two base cases are necessary. First, $R_0(x)$ is a constant polynomial satisfying $R_0(0) = F_3/2 = 1$, so $R_0(x) = 1$ with leading coefficient $1 = (-1)^0/0!$ as claimed. We want to show that, for integers $k > 0$,

$$R_0(k) = 1 = \frac{F_{k+3}}{2^{k+1}} + \sum_{i=1}^{k} \frac{F_i}{2^{i+1}}$$

and, for integers $k < 0$,

$$R_0(k) = 1 = \frac{F_{k+3}}{2^{k+1}} - \sum_{i=1}^{-k-1} \frac{F_{-i}}{2^{-i+1}}.$$

These follow from substituting $x = 2$ and $x = -1/2$, respectively, into (22).

For $n = 1$, the polynomial $R_1(x)$ satisfies $R_1(0) = F_5 = 5$ and $R_1(1) = F_6/2 = 4$, so $R_1(x) = -x + 5$, a linear polynomial with leading coefficient $-1 = (-1)^1/1!$ as claimed. We want to show that, for integers $k > 1$,

$$R_1(k) = -k + 5 = \frac{F_{k+5}}{2^k} - \sum_{i=1}^{k-1} \frac{F_i}{2^{i+1}}(k - i) \tag{26}$$

and, for integers $k < 0$,

$$R_1(k) = -k + 5 = \frac{F_{k+5}}{2^k} + \sum_{i=1}^{-k-1} \frac{F_{-i}}{2^{-i+1}}(k + i). \tag{27}$$

Given $k > 1$, use $x = 2$ in (23) to see

$$\begin{aligned}
\sum_{i=1}^{k-1} \frac{F_i}{2^{i+1}}(k - i) &= \frac{1}{2^k}\mathcal{F}_k'(2) \\
&= \frac{1}{2^k}\big((k+1)2^k - 3\mathcal{F}_k(2) - F_{k+1}\big) \\
&= \frac{1}{2^k}\big((k+1)2^k - 3(2^{k+1} - F_{k+3}) - F_{k+1}\big) \\
&= k - 5 + \frac{1}{2^k}F_{k+5},
\end{aligned}$$

where we apply (15) in the last equality. Rearranging gives (26).

For (27), assume $k < 0$ and let $m = -k$. Using $x = -1/2$ in (23) gives

$$\sum_{i=1}^{-k-1} \frac{F_{-i}}{2^{-i+1}}(k+i)$$

$$= (-1)^{m+1} 2^{m-2} \mathcal{F}'_m\left(-\frac{1}{2}\right)$$

$$= (-1)^m 2^m \left(2\mathcal{F}_m\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^m (m+1) - F_{m+1}\right)$$

$$= (-1)^m 2^m \left((-8)\left(\left(-\frac{1}{2}\right)^{m+1} + \frac{1}{2}F_{m+1} - F_m\right) + \left(-\frac{1}{2}\right)^m (m+1) - F_{m+1}\right)$$

$$= (-1)^m 2^m \left(-F_{m-5} + \left(-\frac{1}{2}\right)^m (m+5)\right),$$

again using (15). Putting the expression back in terms of k establishes (27), which completes the base cases of the induction.

Let $n > 1$ and assume that the theorem holds for all nonnegative integers no greater than n. We prove that $R_{n+1}(x)$ has degree $n+1$ and leading coefficient $(-1)^{n+1}/(n+1)!$, more specifically, that

$$R_{n+1}(k) = 2^{n-k} F_{k+2n+5} + (-1)^{n+1} \sum_{i=1}^{k-n-1} \frac{F_i}{2^{i+1}} \binom{k-i}{n+1} \tag{28}$$

for each integer $k > n+1$ and

$$R_{n+1}(k) = 2^{n-k} F_{k+2n+5} + (-1)^{n} \sum_{i=1}^{-k-1} \frac{F_{-i}}{2^{-i+1}} \binom{k+i}{n+1} \tag{29}$$

for each integer $k < 0$.

We begin with (28); let $k > n+1$. Using $x = 2$ in (24) leads to

$$\mathcal{F}_k^{(n+1)}(2) + 3(n+1)\mathcal{F}_k^{(n)}(2) + n(n+1)\mathcal{F}_k^{(n-1)}(2) = 2^{k-n}(k+1)_{n+1}. \tag{30}$$

The induction hypothesis and (7) with $x = 2$ give

$$R_n(k) = 2^{n-k-1}\left(F_{k+2n+3} + \frac{(-1)^n}{n!}\mathcal{F}_k^{(n)}(2)\right),$$

$$R_{n-1}(k) = 2^{n-k-2}\left(F_{k+2n+1} + \frac{(-1)^{n-1}}{(n-1)!}\mathcal{F}_k^{(n-1)}(2)\right).$$

Substituting these into (30), we have

$$\left(\sum_{i=1}^{k-n-1} 2^{k-n-1-i} F_i (k-i)_{n+1}\right) + 3(n+1)2^{1+k-n} n! (-1)^n \left(R_n(k) - 2^{n-k-1}F_{k+2n+3}\right)$$
$$+ n(n+1)2^{2+k-n}(n-1)!(-1)^{n-1}\left(R_{n-1}(k) - 2^{n-k-2}F_{k+2n+1}\right) = (k+1)_{n+1}.$$

Rearranging and using the identity (16) gives

$$(-1)^{n+1}\binom{k+1}{n+1} + 6R_n(k) - 4R_{n-1}(k) = 2^{n-k}F_{k+2n+5} + (-1)^{n+1}\sum_{i=1}^{k-n-1}\frac{F_i}{2^{i+1}}\binom{k-i}{n+1}.$$

Now define the polynomial $g(x)$ as

$$g(x) = (-1)^{n+1}\binom{x+1}{n+1} + 6R_n(x) - 4R_{n-1}(x).$$

To complete the verification of (28), it suffices to show $g(x) = R_{n+1}(x)$. This will also establish (8).

Notice that $g(x)$ is a degree $n+1$ polynomial with leading coefficient $(-1)^{n+1}/(n+1)!$. It is then enough to verify that $g(k) = 2^{n-k}F_{k+2n+5}$ for $k = 0, \ldots, n+1$. We do this in three cases, each using (16).

- For $0 \le k \le n-1$, we have $R_n(k) = 2^{n-k-1}F_{k+2n+3}$ and $R_{n-1}(k) = 2^{n-k-2}F_{k+2n+1}$. Since $\binom{k+1}{n+1} = 0$,

$$g(k) = 6\cdot 2^{n-k-1}F_{k+2n+3} - 4\cdot 2^{n-k-2}F_{k+2n+1} = 2^{n-k}(3F_{k+2n+3} - F_{k+2n+1}) = 2^{n-k}F_{k+2n+5}.$$

- For $k = n$, we have $R_n(n) = F_{3n+3}/2$ and

$$R_{n-1}(n) = \frac{F_{3n+1}}{2^2} + (-1)^{n-1}\frac{F_1}{2^2}\binom{n-1}{n-1} = \frac{F_{3n+1}}{4} + \frac{(-1)^{n-1}}{4}.$$

  Hence we have

$$g(n) = (-1)^{n+1} + 6\frac{F_{3n+3}}{2} - 4\left(\frac{F_{3n+1}}{4} + \frac{(-1)^{n-1}}{4}\right) = F_{3n+5}.$$

- For $k = n+1$, we have

$$\begin{aligned} R_n(n+1) &= \frac{F_{3n+4}}{4} + \frac{(-1)^n}{4}, \\ R_{n-1}(n+1) &= \frac{F_{3n+2}}{8} + \frac{(-1)^{n-1}}{4}n + \frac{(-1)^{n-1}}{8} \end{aligned}$$

  so that

$$\begin{aligned} g(n+1) &= (-1)^{n+1}(n+2) + 6\left(\frac{F_{3n+4}}{4} + \frac{(-1)^n}{4}\right) - 4\left(\frac{F_{3n+2}}{8} + \frac{(-1)^{n-1}}{4}n + \frac{(-1)^{n-1}}{8}\right) \\ &= \frac{F_{3n+6}}{2}. \end{aligned}$$

It remains to establish (29). Assume $k < 0$ and let $m = -k$. Also, let $m' = m + n$. Recall the identity

$$\binom{a}{b} = (-1)^b \binom{b-a-1}{b} \tag{31}$$

for integers a and b with $b \geq 0$. This allows us to write the sum in (29) as

$$\begin{aligned}
\sum_{i=1}^{-k-1} \frac{F_{-i}}{2^{-i+1}} \binom{k+i}{n+1} &= \sum_{i=1}^{m-1} \frac{F_{-i}}{2^{-i+1}} \binom{-m+i}{n+1} \\
&= (-1)^{n+1} \sum_{i=1}^{m-1} \frac{F_{-i}}{2^{-i+1}} \binom{m+n-i}{n+1} \\
&= \frac{(-1)^{n+1}}{(n+1)!} \sum_{i=1}^{m-1} \frac{(-1)^{i-1}}{2^{-i+1}} F_i (m+n-i)_{n+1} \\
&= \frac{(-1)^{n+1}}{(n+1)!} \sum_{i=1}^{m'-(n+1)} \left(-\frac{1}{2}\right)^{-i+1} F_i (m'-i)_{n+1} \\
&= \frac{(-1)^{n+1}}{(n+1)!} \left(-\frac{1}{2}\right)^{(n+2)-m'} \sum_{i=1}^{m'-(n+1)} \left(-\frac{1}{2}\right)^{m'-(n+1)-i} F_i (m'-i)_{n+1} \\
&= \frac{(-1)^{m+n+1}}{2^{2-m}(n+1)!} \mathcal{F}_{m+n}^{(n+1)} \left(-\frac{1}{2}\right).
\end{aligned}$$

