Journal of Integer Sequences, Vol. 25 (2022),
Article 22.4.8

Precious Metal Sequences and Sierpiński-Type Graphs

Andreas M. Hinz[1]
Department of Mathematics
Ludwig-Maximilians-Universität München
Theresienstraße 39
80333 Munich
Germany
hinz@math.lmu.de

Paul K. Stockmeyer
Department of Computer Science
The College of William & Mary
P.O. Box 8795
Williamsburg, VA 23187-8795
USA
pkstockmeyer@cox.net

Abstract

Sierpiński graphs S_p^n and Sierpiński triangle graphs $\widehat{S}_p^n$ form two-parametric families of connected simple graphs which are related, for $p = 3$, to the Tower of Hanoi with n discs and for $n \to \infty$ to the Sierpiński triangle fractal. The vertices of minimal degree play a special role as extreme vertices in S_p^n and primitive vertices in $\widehat{S}_p^n$. The key concept of this note is that of an m-key vertex whose distance to one of the extreme or primitive vertices, respectively, is m times the distance to another one. The number of such vertices and the distances occurring lead to integer sequences with respect to parameter n like, e.g., the Fibonacci sequence (golden) for $p = 3$ and the Pell sequence (silver) for $p = 4$. The elements of most of these sequences form self-generating sets. We discuss the cases $m = 1, 2, 3, 4$ in detail.

[1]Corresponding author.

1 Introduction

The vertices of *Sierpiński graph* S_p^n with *base* $p \in \mathbb{N}$, $p \geq 2$, and *exponent* $n \in \mathbb{N}_0$ are n-tuples of the set $P := [p]_0 := \{0, \ldots, p-1\}$ and written as $s = s_n \ldots s_1$ with $s_d \in P$ for $d \in [n] := \{1, \ldots, n\}$. The edge set is given by

$$E(S_p^n) = \left\{ \{\underline{s}ij^{d-1}, \underline{s}ji^{d-1}\} \mid \{i,j\} \in \binom{P}{2},\ d \in [n],\ \underline{s} = s_n \ldots s_{d+1} \in P^{n-d} \right\}. \tag{1}$$

Note that $S_p^0 = (\{\epsilon\}, \emptyset) \cong K_1$ with the *empty word* ϵ, $S_p^1 \cong K_p$ and that S_2^n is a path on 2^n vertices. Sierpiński graphs were introduced in the 1990s as mathematical models for the famous *Tower of Hanoi* ($p = 3$) and the *Chinese Rings* ($p = 2$); see [8, Chapter 4]. In the past two decades they developed a life on their own as can be seen in a recent survey [9]. Apart from many graph parameters determined, metric properties have been investigated, and the graphs S_3^n were used to approximate the fractal structure of the *Sierpiński triangle* (see [10]).

If we concatenate $s_{n+1} = k \in P$ to the left of all vertices of S_p^n we get what we may call the graph $kS_p^n \cong S_p^n$ as a subgraph of S_p^{1+n}. These p copies of S_p^n are mutually linked in S_p^{1+n} by the so-called *critical edges* $\{ij^n, ji^n\}$, $\{i,j\} \in \binom{P}{2}$, according to (1). This shows that Sierpiński graphs S_p^n are connected and therefore endowed with the canonical distance function δ where $\delta(s,t)$ is the length of a shortest s,t-path in S_p^n. The importance of the critical edges lies in the fact that for $p \geq 3$ a shortest $i\overline{s}, j\overline{t}$-path may either run through critical edge $\{ij^n, ji^n\}$ (*direct path*) or via two critical edges, namely $\{ik^n, ki^n\}$ and $\{kj^n, jk^n\}$ for some (but only one) $k \in P \setminus \{i,j\}$ (*indirect path*). The decision whether the direct or an indirect path is shortest (or both are) and for which k, is not easy and has been analyzed and solved with an algorithm by Hinz and Holz auf der Heide [6]. The decisive ingredient is the distance $\delta(s, j^n)$ of an arbitrary vertex $s \in P^n$ to a so-called *extreme vertex* j^n in S_p^n. It is given (see [8, Theorem 4.5]) by the formula

$$\forall\, s \in P^n\ \forall\, j \in P : \delta(s, j^n) = \sum_{d=1}^{n} (s_d \neq j) \cdot 2^{d-1}, \tag{2}$$

where we make use of the *Iverson bracket* (or *Iverson convention*) which assigns a numerical (binary) value $(\mathcal{A})$ to a statement $\mathcal{A}$; it is defined by $(\mathcal{A}) = 1$, if $\mathcal{A}$ is true, and $(\mathcal{A}) = 0$, if $\mathcal{A}$ is false. Obviously, $\delta(s, j^n) \leq 2^n - 1$ and putting $s = i^n$ for some $i \in P \setminus \{j\}$ we have $\mathrm{diam}(S_p^n) = \delta(i^n, j^n) = 2^n - 1$. Another immediate consequence of (2) is the following invariant:

$$\forall\, s \in P^n : \sum_{j=0}^{p-1} \delta(s, j^n) = (p-1) \cdot (2^n - 1). \tag{3}$$