Using (24) with $x = -1/2$, we write the last expression as

$$\begin{aligned}
&\frac{(-1)^{m+n+1}}{2^{2-m}(n+1)!} \\
&\times \left(-8(n+1)\mathcal{F}_{m+n}^{(n)} \left(-\frac{1}{2}\right) + 4n(n+1)\mathcal{F}_{m+n}^{(n-1)} \left(-\frac{1}{2}\right) - 4 \left(-\frac{1}{2}\right)^m (m+n+1)_{n+1} \right).
\end{aligned} \tag{32}$$

Next, we determine $\mathcal{F}_{m+n}^{(n)}(-1/2)$ and $\mathcal{F}_{m+n}^{(n-1)}(-1/2)$. Using (21) with $x = -1/2$, identity (31), and the induction hypothesis gives

$$\begin{aligned}
\mathcal{F}_{m+n}^{(n)} \left(-\frac{1}{2}\right) &= \sum_{i=1}^{m} \left(-\frac{1}{2}\right)^{m-i} F_i (m+n-i)_n \\
&= n! \sum_{i=1}^{(m+1)-1} \left(-\frac{1}{2}\right)^{m-i} (-1)^{n+i+1} F_{-i} \binom{-(m+1)+i}{n} \\
&= \frac{(-1)^{m+n+1} n!}{2^{m-1}} \sum_{i=1}^{(m+1)-1} \frac{F_{-i}}{2^{-i+1}} \binom{-(m+1)+i}{n} \\
&= \frac{(-1)^m n!}{2^{m-1}} \left(R_n(-m-1) - 2^{m+n} F_{2n-m+2} \right).
\end{aligned}$$

Similarly,

$$\mathcal{F}_{m+n}^{(n-1)}\left(-\frac{1}{2}\right) = \frac{(-1)^{m+1}(n-1)!}{2^m}\left(R_{n-1}(-m-2) - 2^{m+n}F_{2n-m-1}\right).$$

Substituting these expressions into (32) gives

$$\begin{aligned}
&\frac{(-1)^{m+n+1}}{2^{2-m}(n+1)!}\Bigg(-8(n+1)\frac{(-1)^m n!}{2^{m-1}}\left(R_n(-m-1) - 2^{m+n}F_{2n-m+2}\right)\\
&\qquad + 4n(n+1)\frac{(-1)^{m+1}(n-1)!}{2^m}\left(R_{n-1}(-m-2) - 2^{m+n}F_{2n-m-1}\right)\\
&\qquad - 4\left(-\frac{1}{2}\right)^m (m+n+1)_{n+1}\Bigg)\\
&= (-1)^{n+1}2^{m+n}F_{2n-m+5} + (-1)^n\left(4R_n(-m-1) + R_{n-1}(-m-2)\right) - \binom{-m-1}{n+1},
\end{aligned}$$

using (17). Now define $j(x)$ as

$$j(x) = (-1)^{n+1}\binom{x-1}{n+1} + 4R_n(x-1) + R_{n-1}(x-2).$$

Establishing $j(x) = R_{n+1}(x)$ will verify (29) and also show (9). As with $g(x)$ above, it suffices to show $j(k) = 2^{n-k}F_{k+2n+5}$ for $k = 0, \ldots, n+1$. The three cases here, presented in less detail, all use (17).

- For $k = 0$,

$$j(0) = (-1)^{n+1}\binom{-1}{n+1} + 4R_n(-1) + R_{n-1}(-2) = 2^n(4F_{2n+2} + F_{2n-1}) = 2^n F_{2n+5}.$$

- For $k = 1$,

$$j(1) = (-1)^{n+1}\binom{0}{n+1} + 4R_n(0) + R_{n-1}(-1) = 2^{n-1}(4F_{2n+3} + F_{2n}) = 2^{n-1}F_{2n+6}.$$

- For $2 \le k \le n+1$,

$$\begin{aligned}
j(k) &= (-1)^{n+1}\binom{k-1}{n+1} + 4R_n(k-1) + R_{n-1}(k-2) = 2^{k-n}(4F_{k+2n+2} + F_{k+2n-1})\\
&= 2^{k-n}F_{k+2n+5}.
\end{aligned}$$

Finally, we show that each of the recurrences (8) and (9) with some initial conditions determines the polynomials $R_n(x)$ completely. That is, the polynomial sequences given by

$$U_0(x) = 1, U_1(x) = -x + 5, \text{ and}$$
$$U_{n+1}(x) = 6R_n(x) - 4R_{n-1}(x) + (-1)^{n+1}\binom{x+1}{n+1} \text{ for } n \geq 1,$$
$$V_0(x) = 1, V_1(x) = -x + 5, \text{ and}$$
$$V_{n+1}(x) = 4V_n(x-1) + V_{n-1}(x-2) + (-1)^{n+1}\binom{x-1}{n+1} \text{ for } n \geq 1$$

satisfy $U_n(x) = V_n(x) = R_n(x)$ for all n. We prove this for $V_n(x)$; the proof for $U_n(x)$ is similar.

By inspection, $V_0(x) = R_0(x)$ and $V_1(x) = R_1(x)$. Assume that $V_n(x) = R_n(x)$ and $V_{n-1}(x) = R_{n-1}(x)$ for some $n \geq 1$. By its defining recurrence, $V_{n+1}(x)$ has degree $n+1$ and leading coefficient $(-1)^{n+1}/(n+1)$, as does $R_{n+1}(x)$, so it is enough to show $V_{n+1}(j) = R_{n+1}(j)$ for $0 \leq j \leq n$. The following verification makes frequent use of (17).

- For $j = 0$, from (7) we have

$$\begin{aligned} V_{n+1}(0) &= 4V_n(-1) + V_{n-1}(-2) + (-1)^{n+1}\binom{-1}{n+1} \\ &= 4R_n(-1) + R_{n-1}(-2) + (-1)^{n+1}(-1)^{n+1} \\ &= 4 \cdot 2^n F_{2n+2} + \left(2^n F_{2n-1} + (-1)^n F_{-1}\binom{-1}{n-1}\right) + 1 \\ &= 2^n F_{2n+5}, \end{aligned}$$

  which matches $R_{n+1}(0)$.

- For $j = 1$, from the definition of $R_n(x)$ and (7) we have

$$\begin{aligned} V_{n+1}(1) &= 4V_n(0) + V_{n-1}(-1) + (-1)^{n+1}\binom{0}{n+1} \\ &= 4R_n(0) + R_{n-1}(-1) \\ &= 4 \cdot 2^{n-1} F_{2n+3} + 2^{n-1} F_{2n} \\ &= 2^{n-1} F_{2n+6}, \end{aligned}$$

  which matches $R_{n+1}(1)$.

- For $2 \leq j \leq n$, from the definition of $R_n(x)$ we have

$$\begin{aligned}V_{n+1}(j) &= 4V_n(j-1) + V_{n-1}(j-2) + (-1)^{n+1}\binom{j-1}{n+1}\\ &= 4R_n(j-1) + R_{n-1}(j-2)\\ &= 4\cdot 2^{n-j}F_{2n+j+2} + 2^{n-j}F_{2n+j-1}\\ &= 2^{n-j}F_{2n+j+5},\end{aligned}$$

which matches $R_{n+1}(j)$ for $2 \le j \le n$. □

It is apparent from the statements of Theorems 4 and 5 that these are more complicated results, with different interpolation points depending on the parity of the degree n. Nonetheless, the $S_n(x)$ polynomials are connected between even and odd values of n by relations including (11). The $T_n(x)$ polynomials have more connections between parities, e.g., (13) and (14). We detail the first of these two similar proofs.

Proof of Theorem 4. We proceed by induction on n with two base cases. First, $S_0(x)$ is a constant polynomial satisfying $S_0(0) = -L_0/2 = -2/2 = -1$, i.e., $S_0(x) = -1$ with leading coefficient $-1 = -5^{0/2}/0!$ as claimed. We want to show that, for integers $k > 0$,

$$S_0(k) = -1 = \frac{L_k}{(-2)^{1+k}} + \frac{5}{2}\sum_{i=1}^{k}\frac{F_i}{(-2)^i}$$

and, for integers $k < 0$,

$$S_0(k) = -1 = \frac{L_k}{(-2)^{1+k}} + 5\sum_{i=1}^{-k-1}(-2)^{i-1}F_{-i}.$$

These follow from substituting $x = -2$ and $x = 1/2$, respectively, into (22).

For $n = 1$, the polynomial S_1 satisfies $S_1(0) = 2^0F_1 = 1$ and $S_1(1) = 2^{-1}F_0 = 0$, so $S_1(x) = -x + 1$, a linear polynomial with leading coefficient $-1 = -5^{(1-1)/2}/1!$ as claimed. We want to establish that, for integers $k > 1$,

$$S_1(k) = -k + 1 = \frac{F_{k-1}}{(-2)^k} - 5\sum_{i=1}^{k-1}\frac{F_i}{(-2)^{i+1}}(k-i) \tag{33}$$

and, for integers $k < 0$,

$$S_1(k) = -k + 1 = \frac{F_{k-1}}{(-2)^k} + 5\sum_{i=1}^{-k-1}(-2)^{i-1}F_{-i}(k+i). \tag{34}$$

Given $k > 1$, using $x = -2$ in the equations of Lemma 7 yields

$$\mathcal{F}_k'(-2) = \sum_{i=1}^{k-1} (-2)^{k-1-i} F_i (k-i),$$
$$5\mathcal{F}_k(-2) = (-2)^{k+1} + 2F_{k+1} - F_k,$$
$$5\mathcal{F}_k'(-2) = (k+1)(-2)^k + 5\mathcal{F}_k(-2) - F_{k+1}.$$

Hence we have

$$\begin{aligned}\sum_{i=1}^{k-1} \frac{F_i}{(-2)^{i+1}}(k-i) &= \frac{1}{(-2)^k}\mathcal{F}_k'(-2) \\ &= \frac{1}{(-2)^k}\left(\frac{1}{5}(k+1)(-2)^k + \mathcal{F}_k(-2) - \frac{1}{5}F_{k+1}\right) \\ &= \frac{1}{5(-2)^k}\left((k+1)(-2)^k + (-2)^{k+1} + 2F_{k+1} - F_k - F_{k+1}\right) \\ &= \frac{1}{5}\left(k - 1 + \frac{F_{k-1}}{(-2)^k}\right).\end{aligned}$$

Rearranging gives (33).

For (34), assume $k < 0$ and write $k = -m$. Substituting $x = 1/2$ into (21) and (23) gives

$$\mathcal{F}_m'\left(\frac{1}{2}\right) = \sum_{i=1}^{m-1} \left(\frac{1}{2}\right)^{m-1-i} F_i(m-i),$$
$$-\frac{5}{4}\mathcal{F}_m'\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^m (m+1) - F_{m+1}.$$

Hence

$$\begin{aligned}\sum_{i=1}^{-k-1} (-2)^{i-1} F_{-i}(k+i) = -\sum_{i=1}^{m-1} (-2)^{i-1} F_{-i}(m-i) &= -2^{m-2}\mathcal{F}_m'\left(\frac{1}{2}\right) \\ = \frac{1}{5}(m + 1 - 2^m F_{m+1}) &= \frac{1}{5}\left(-k + 1 - \frac{F_{k-1}}{(-2)^k}\right).\end{aligned}$$

This establishes (34), completing the base cases.

Let $n > 1$ and assume that the theorem holds for all nonnegative integers no greater than n. We want to prove that, if $n + 1$ is even, then $S_{n+1}(x)$ has degree $n + 1$ and leading coefficient $-5^{(n+1)/2}/(n + 1)!$ and, for each integer $k > n + 1$,

$$S_{n+1}(k) = \frac{L_{k-n-1}}{(-2)^{k-n}} - 5^{\frac{n+3}{2}} \sum_{i=1}^{k-n-1} \frac{F_i}{(-2)^{i+1}}\binom{k-i}{n+1} \tag{35}$$

and, for each integer $k < 0$,

$$S_{n+1}(k) = \frac{L_{k-n-1}}{(-2)^{k-n}} + 5^{\frac{n+3}{2}} \sum_{i=1}^{-k-1} (-2)^{i-1} F_{-i} \binom{k+i}{n+1}. \tag{36}$$

If $n+1$ is odd, then $S_{n+1}(x)$ has degree $n+1$ and leading coefficient $-5^{n/2}/(n+1)!$ and, for each integer $k > n+1$,

$$S_{n+1}(k) = \frac{F_{k-n-1}}{(-2)^{k-n}} - 5^{\frac{n+2}{2}} \sum_{i=1}^{k-n-1} \frac{F_i}{(-2)^{i+1}} \binom{k-i}{n+1} \tag{37}$$

and, for each integer $k < 0$,

$$S_{n+1}(k) = \frac{F_{k-n-1}}{(-2)^{k-n}} + 5^{\frac{n+2}{2}} \sum_{i=1}^{-k-1} (-2)^{i-1} F_{-i} \binom{k+i}{n+1}. \tag{38}$$

We begin with odd n, thus (35) and (36). Let $k > n+1$. Using $x = -2$ in (24) gives

$$5\mathcal{F}_k^{(n+1)}(-2) - 5(n+1)\mathcal{F}_k^{(n)}(-2) + n(n+1)\mathcal{F}_k^{(n-1)}(-2) = (-2)^{k-n}(k+1)_{n+1}. \tag{39}$$

By the induction hypothesis, $S_n(x)$ has degree n and leading coefficient $-5^{(n-1)/2}/n!$ while $S_{n-1}(x)$, since $n-1$ is even, has degree $n-1$ and leading coefficient $-5^{(n-1)/2}/(n-1)!$. Also,

$$S_n(k) = \frac{F_{k-n}}{(-2)^{1+k-n}} - \frac{5^{\frac{n+1}{2}}}{(-2)^{1+k-n} n!} \mathcal{F}_k^{(n)}(-2),$$

$$S_{n-1}(k) = \frac{L_{k-n+1}}{(-2)^{2+k-n}} - \frac{5^{\frac{n+1}{2}}}{(-2)^{2+k-n}(n-1)!} \mathcal{F}_k^{(n-1)}(-2).$$

By (21) with $x = -2$ and (39),

$$5 \sum_{i=1}^{k-n-1} (-2)^{k-n-1-i} F_i (k-i)_{n+1} - 5(n+1)(-2)^{1+k-n} n! 5^{-\frac{n+1}{2}} \left(\frac{F_{k-n}}{(-2)^{1+k-n}} - S_n(k) \right)$$
$$+ n(n+1)(-2)^{2+k-n}(n-1)! 5^{-\frac{n+1}{2}} \left(\frac{L_{k-n+1}}{(-2)^{2+k-n}} - S_{n-1}(k) \right) = (-2)^{k-n}(k+1)_{n+1}.$$

Rearranging and (19) give

$$-4S_{n-1}(k) - 10S_n(k) - 5^{\frac{n+1}{2}} \binom{k+1}{n+1} = \frac{L_{k-n-1}}{(-2)^{k-n}} - 5^{\frac{n+3}{2}} \sum_{i=1}^{k-n-1} \frac{F_i}{(-2)^{i+1}} \binom{k-i}{n+1}.$$

To prove (35), let

$$g(x) = -4S_{n-1}(x) - 10S_n(x) - 5^{\frac{n+1}{2}} \binom{x+1}{n+1}.$$

We show that $g(x) = S_{n+1}(x)$. Notice that $-5^{(n+1)/2}/(n+1)!$ is the leading coefficient of $g(x)$. Thus it suffices to verify that $g(k) = L_{k-n-1}/(-2)^{k-n}$ for $k = 0, \ldots, n+1$. We do this in three cases.

- For $0 \le k \le n-1$, we have $S_n(k) = F_{k-n}/(-2)^{1+k-n}$ and $S_{n-1}(k) = L_{k-n+1}/(-2)^{2+k-n}$ from the definition. Also, $\binom{k+1}{n+1} = 0$, so

$$\begin{aligned} g(k) &= -4\frac{L_{k-n+1}}{(-2)^{2+k-n}} - 10\frac{F_{k-n}}{(-2)^{1+k-n}} - 5^{\frac{n+1}{2}}\binom{k+1}{n+1} \\ &= \frac{1}{(-2)^{k-n}}(-L_{k-n+1} + 5F_{k-n}) = \frac{L_{k-n-1}}{(-2)^{k-n}}. \end{aligned}$$

- For $k = n$, we have $S_n(n) = -F_0/2 = 0$ and, by the induction hypothesis,

$$S_{n-1}(n) = \frac{L_1}{(-2)^2} - 5^{\frac{n+1}{2}}\frac{F_1}{(-2)^2}\binom{n-1}{n-1} = \frac{1}{4}\left(1 - 5^{\frac{n+1}{2}}\right).$$

  Also, $\binom{k+1}{n+1} = \binom{n+1}{n+1} = 1$, so $g(k) = -1 + 5^{\frac{n+1}{2}} - 5^{\frac{n+1}{2}} = -1 = L_{-1}$.

- For $k = n+1$, by the induction hypothesis,

$$\begin{aligned} S_n(n+1) &= \frac{F_1}{(-2)^2} - 5^{\frac{n+1}{2}}\frac{F_1}{(-2)^2}\binom{n}{n} \\ &= \frac{1}{4}\left(1 - 5^{\frac{n+1}{2}}\right), \\ S_{n-1}(n+1) &= \frac{L_2}{(-2)^3} - 5^{\frac{n+1}{2}}\left(\frac{F_1}{(-2)^2}\binom{n}{n-1} + \frac{F_2}{(-2)^3}\binom{n-1}{n-1}\right) \\ &= -\frac{1}{8}\left(3 + 5^{\frac{n+1}{2}}(2n-1)\right). \end{aligned}$$

  Also, $\binom{k+1}{n+1} = \binom{n+2}{n+1} = n+2$. Altogether,

$$g(k) = \frac{3}{2} + 5^{\frac{n+1}{2}}\left(n - \frac{1}{2}\right) - \frac{5}{2}\left(1 - 5^{\frac{n+1}{2}}\right) - 5^{\frac{n+1}{2}}(n+2) = -1 = -\frac{L_1}{2}.$$

Thus we have established (35) and the polynomial recurrence, for n odd,

$$S_{n+1}(x) + 10S_n(x) + 4S_{n-1}(x) = -5^{\frac{n+1}{2}}\binom{x+1}{n+1}. \tag{40}$$

For (36), let $k < 0$ and write $k = -m$. Also, let $m' = m + n$. Using (31), we can write the sum in (36) as

$$\sum_{i=1}^{-k-1}(-2)^{i-1}F_{-i}\binom{k+i}{n+1} = \sum_{i=1}^{m-1}(-2)^{i-1}F_{-i}\binom{-m+i}{n+1}$$

$$= (-1)^{n+1} \sum_{i=1}^{m-1} (-2)^{i-1} F_{-i} \binom{m+n-i}{n+1}$$
$$= \frac{(-1)^{n+1}}{(n+1)!} \sum_{i=1}^{m-1} (-1)^{i-1} (-2)^{i-1} F_i (m+n-i)_{n+1}$$
$$= \frac{(-1)^{n+1}}{(n+1)!} \sum_{i=1}^{m'-(n+1)} \left(\frac{1}{2}\right)^{-i+1} F_i (m'-i)_{n+1}$$
$$= \frac{(-1)^{n+1}}{(n+1)!} \left(\frac{1}{2}\right)^{(n+2)-m'} \sum_{i=1}^{m'-(n+1)} \left(\frac{1}{2}\right)^{m'-(n+1)-i} F_i (m'-i)_{n+1}$$
$$= \frac{(-1)^{n+1}}{2^{2-m}(n+1)!} \mathcal{F}_{m+n}^{(n+1)} \left(\frac{1}{2}\right).$$

Using (24) with $x = 1/2$, we write the last expression as

$$\frac{(-1)^{n+1}}{2^{2-m}(n+1)!} \left(\frac{4}{5} n(n+1) \mathcal{F}_{m+n}^{(n-1)} \left(\frac{1}{2}\right) - \frac{4}{5} \left(\frac{1}{2}\right)^m (m+n+1)_{n+1} \right). \tag{41}$$

Next, we determine $\mathcal{F}_{m+n}^{(n-1)}(1/2)$. Using (21), (31), and the induction hypothesis gives

$$\mathcal{F}_{m+n}^{(n-1)} \left(\frac{1}{2}\right) = \sum_{i=1}^{m+1} \left(\frac{1}{2}\right)^{m+1-i} F_i (m+n-i)_{n-1}$$
$$= (n-1)! \sum_{i=1}^{(m+2)-1} \left(\frac{1}{2}\right)^{m+1-i} (-1)^{n+i} F_{-i} \binom{-(m+2)+i}{n-1}$$
$$= \frac{(-1)^{n+1}(n-1)!}{2^m} \sum_{i=1}^{(m+2)-1} (-2)^{i-1} F_{-i} \binom{-(m+2)+i}{n-1}$$
$$= (n-1)! 2^{-m} 5^{-\frac{n+1}{2}} \left(S_{n-1}(-m-2) + 2^{m+n} L_{m+n+1} \right).$$

Substituting this expression into (41) gives

$$\frac{(-1)^{n+1}}{2^{2-m}(n+1)!} \Bigg(\frac{4}{5} n(n+1)(n-1)! 2^{-m} 5^{-\frac{n+1}{2}} \left(S_{n-1}(-m-2) + 2^{m+n} L_{m+n+1} \right)$$
$$- \frac{4}{5} \left(\frac{1}{2}\right)^m (m+n+1)_{n+1} \Bigg)$$
$$= 5^{-\frac{n+3}{2}} (S_{n-1}(-m-2) + 2^{m+n} L_{m+n+1}) - \frac{1}{5} \binom{m+n+1}{n+1}.$$

This implies

$$S_{n-1}(k-2) - 5^{\frac{n+1}{2}} \binom{k-1}{n+1} = \frac{L_{k-n-1}}{(-2)^{k-n}} + 5^{\frac{n+3}{2}} \sum_{i=1}^{-k-1} (-2)^{i-1} F_{-i} \binom{k+i}{n+1}.$$

Now define $j(x)$ as

$$j(x) = S_{n-1}(x-2) - 5^{\frac{n+1}{2}} \binom{x-1}{n+1}.$$

Establishing $j(x) = S_{n+1}(x)$ will verify (36). It suffices to show that $j(k) = -2^{n-k} L_{n-k+1}$ for $k = 0, \ldots, n+1$. We consider three cases.

- For $k = 0$,

$$\begin{aligned} j(0) &= S_{n-1}(-2) - 5^{\frac{n+1}{2}} \binom{-1}{n+1} \\ &= \frac{L_{-1-n}}{(-2)^{-n}} + 5^{\frac{n+1}{2}} F_{-1} \binom{-1}{n+1} - 5^{\frac{n+1}{2}} \binom{-1}{n+1} \\ &= -2^n L_{n+1}. \end{aligned}$$

- For $k = 1$,

$$j(1) = S_{n-1}(-1) - 5^{\frac{n+1}{2}} \binom{0}{n+1} = -2^{n-1} L_n.$$

- For $2 \le k \le n+1$, since $\binom{k-1}{n+1} = 0$, we have

$$j(k) = S_{n-1}(k-2) - 5^{\frac{n+1}{2}} \binom{k-1}{n+1} = -2^{n-k} L_{n-k+1}.$$

Thus we have established (36) and the polynomial recurrence, for n odd,

$$S_{n+1}(x) - S_{n-1}(x-2) = -5^{\frac{n+1}{2}} \binom{x-1}{n+1}. \tag{42}$$

For n even, the proofs of (37) and (38) are completely analogous to the odd case. The polynomial recurrences corresponding to (40) and (42) that arise for n even are

$$S_{n+1}(x) + 2S_n(x) + 4S_{n-1}(x) = -5^{\frac{n}{2}} \binom{x+1}{n+1}, \tag{43}$$

$$S_{n+1}(x) - S_{n-1}(x-2) = -5^{\frac{n}{2}} \binom{x-1}{n+1}. \tag{44}$$

Combining (40) and (43) gives (11), likewise (42) and (44) for (12).

Finally, we want to show that each of the recurrences (11) and (12) with some initial conditions determines the polynomials $S_n(x)$ completely. That is, the polynomial sequences given by

$$U_0(x) = -1, U_1(x) = -x + 1, \text{ and}$$
$$U_{n+1}(x) = -2 \cdot 5^{\frac{1-(-1)^n}{2}} U_n(x) - 4U_{n-1}(x) - 5^{\lfloor \frac{n+1}{2} \rfloor} \binom{x+1}{n+1} \text{ for } n \geq 1,$$
$$V_0(x) = -1, V_1(x) = -x + 1, \text{ and}$$
$$V_{n+1}(x) = V_{n-1}(x-2) - 5^{\lfloor \frac{n+1}{2} \rfloor} \binom{x-1}{n+1} \text{ for } n \geq 1$$

satisfy $U_n(x) = V_n(x) = S_n(x)$ for all n. We prove this for $U_n(x)$; the proof for $V_n(x)$ is similar.

By inspection, $U_0(x) = S_0(x)$ and $U_1(x) = S_1(x)$. Assume that $U_n(x) = S_n(x)$ and $U_{n-1}(x) = S_{n-1}(x)$ for some $n \geq 1$. By its defining recurrence, $U_{n+1}(x)$ has degree $n+1$ and leading coefficient $-5^{\lfloor (n+1)/2 \rfloor}$, as does $S_{n+1}(x)$, so it is enough to show $U_{n+1}(j) = S_{n+1}(j)$ for $0 \leq j \leq n$. We need to consider n even or odd separately, each with two cases.

For n even:

- For $0 \leq j \leq n-1$,

$$\begin{aligned} U_{n+1}(j) &= -2 \cdot 5^{\frac{1-(-1)^n}{2}} S_n(j) - 4S_{n-1}(j) - 5^{\lfloor \frac{n+1}{2} \rfloor} \binom{j+1}{n+1} \\ &= -2 \cdot 5^0(-2^{n-1-j} L_{n-j}) - 4(-2)^{n-2-j} F_{j-n+1} \\ &= 2^{n-j}(F_{n-j+1} + F_{n-j-1}) - (-2)^{n-j} F_{j-n+1} \\ &= (-2)^{n-j} F_{j-n+1}, \end{aligned}$$

  which matches $S_{n+1}(j)$.

- For $j = n$, we have $U_{n+1}(n) = -2(-1) - (1 - 5^{\frac{n}{2}}) - 5^{\frac{n}{2}} = 1$, which matches $S_{n+1}(n)$.

For n odd:

- For $0 \leq j \leq n-1$,

$$\begin{aligned} U_{n+1}(j) &= -2 \cdot 5^1((-2)^{n-1-j} F_{j-n}) - 4(-2^{n-2-j}) L_{n-1-j} - 0 \\ &= -2^{n-j}(L_{n-j+1} + L_{n-j-1}) + 2^{n-j} L_{n-j-1} \\ &= -2^{n-j} L_{n-j+1}, \end{aligned}$$

  which matches $S_{n+1}(j)$.

- For $j = n$, we have $U_{n+1}(n) = -2 \cdot 5 \cdot 0 - (1 - 5^{\frac{n+1}{2}}) - 5^{\frac{n+1}{2}} = -1$, which matches $S_{n+1}(n)$. □

Proof of Theorem 5. The proof is analogous to that of Theorem 4. □

4 Expansion in binomial polynomials

The binomial polynomials $\binom{x}{n}$ provide an alternative basis for polynomials: For any degree n polynomial $h(x)$ with complex coefficients, there exist unique complex numbers $a_0, \ldots, a_n$ such that $h(x) = \sum_{i=0}^{n} a_i \binom{x}{i}$. Moreover, the coefficients $a_0, \ldots, a_n$ can be determined by

$$a_i = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} h(k) \tag{45}$$

for $i = 0, \ldots, n$. Cahen and Chabert [1, 2] provide background on integer-valued polynomials such as these, including historical notes mentioning Newton and Pólya.

In this section we determine the binomial polynomial expansions for the polynomials of Section 2. This will contribute to the identities of Section 5. As in Section 3, we provide complete proofs for two of the five results, as the remaining proofs are very similar.

Theorem 8. *The polynomials $P_n(x)$ defined in Theorem 1 satisfy*

$$P_n(x) = \sum_{i=0}^{n} F_{n-i+2} \binom{x}{i}. \tag{46}$$

Proof. Multiplying (4) by t^n and forming the formal power series over n gives

$$\sum_{n=1}^{\infty} t^n P_{n+1}(x) - \sum_{n=1}^{\infty} t^n P_n(x) - \sum_{n=1}^{\infty} t^n P_{n-1}(x) = \sum_{n=1}^{\infty} t^n \binom{x+1}{n+1}.$$

Letting $g(t,x) = \sum_{n=0}^{\infty} t^n P_n(x)$, we have

$$\frac{1}{t}(g(t,x) - P_0(x) - tP_1(x)) - (g(t,x) - P_0(x)) - tg(t,x) = \sum_{n=1}^{\infty} t^n \binom{x+1}{n+1}.$$

Putting in $P_0(x) = 1$, $P_1(x) = x + 2$, recognizing the generating function for F_{n+1}, and solving for $g(t,x)$ yields

$$\begin{aligned}
g(t,x) &= \frac{1}{1-t-t^2} \left(1 + (x+1)t + \sum_{n=2}^{\infty} t^n \binom{x+1}{n} \right) \\
&= \left(\sum_{n=0}^{\infty} F_{n+1} t^n \right) \left(\sum_{n=0}^{\infty} t^n \binom{x+1}{n} \right) \\
&= \sum_{n=0}^{\infty} t^n \sum_{i=0}^{n} F_{n-i+1} \binom{x+1}{i}
\end{aligned}$$

so that

$$P_n(x) = \sum_{i=0}^{n} F_{n-i+1}\binom{x+1}{i} \tag{47}$$

$$= \sum_{i=0}^{n} F_{n-i+1}\binom{x}{i} + \sum_{i=1}^{n} F_{n-i+1}\binom{x}{i-1}$$

$$= \sum_{i=0}^{n} F_{n-i+2}\binom{x}{i}. \qquad \square$$

The analogous results for $Q_n(x)$ and $R_n(x)$ have very similar proofs.

Theorem 9. *The polynomials $Q_n(x)$ defined in Theorem 2 satisfy*

$$Q_n(x) = \sum_{i=0}^{n} (-1)^i F_{2n-2i+1}\binom{x}{i}. \tag{48}$$

Theorem 10. *The polynomials $R_n(x)$ defined in Theorem 3 satisfy*

$$R_n(x) = \sum_{i=0}^{n} (-1)^i 2^{n-i-1} F_{2n-2i+3}\binom{x}{i}. \tag{49}$$

We record the equations analogous to (47):

$$Q_n(x) = \sum_{i=0}^{n} (-1)^i F_{2n-2i+2}\binom{x+1}{i}, \tag{50}$$

$$R_n(x) = \sum_{i=0}^{n} (-1)^i 2^{n-i} F_{2n-2i+2}\binom{x+1}{i}. \tag{51}$$

Again, $S_n(x)$ and $T_n(x)$ have more complicated results. We provide the complete proof for $T_n(x)$; the proof for $S_n(x)$ is very similar.

Theorem 11. *The polynomials $S_n(x)$ defined in Theorem 4 satisfy,*

- *for $n \geq 2$,*

$$S_{n+2}(x) - 12S_n(x) + 16S_{n-2}(x) = -5^{\lfloor \frac{n}{2} \rfloor}\left(5\binom{x+1}{n+2} - 10\binom{x+1}{n+1} + 4\binom{x+1}{n}\right);$$

- *for positive even n,*

$$S_n(x) = -\sum_{i=0}^{n/2} 5^{i-1} 2^{n-2i} F_{n-2i+2}\left(5\binom{x+1}{2i} - 10\binom{x+1}{2i-1} + 4\binom{x+1}{2i-2}\right), \tag{52}$$

$$S_n(x) = \sum_{i=0}^{n} (-1)^{i+1} 5^{\lfloor \frac{i+1}{2} \rfloor} 2^{n-i-1} b_{n-i}\binom{x}{i} \quad \text{with } b_k \text{ as defined in (2);} \tag{53}$$

- *for positive odd n,*

$$S_n(x) = -\sum_{i=0}^{(n+1)/2} 5^{i-2}2^{n-2i+1}F_{n-2i+3}\left(5\binom{x+1}{2i-1} - 10\binom{x+1}{2i-2} + 4\binom{x+1}{2i-3}\right), \quad (54)$$

$$S_n(x) = \sum_{i=0}^{n}(-1)^i 5^{\lfloor \frac{i}{2} \rfloor}2^{n-i-1}c_{n-i}\binom{x}{i} \quad \text{with } c_k \text{ as defined in (3).} \quad (55)$$

Theorem 12. *The polynomials $T_n(x)$ defined in Theorem 5 satisfy,*

- *for $n \geq 2$,*

$$T_{n+2}(x) - 27T_n(x) + 81T_{n-2}(x) = (-1)^n 5^{\lfloor \frac{n}{2} \rfloor}\left(5\binom{x+1}{n+2} - 15\binom{x+1}{n+1} + 9\binom{x+1}{n}\right); \quad (56)$$

- *for positive even n,*

$$T_n(x) = \sum_{i=0}^{n/2} 5^{i-1}3^{n-2i}F_{n-2i+2}\left(5\binom{x+1}{2i} - 15\binom{x+1}{2i-1} + 9\binom{x+1}{2i-2}\right), \quad (57)$$

$$T_n(x) = \sum_{i=0}^{n}(-1)^i 5^{\lfloor \frac{i+1}{2} \rfloor}3^{n-i-1}b_{n+2-i}\binom{x}{i} \quad \text{with } b_k \text{ as defined in (2);} \quad (58)$$

- *for positive odd n,*

$$T_n(x) = -\sum_{i=0}^{(n+1)/2} 5^{i-2}3^{n-2i+1}F_{n-2i+3}\left(5\binom{x+1}{2i-1} - 15\binom{x+1}{2i-2} + 9\binom{x+1}{2i-3}\right), \quad (59)$$

$$T_n(x) = \sum_{i=0}^{n}(-1)^i 5^{\lfloor \frac{i}{2} \rfloor}3^{n-i-1}c_{n+2-i}\binom{x}{i} \quad \text{with } c_k \text{ as defined in (3).} \quad (60)$$

Proof. To prove (56), we consider the even and odd cases separately.

For n even, (13) gives

$$T_{n+1}(x) - 3T_n(x) + 9T_{n-1}(x) = -5^{\frac{n}{2}}\binom{x+1}{n+1},$$
$$T_n(x) - 15T_{n-1}(x) + 9T_{n-2}(x) = 5^{\frac{n}{2}}\binom{x+1}{n},$$

and a similar equation with initial term $T_{n+2}(x)$. These combine to give

$$T_{n+2}(x) - 27T_n(x) + 81T_{n-2}(x) = 5^{\frac{n}{2}}\left(5\binom{x+1}{n+2} - 15\binom{x+1}{n+1} + 9\binom{x+1}{n}\right). \quad (61)$$

The case of n odd similarly leads to

$$T_{n+2}(x) - 27T_n(x) + 81T_{n-2}(x) = -5^{\frac{n-1}{2}} \left(5\binom{x+1}{n+2} - 15\binom{x+1}{n+1} + 9\binom{x+1}{n}\right). \quad (62)$$

With the integer floor notation, (61) and (62) can be combined as (56).

For (57) and (59), we introduce generating functions, as in the proof of Theorem 8. By (56),

$$\sum_{n=2}^{\infty} t^n T_{n+2}(x) - 27\sum_{n=2}^{\infty} t^n T_n(x) + 81\sum_{n=2}^{\infty} t^n T_{n-2}(x)$$
$$= \sum_{n=2}^{\infty} t^n (-1)^n 5^{\lfloor \frac{n}{2} \rfloor} \left(5\binom{x+1}{n+2} - 15\binom{x+1}{n+1} + 9\binom{x+1}{n}\right).$$

Writing $g(t,x) = \sum_{n=0}^{\infty} t^n T_n(x)$ and using initial $T_n(x)$ values from Table 5, we find

$$g(t,x) = \frac{1}{1-27t^2+81t^4} \sum_{n=0}^{\infty} t^n (-1)^n 5^{\lfloor \frac{n-2}{2} \rfloor} \left(5\binom{x+1}{n} - 15\binom{x+1}{n-1} + 9\binom{x+1}{n-2}\right).$$

From the identity (16) and weighting the Fibonacci numbers by powers of 3, we have the generating function

$$\frac{1}{1-27t^2+81t^4} = \sum_{n=0}^{\infty} 3^{2n} F_{2n+2} t^{2n} = \sum_{n=0}^{\infty} d_n t^n,$$

where

$$d_n = \begin{cases} 3^n F_{n+2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(This d_n allows us to avoid some separate parity cases.) Therefore,

$$g(t,x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} t^n (-1)^k 5^{\lfloor \frac{k-2}{2} \rfloor} d_{n-k} \left(5\binom{x+1}{k} - 15\binom{x+1}{k-1} + 9\binom{x+1}{k-2}\right)$$

and comparing coefficients of t^n gives

$$T_n(x) = \sum_{k=0}^{n} (-1)^k 5^{\lfloor \frac{k-2}{2} \rfloor} d_{n-k} \left(5\binom{x+1}{k} - 15\binom{x+1}{k-1} + 9\binom{x+1}{k-2}\right),$$

a combined expression for (57) and (59).

Finally, for (58) and (60), let $T_n(x) = \sum_{i=0}^{n} a_i \binom{x}{i}$ and apply (45).

If n is even, then for $0 \le i \le n$ we have, using (20) and the binomial theorem,

$$\begin{aligned}
a_i &= \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} T_n(k) \\
&= \sum_{k=0}^{i} (-1)^{i+k} 3^{n-k-1} L_{n+k+2} \binom{i}{k} \\
&= (-1)^i 3^{n-1} \alpha^{n+2} \sum_{k=0}^{i} \left(-\frac{\alpha}{3}\right)^k \binom{i}{k} + (-1)^i 3^{n-1} \beta^{n+2} \sum_{k=0}^{i} \left(-\frac{\beta}{3}\right)^k \binom{i}{k} \\
&= (-1)^i 3^{n-1} \alpha^{n+2} \left(1 - \frac{\alpha}{3}\right)^i + (-1)^i 3^{n-1} \beta^{n+2} \left(1 - \frac{\beta}{3}\right)^i \\
&= (-1)^i 3^{n-i-1} 5^{\frac{i}{2}} (\alpha^{n-i+2} + (-1)^i \beta^{n-i+2}) \\
&= \begin{cases} 3^{n-i-1} 5^{\frac{i}{2}} L_{n-i+2}, & \text{if } i \text{ is even;} \\ -3^{n-i-1} 5^{\frac{i+1}{2}} F_{n-i+2}, & \text{if } i \text{ is odd.} \end{cases}
\end{aligned}$$

Given the definition of the b_k, this is (58).

The case of n odd is analogous using (18):

$$\begin{aligned}
a_i &= \sum_{k=0}^{i} (-1)^{i+k} 3^{n-k-1} F_{n+k+2} \binom{i}{k} \\
&= (-1)^i 3^{n-1} 5^{-\frac{1}{2}} \alpha^{n+2} \left(1 - \frac{\alpha}{3}\right)^i - (-1)^i 3^{n-1} 5^{-\frac{1}{2}} \beta^{n+2} \left(1 - \frac{\beta}{3}\right)^i \\
&= \begin{cases} 3^{n-i-1} 5^{\frac{i}{2}} F_{n-i+2}, & \text{if } i \text{ is even;} \\ -3^{n-i-1} 5^{\frac{i-1}{2}} L_{n-i+2}, & \text{if } i \text{ is odd.} \end{cases}
\end{aligned}$$

Given the definition of the c_k, this is (60). □

5 Applications

We conclude with several identities involving Fibonacci numbers, Lucas numbers, binomial coefficients, and the harmonic numbers that follow from the preceding results.

The first results follow from the theorems of Section 2 concerning the leading coefficients of the various polynomials.

Corollary 13. *For every nonnegative integer n,*

(a) $\displaystyle\sum_{i=0}^{n} (-1)^i F_{2n-i+1} \binom{n}{i} = 1,$

(b) $\displaystyle\sum_{i=0}^{n}\left(-\frac{1}{2}\right)^{i}F_{2n+i+3}\binom{n}{i}=\left(\frac{1}{2}\right)^{n-1}.$

For every even nonnegative n,

(c) $\displaystyle\sum_{i=0}^{n-1}\frac{L_{i-n}}{2^{i}}\binom{n}{i}=\frac{5^{n/2}-1}{2^{n-1}},$

(d) $\displaystyle\sum_{i=0}^{n}\left(-\frac{1}{3}\right)^{i}L_{n+i+2}\binom{n}{i}=\frac{5^{n/2}}{3^{n-1}}.$

For every odd positive n,

(e) $\displaystyle\sum_{i=0}^{n}\frac{F_{i-n}}{2^{i}}\binom{n}{i}=\frac{5^{(n-1)/2}-1}{2^{n-1}},$

(f) $\displaystyle\sum_{i=0}^{n}\left(-\frac{1}{3}\right)^{i}F_{n+i-2}\binom{n}{i}=\frac{5^{(n-1)/2}}{3^{n-1}}.$

Proof. These identities follow from equating the leading coefficients of the polynomials in Section 2 with their alternative forms obtained from the Lagrange interpolation formula, e.g., (25). In particular, (a) follows from Theorem 2 and (b) from Theorem 3. For even n, (c) follows from Theorem 4 and (d) from Theorem 5. Similarly, for odd n, (e) and (f) also follow from Theorems 4 and 5, respectively. □

Next, we give a two-parameter result generalizing Corollary 13(a).

Corollary 14. *For all nonnegative integers n and k,*

$$\sum_{i=0}^{n}(-1)^{i}F_{2n-i+k}\binom{n}{i}=F_{k}.$$

Proof. We proceed by induction. Since $F_1=1$, Corollary 13(a) is the $k=1$ case. The $k=2$ case,

$$\sum_{i=0}^{n}(-1)^{i}F_{2n-i+2}\binom{n}{i}=1=F_{2}, \tag{63}$$

was established as Corollary 3.2 in our previous article [3]. For $k\geq 3$, the result follows using the Fibonacci recurrence. For $k=0$, subtracting Corollary 13(a) from (63) leaves

$$\sum_{i=0}^{n}(-1)^{i}F_{2n-i}\binom{n}{i}=0=F_{0}.$$ □

There are several direct applications of the results in Section 4.

Corollary 15. *For positive integers n,*

$$\text{(a)}\quad \sum_{i=0}^{n} F_{n-i+2}\binom{n}{i} = F_{2n+2},$$

$$\text{(b)}\quad \sum_{i=0}^{n} (-1)^i F_{2n-2i+1}\binom{n}{i} = F_{n+1},$$

$$\text{(c)}\quad \sum_{i=0}^{n} (-1)^i 2^{n-i-1} F_{2n-2i+3}\binom{n-1}{i} = F_{3n+2}.$$

Also, for n even,

$$\text{(d)}\quad \sum_{i=0}^{n/2} 5^{i-1}2^{n-2i}F_{n-2i+2}\left(5\binom{n}{2i} - 10\binom{n}{2i-1} + 4\binom{n}{2i-2}\right) = 1,$$

$$\text{(e)}\quad \sum_{i=0}^{n/2} 5^{i-1}3^{n-2i}F_{n-2i+2}\left(5\binom{n}{2i} - 15\binom{n}{2i-1} + 9\binom{n}{2i-2}\right) = L_{2n+1}$$

and, for n odd,

$$\text{(f)}\quad -\sum_{i=0}^{(n+1)/2} 5^{i-2}2^{n-2i+1}F_{n-2i+3}\left(5\binom{n}{2i-1} - 10\binom{n}{2i-2} + 4\binom{n}{2i-3}\right) = 1,$$

$$\text{(g)}\quad -\sum_{i=0}^{(n+1)/2} 5^{i-2}3^{n-2i+1}F_{n-2i+3}\left(5\binom{n}{2i-1} - 15\binom{n}{2i-2} + 9\binom{n}{2i-3}\right) = F_{2n+1}.$$

Proof. For (a), let $x = n$ in (46). For (b), let $x = n$ in (48). For (c), let $x = n-1$ in (49). For (d), let $x = n-1$ in (52) and, for (e), let $x = n-1$ in (57). For (f), let $x = n-1$ in (54) and, for (g), let $x = n-1$ in (59). □

The following several corollaries use the derivatives of the binomial polynomial expansions found in Section 4. We provide one detailed proof and notes for the others. Note that the terms $\delta_{n,\ell}$ vary slightly for each result.

Corollary 16. *For every integer $n \geq 2$ and $\ell \geq 0$,*

$$\sum_{i=\ell}^{n+\ell} H_i F_{i-\ell+2}\left(\binom{n+\ell}{i} - 3\binom{n+\ell-1}{i} + \binom{n+\ell-2}{i}\right)$$

$$
\begin{aligned}
&= \frac{F_{2n+\ell+2}}{n+\ell} - \frac{F_{2n+\ell-2}}{n+\ell-1} - \delta_{n,\ell} \\
&\quad - \sum_{j=0}^{\ell-1} \left(\left(H_{n+\ell} \left(\frac{n+j}{n} \right) \left(\frac{n+j-1}{n-1} \right) - 3H_{n+\ell-1} \left(\frac{n+j-1}{n-1} \right) + H_{n+\ell-2} \right) \right. \\
&\qquad\qquad \left. \times F_{\ell-j} \binom{n+j-2}{j} \right),
\end{aligned}
$$

where $\delta_{n,0} = \frac{1}{n}$ *and* $\delta_{n,\ell} = \binom{n+\ell-1}{n}(H_{n+\ell-1} - H_{\ell-1})$ *for* $\ell > 0$.

Proof. Differentiating (46) with respect to x and evaluating at $x = n + \ell$ gives

$$
\begin{aligned}
P_n'(n+\ell) &= \sum_{i=0}^{n} F_{n-i+2} \binom{n+\ell}{i} \sum_{j=0}^{i-1} \frac{1}{n+\ell-j} \\
&= \sum_{i=0}^{n} F_{n-i+2} \binom{n+\ell}{i} (H_{n+\ell} - H_{n+\ell-i}) \\
&= H_{n+\ell} \sum_{i=0}^{n} F_{n-i+2} \binom{n+\ell}{i} - \sum_{i=0}^{n} H_{n+\ell-i} F_{n-i+2} \binom{n+\ell}{i} \\
&= H_{n+\ell} P_n(n+\ell) - \sum_{i=0}^{n} H_{n+\ell-i} F_{n-i+2} \binom{n+\ell}{n+\ell-i} \\
&= H_{n+\ell} \left(F_{2n+\ell+2} - \sum_{i=1}^{\ell} F_i \binom{n+\ell-i}{n} \right) - \sum_{j=\ell}^{n+\ell} H_j F_{j-\ell+2} \binom{n+\ell}{j},
\end{aligned}
\tag{64}
$$

where the last equality uses Theorem 1.

Differentiating (5) with respect to x and evaluating at $x = n + \ell$ gives

$$
P_n'(n+\ell) - 3P_{n-1}'(n-1+\ell) + P_{n-2}'(n-2+\ell) = \binom{x-1}{n} \sum_{j=0}^{n-1} \frac{1}{x-1-j} \Bigg|_{x=n+\ell} \tag{65}
$$

which, for $\ell = 0$, becomes

$$
\lim_{x \to n} \binom{x-1}{n} \frac{1}{x-1-(n-1)} = \frac{1}{n}
$$

while, for $\ell > 0$, is

$$
\binom{n+\ell-1}{n} (H_{n+\ell-1} - H_{\ell-1}),
$$

matching $\delta_{n,\ell}$ as defined in the corollary statement.

Using (64) for P' terms in (65) and rearranging gives

$$\sum_{i=\ell}^{n+\ell} H_i F_{i-\ell+2}\left(\binom{n+\ell}{i} - 3\binom{n+\ell-1}{i} + \binom{n+\ell-2}{i}\right)$$
$$= (H_{n+\ell}F_{2n+2+\ell} - 3H_{n-1+\ell}F_{2n+\ell} + H_{n-2+\ell}F_{2n-2+\ell}) - \left(H_{n+\ell}\sum_{i=1}^{\ell} F_i\binom{n+\ell-i}{n}\right.$$
$$\left. - 3H_{n-1+\ell}\sum_{i=1}^{\ell} F_i\binom{n-1+\ell-i}{n-1} + H_{n-2+\ell}\sum_{i=1}^{\ell} F_i\binom{n-2+\ell-i}{n-2}\right) - \delta_{n,\ell}. \quad (66)$$

The first grouped expression in (66) simplifies as

$$\begin{aligned} &H_{n+\ell}F_{2n+2+\ell} - 3H_{n-1+\ell}F_{2n+\ell} + H_{n-2+\ell}F_{2n-2+\ell} \\ &\quad= \left(H_{n-2+\ell} + \frac{1}{n+\ell-1} + \frac{1}{n+\ell}\right)F_{2n+2+\ell} \\ &\qquad - 3\left(H_{n-2+\ell} + \frac{1}{n+\ell-1}\right)F_{2n+\ell} + H_{n-2+\ell}F_{2n-2+\ell} \\ &\quad= H_{n-2+\ell}(F_{2n+2+\ell} - 3F_{2n+\ell} + F_{2n-2+\ell}) + \frac{F_{2n+2+\ell} - 3F_{2n+\ell}}{n+\ell-1} + \frac{F_{2n+2+\ell}}{n+\ell} \\ &\quad= \frac{F_{2n+\ell+2}}{n+\ell} - \frac{F_{2n+\ell-2}}{n+\ell-1}, \end{aligned}$$

where the last equality uses (16).

The expression with summations in (66), using the identity $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, becomes

$$\begin{aligned} &H_{n+\ell}\sum_{i=1}^{\ell} F_i\binom{n+\ell-i}{n} - 3H_{n-1+\ell}\sum_{i=1}^{\ell} F_i\binom{n-1+\ell-i}{n-1} + H_{n-2+\ell}\sum_{i=1}^{\ell} F_i\binom{n-2+\ell-i}{n-2} \\ &\quad= \sum_{i=1}^{\ell}\left(\left(H_{n+\ell}\cdot\frac{n+\ell-i}{n}\cdot\frac{n-1+\ell-i}{n-1} - 3H_{n-1+\ell}\cdot\frac{n-1+\ell-i}{n-1} + H_{n-2+\ell}\right)\right. \\ &\qquad\left. \times F_i\binom{n-2+\ell-i}{n-2}\right). \end{aligned}$$

Using these two expressions in (66) and reindexing by $j = \ell - i$ leads to the desired identity. □

Corollary 17. *For every integer $n \geq 3$ and $\ell \geq 0$,*

$$\sum_{i=\ell+1}^{n+\ell} H_i F_{i-\ell} \left(\binom{n+\ell}{i} - 3\binom{n+\ell-1}{i} + \binom{n+\ell-2}{i} \right)$$
$$= \frac{F_{2n+\ell}}{n+\ell} - \frac{F_{2n+\ell-4}}{n+\ell-1} - \delta_{n,\ell}$$
$$- \sum_{j=0}^{\ell-1} \left(\left(H_{n+\ell} \left(\frac{n+j-1}{n-1} \right) \left(\frac{n+j-2}{n-2} \right) - 3H_{n+\ell-1} \left(\frac{n+j-2}{n-2} \right) + H_{n+\ell-2} \right) \right.$$
$$\left. \times F_{\ell-j} \binom{n+j-3}{j} \right),$$

where $\delta_{n,0} = \frac{1}{n-1}$ and $\delta_{n,\ell} = \binom{n+\ell-2}{n-1}(H_{n+\ell-2} - H_{\ell-1})$ for $\ell > 0$.

Proof. Here, differentiate (47), otherwise the proof follows the same structure as that of Corollary 16. □

Corollary 18. *For every integer $n \geq 2$ and $\ell \geq 0$,*

$$(-1)^\ell \sum_{i=\ell}^{n+\ell} (-1)^i H_i F_{2i-2\ell+1} \left(\binom{n+\ell}{i} + \binom{n+\ell-1}{i} - \binom{n+\ell-2}{i} \right)$$
$$= (-1)^n \left(\frac{F_{n-\ell+1}}{n+\ell} + \frac{F_{n-\ell-1}}{n+\ell-1} \right) - \delta_{n,\ell}$$
$$+ \sum_{i=0}^{\ell-1} \left(\left(H_{n+\ell} \left(\frac{n+i}{n} \right) \left(\frac{n+i-1}{n-1} \right) + H_{n+\ell-1} \left(\frac{n+i-1}{n-1} \right) - H_{n+\ell-2} \right) \right.$$
$$\left. \times F_{i-\ell} \binom{n+i-2}{i} \right),$$

where $\delta_{n,0} = \frac{1}{n}$ and $\delta_{n,\ell} = \binom{n+\ell-1}{n}(H_{n+\ell-1} - H_{\ell-1})$ for $\ell > 0$.

Proof. The proof is similar to that of Corollary 16, incorporating (48) and (6). □

Corollary 19. *For every integer $n \geq 3$ and $\ell \geq 0$,*

$$(-1)^\ell \sum_{i=\ell+1}^{n+\ell} (-1)^i H_i F_{2i-2\ell} \left(\binom{n+\ell}{i} + \binom{n+\ell-1}{i} - \binom{n+\ell-2}{i} \right)$$
$$= (-1)^n \left(\frac{F_{n-\ell}}{n+\ell} + \frac{F_{n-\ell-2}}{n+\ell-1} \right) + \delta_{n,\ell}$$

$$
-\sum_{i=0}^{\ell-1}\left(\left(H_{n+\ell}\left(\frac{n+i-1}{n-1}\right)\left(\frac{n+i-2}{n-2}\right)+H_{n+\ell-1}\left(\frac{n+i-2}{n-2}\right)-H_{n+\ell-2}\right)\right.
$$

$$
\left.\times F_{i-\ell}\binom{n+i-3}{i}\right),
$$

where $\delta_{n,0}=\frac{1}{n-1}$ *and* $\delta_{n,\ell}=\binom{n+\ell-2}{n-1}(H_{n+\ell-2}-H_{\ell-1})$ *for* $\ell>0$.

Proof. The proof is similar to that of Corollary 16, incorporating (50) and (6). □

Corollary 20. *For every integer* $n\geq 2$ *and* $\ell\geq 0$,

$$
(-1)^{\ell}\sum_{i=\ell}^{n+\ell}(-1)^{i}2^{i-\ell-1}H_iF_{2i-2\ell+3}\left(\binom{n+\ell}{i}+4\binom{n+\ell-1}{i}-\binom{n+\ell-2}{i}\right)
$$

$$
=\frac{(-1)^n}{2^{\ell+1}}\left(\frac{F_{3n+\ell+3}}{n+\ell}+\frac{F_{3n+\ell-3}}{n+\ell-1}\right)-\delta_{n,\ell}
$$

$$
+\sum_{i=0}^{\ell-1}\left(\left(H_{n+\ell}\left(\frac{n+i}{n}\right)\left(\frac{n+i-1}{n-1}\right)+4H_{n+\ell-1}\left(\frac{n+i-1}{n-1}\right)-H_{n+\ell-2}\right)\right.
$$

$$
\left.\times\frac{F_{\ell-i}}{2^{\ell-i+1}}\binom{n+i-2}{i}\right),
$$

where $\delta_{n,0}=\frac{1}{n}$ *and* $\delta_{n,\ell}=\binom{n+\ell-1}{n}(H_{n+\ell-1}-H_{\ell-1})$ *for* $\ell>0$.

Proof. The proof is similar to that of Corollary 16, incorporating (49) and (9). □

Corollary 21. *For every integer* $n\geq 3$ *and* $\ell\geq 0$,

$$
(-1)^{\ell}\sum_{i=\ell+1}^{n+\ell}(-1)^{i}2^{i-\ell-1}H_iF_{2i-2\ell}\left(\binom{n+\ell}{i}+4\binom{n+\ell-1}{i}-\binom{n+\ell-2}{i}\right)
$$

$$
=\frac{(-1)^n}{2^{\ell+1}}\left(\frac{F_{3n+\ell}}{n+\ell}+\frac{F_{3n+\ell-6}}{n+\ell-1}\right)+\delta_{n,\ell}
$$

$$
-\sum_{i=0}^{\ell-1}\left(\left(H_{n+\ell}\left(\frac{n+i-1}{n-1}\right)\left(\frac{n+i-2}{n-2}\right)+4H_{n+\ell-1}\left(\frac{n+i-2}{n-2}\right)-H_{n+\ell-2}\right)\right.
$$

$$
\left.\times\frac{F_{\ell-i}}{2^{\ell-i+1}}\binom{n+i-3}{i}\right),
$$

where $\delta_{n,0}=\frac{1}{n-1}$ *and* $\delta_{n,\ell}=\binom{n+\ell-2}{n-1}(H_{n+\ell-2}-H_{\ell-1})$ *for* $\ell>0$.

Proof. The proof is similar to that of Corollary 16, incorporating (51) and (9). □

We record a few of the simpler particular identities that follow from the previous corollaries.

Corollary 22. *For every integer $n \geq 2$,*

(a) $$\sum_{i=1}^{n} H_i F_{i+2} \left(\binom{n}{i} - 3\binom{n-1}{i} + \binom{n-2}{i} \right) = \frac{F_{2n+2}-1}{n} - \frac{F_{2n-2}}{n-1},$$

(b) $$\sum_{i=1}^{n} H_i F_{i} \left(\binom{n}{i} - 3\binom{n-1}{i} + \binom{n-2}{i} \right) = \frac{F_{2n}}{n} - \frac{F_{2n-4}-1}{n-1},$$

(c) $$\sum_{i=1}^{n} (-1)^i H_i F_{2i+1} \left(\binom{n}{i} + \binom{n-1}{i} - \binom{n-2}{i} \right) = (-1)^n \left(\frac{F_{n-1}}{n-1} + \frac{F_{n+1}}{n} \right) - \frac{1}{n},$$

(d) $$\sum_{i=1}^{n} (-1)^i H_i F_{2i} \left(\binom{n}{i} + \binom{n-1}{i} - \binom{n-2}{i} \right) = (-1)^n \left(\frac{F_{n}}{n} + \frac{F_{n-2}}{n-1} \right) + \frac{1}{n-1},$$

(e) $$\sum_{i=1}^{n} (-2)^i H_i F_{2i+3} \left(\binom{n}{i} + 4\binom{n-1}{i} - \binom{n-2}{i} \right) = (-1)^n \left(\frac{F_{3n-3}}{n-1} + \frac{F_{3n+3}}{n} \right) - \frac{2}{n},$$

(f) $$\sum_{i=1}^{n} (-2)^i H_i F_{2i} \left(\binom{n}{i} + 4\binom{n-1}{i} - \binom{n-2}{i} \right) = (-1)^n \left(\frac{F_{3n}}{n} + \frac{F_{3n-6}}{n-1} \right) + \frac{2}{n-1},$$

(g) $$\sum_{i=1}^{n} H_{i+1} F_{i} \left(\binom{n+1}{i+1} - 3\binom{n}{i+1} + \binom{n-1}{i+1} \right) = \frac{F_{2n+1}-1}{n+1} - \frac{F_{2n-3}+2}{n},$$

(h) $$\sum_{i=1}^{n} H_{i+2} F_{i} \left(\binom{n+2}{i+2} - 3\binom{n+1}{i+2} + \binom{n}{i+2} \right) = \frac{F_{2n+2}+1}{n+2} - \frac{F_{2n-2}-3}{n+1} + n + 1.$$

Proof. Identities (a)–(f) follow from Corollaries 16 through 21 with $\ell = 0$, respectively. Identities (g) and (h) follow from Corollary 17 with $\ell = 1$ and $\ell = 2$, respectively. □

In fact, each value of ℓ in each of Corollaries 16 through 21 produces an identity.

We conclude this section with results particular to the polynomials $S_n(x)$ of Theorem 4. Notice that the recurrence (12) combines $S_{n+1}(x)$ and $S_{n-1}(x-2)$ into a scaled binomial polynomial; this simplification is the core of the following results. The analogous (14) does not combine as nicely, so there are not similar results based on $T_n(x)$.

We rewrite (12) as

$$S_{k+1}(x+1) - S_{k-1}(x-1) = -5^{\lfloor \frac{k+1}{2} \rfloor} \binom{x}{k+1}$$

and differentiate with respect to x on both sides,

$$S'_{k+1}(x+1) - S'_{k-1}(x-1) = -5^{\lfloor \frac{k+1}{2} \rfloor} \binom{x}{k+1} \sum_{i=0}^{k} \frac{1}{x-i}.$$

Letting $x = k$ gives

$$S'_{k+1}(k+1) - S'_{k-1}(k-1) = -\frac{5^{\lfloor \frac{k+1}{2} \rfloor}}{k+1}. \tag{67}$$

That equation informs our last corollary.

Corollary 23. *With b_k and c_k as defined in (2) and (3), and for every $n \geq 1$,*

(a) $\displaystyle\sum_{k=1}^{2n} (-2)^k 5^{n-\lfloor \frac{k}{2} \rfloor} b_k H_k \binom{2n}{k} = 2H_{2n} - \sum_{k=1}^{n} \frac{5^k}{k},$

(b) $\displaystyle\sum_{k=1}^{2n+1} (-2)^{k-1} 5^{n-\lfloor \frac{k}{2} \rfloor} b_k H_k \binom{2n+1}{k} = \sum_{k=0}^{n} \frac{5^k}{2k+1},$

(c) $\displaystyle\sum_{k=1}^{2n} (-2)^{k-1} 5^{n-\lfloor \frac{k-1}{2} \rfloor} c_k H_k \binom{2n}{k} = -\sum_{k=1}^{n} \frac{5^k}{2k-1},$

(d) $\displaystyle\sum_{k=1}^{2n+1} (-2)^k 5^{n-\lfloor \frac{k}{2} \rfloor + \frac{1+(-1)^k}{2}} c_k H_k \binom{2n+1}{k} = -2H_{2n+1} + \sum_{k=1}^{n} \frac{5^k}{k}.$

Proof. For (a), summing (67) evaluated at $k = 1, 3, \ldots, 2n-1$ gives

$$S'_{2n}(2n) = S'_{2n}(2n) - S'_0(0) = -\frac{5}{2} - \frac{5^2}{4} - \cdots - \frac{5^n}{2n} = -\frac{1}{2} \sum_{k=1}^{n} \frac{5^k}{k}. \tag{68}$$

Next, we find a different expression for $S'_{2n}(2n)$. Differentiating (53) with respect to x and evaluating at $x = 2n$ gives

$$\begin{aligned} S'_{2n}(2n) &= \sum_{i=0}^{2n} (-1)^{i+1} 5^{\lfloor \frac{i+1}{2} \rfloor} 2^{2n-i-1} b_{2n-i} \binom{2n}{i} \sum_{j=0}^{i-1} \frac{1}{2n-j} \\ &= \sum_{i=0}^{2n} (-1)^{i+1} 5^{\lfloor \frac{i+1}{2} \rfloor} 2^{2n-i-1} b_{2n-i} \binom{2n}{i} (H_{2n} - H_{2n-i}) \end{aligned}$$

$$
\begin{aligned}
&= H_{2n} \sum_{i=0}^{2n} (-1)^{i+1} 5^{\lfloor \frac{i+1}{2} \rfloor} 2^{2n-i-1} b_{2n-i} \binom{2n}{i} \\
&\qquad - \sum_{i=0}^{2n} (-1)^{i+1} 5^{\lfloor \frac{i+1}{2} \rfloor} 2^{2n-i-1} b_{2n-i} H_{2n-i} \binom{2n}{i} \\
&= H_{2n} S_{2n}(2n) - \sum_{i=0}^{2n} (-1)^{i+1} 5^{\lfloor \frac{i+1}{2} \rfloor} 2^{2n-i-1} b_{2n-i} H_{2n-i} \binom{2n}{2n-i} \\
&= -H_{2n} - \sum_{k=0}^{2n} (-1)^{2n-k+1} 5^{\lfloor \frac{2n-k+1}{2} \rfloor} 2^{k-1} b_k H_k \binom{2n}{k} \\
&= -H_{2n} + \frac{1}{2} \sum_{k=0}^{2n} (-2)^k 5^{n-\lfloor \frac{k}{2} \rfloor} b_k H_k \binom{2n}{k}.
\end{aligned}
$$

Combining this with (68) establishes (a).

The verification of (b) is very similar: Sum (67) evaluated at $k = 2, 4, \ldots, 2n$ and use (55) for the other expression of $S'_{2n}(2n)$.

For (c), differentiating (52) with respect to x and evaluating at $x = 2n$ gives

$$
\begin{aligned}
S'_{2n}(2n) &= -\sum_{i=0}^{n} 5^{i-1} 2^{2n-2i} F_{2n-2i+2} \Bigg(5\binom{2n+1}{2i} (H_{2n+1} - H_{2n-2i+1}) \\
&\qquad - 10\binom{2n+1}{2i-1} (H_{2n+1} - H_{2n-2i+2}) + 4\binom{2n+1}{2i-2} (H_{2n+1} - H_{2n-2i+3}) \Bigg) \\
&= H_{2n+1} S_{2n}(2n) + \sum_{i=0}^{n} 5^{i} 2^{2n-2i} F_{2n-2i+2} H_{2n-2i+1} \binom{2n+1}{2n-2i+1} \\
&\qquad - \sum_{i=0}^{n} 5^{i} 2^{2n-2i+1} F_{2n-2i+2} H_{2n-2i+2} \binom{2n+1}{2n-2i+2} \\
&\qquad + \sum_{i=0}^{n} 5^{i-1} 2^{2n-2i+2} F_{2n-2i+2} H_{2n-2i+3} \binom{2n+1}{2n-2i+3} \\
&= -H_{2n+1} + \sum_{k=0}^{n} 5^{n-k} 2^{2k} F_{2k+2} H_{2k+1} \binom{2n+1}{2k+1} \\
&\qquad - \sum_{k=0}^{n} 5^{n-k+1} 2^{2k-1} F_{2k} H_{2k} \binom{2n+1}{2k} \\
&\qquad + \sum_{k=0}^{n} 5^{n-k} 2^{2k} F_{2k} H_{2k+1} \binom{2n+1}{2k+1}
\end{aligned}
$$

$$= -H_{2n+1} + \sum_{k=0}^{n} 5^{n-k} 2^{2k} (F_{2k} + F_{2k+2}) H_{2k+1} \binom{2n+1}{2k+1}$$
$$- \sum_{k=0}^{n} 5^{n-k+1} 2^{2k-1} F_{2k} H_{2k} \binom{2n+1}{2k}$$
$$= -H_{2n+1} + \sum_{k=0}^{n} 5^{n-k} 2^{2k} L_{2k+1} H_{2k+1} \binom{2n+1}{2k+1}$$
$$- \sum_{k=0}^{n} 5^{n-k+1} 2^{2k-1} F_{2k} H_{2k} \binom{2n+1}{2k}$$

using (19). Doubling both sides, recalling the definition of the c_k, and combining with (68) gives (c).

The verification of (d) is similar, differentiating (54) and evaluating at $x = 2n + 1$. □

6 Further work

We believe that the coefficients of the various polynomials, with initial values given in the tables of Section 2, merit further study.

Also, the sum in Corollary 22(b) can be rewritten

$$\sum_{i=1}^{n} H_i F_i \left(\binom{n}{i} - 3\binom{n-1}{i} + \binom{n-2}{i} \right)$$
$$= \sum_{i=1}^{n} H_i F_i \binom{n}{i} - 3 \sum_{i=1}^{n-1} H_i F_i \binom{n-1}{i} + \sum_{i=1}^{n-2} H_i F_i \binom{n-2}{i}.$$

With the other side of Corollary 22(b), we have the sum $W_n = \sum_{i=1}^{n} H_i F_i \binom{n}{i}$ satisfying the recurrence

$$W_n - 3W_{n-1} + W_{n-2} = \frac{F_{2n}}{n} - \frac{F_{2n-4} - 1}{n - 1} \tag{69}$$

for all $n \geq 2$. A solution to the nonhomogeneous second order linear recurrence (69), with initial condition $W_0 = 0$ and $W_1 = 1$, would give us a closed form for $\sum_{i=1}^{n} H_i F_i \binom{n}{i}$. Each identity in Corollary 22 could give rise to a similar recurrence.

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