Sierpiński graphs S_3^n are isomorphic to *Hanoi graphs* H_3^n; see [8, pp. 177ff]. For these, the number of *2-key vertices* with $\delta(s, 2^n) = 2 \cdot \delta(s, 0^n)$ has been found to be Fibonacci number F_{n-1} in [11, Theorem 3.1]. Here, for $n \geq 3$, the *2-key distances* $\delta(s, 0^n)$ have the form $2\beta + 2^{n-2} + 1$ with β running through the set of $(n-3)$-bit numbers without consecutive

0s; see [11, Lemma 3.2]. The fact that all 2-key distances are odd follows also from the observation from (2) that exactly one of the distances $\delta(s, j^n)$ is even, namely for $j = s_1$ in S_3^n and for $j = s_n \vartriangle \cdots \vartriangle s_1$ in H_3^n, where the operation given by $i \vartriangle k = i + (3 - 2i - k)(i \neq k)$ for $i, k \in \{0, 1, 2\}$ has to be evaluated from the right; see [8, (2.8)].

In the present note we want to extend these results in three ways. We will consider Sierpiński graphs of any base $p \geq 2$ (Section 2), thereby looking at *m-key vertices*, i.e., those $s \in P^n$ for which $\delta(s, (p-1)^n) = m \cdot \delta(s, 0^n)$ and their respective *m-key distances* for $m = 1$ (Section 2.1), $m = 2$ (Section 2.2), $m = 3$ (Section 2.3), and $m = 4$ (Section 2.4). Finally, we will consider the corresponding questions for *Sierpiński triangle graphs* $\widehat{S}_p^n$ (Section 3). These are graphs which have often been mistaken for Sierpiński graphs and even been called so (see [9] for a discussion), but whose metric properties are somewhat more difficult to access (see [7]). Our focus will be on integer sequences emerging from these considerations. Some of the sequences come from the so-called *self-generating sets*, like, e.g., the *Mersenne sequence* $M_n = 2^n - 1$ (A000225, referring to the *On-Line Encyclopedia of Integer Sequences* (OEIS)) with $\alpha = 1$ and $\mathcal{F} = \{k \mapsto 2k + 1\}$ in the following lemma.

Lemma 1. *Let* $\alpha \in \mathbb{N}$ *and* $\mathcal{F}$ *be a finite set of functions from* $\mathbb{N}$ *to* $\mathbb{N}$ *with*

$$\forall\, f \in \mathcal{F}\ \forall\, x \in \mathbb{N} : f(x) > x. \tag{4}$$

We say that $\Gamma \subset \mathbb{N}$ *fulfills property* $\mathrm{SG}(\alpha, \mathcal{F})$*, iff* $\{\alpha\} \cup \bigcup\{f(\Gamma) \mid f \in \mathcal{F}\} \subset \Gamma$.
Then the following are equivalent:

1. $\mathbb{N} \supset C = \{\alpha\} \cup \bigcup\{f(C) \mid f \in \mathcal{F}\}$,
2. $C = \{c_k \circ \cdots \circ c_1(\alpha) \mid c_\ell \in \mathcal{F},\ \ell \in [k],\ k \in \mathbb{N}_0\}$,
3. $C = \bigcap\{\Gamma \subset \mathbb{N} \mid \Gamma \text{ fulfills } \mathrm{SG}(\alpha, \mathcal{F})\}$,
4. C *is the smallest subset of* $\mathbb{N}$ *(w.r.t. "$\subset$") that fulfills* $\mathrm{SG}(\alpha, \mathcal{F})$.

Such a C *is called a* self-generating set*,* α *is its* seed *and* $\mathcal{F}$ *is its* generating function set*. Points* 2 *to* 4 *guarantee that* C *is defined uniquely by* 1.

Proof. 1. ⇒ 2. Let $\mathbb{N} \supset C = \{\alpha\} \cup \bigcup\{f(C) \mid f \in \mathcal{F}\}$ and define $C' := \bigcup\{C_k \mid k \in \mathbb{N}_0\}$ with $C_k := \{c_k \circ \cdots \circ c_1(\alpha) \mid c_\ell \in \mathcal{F},\ \ell \in [k]\}$. We prove $C_k \subset C$ by induction on k. $C_0 = \{\alpha\} \subset C$. If $x = c_{k+1} \circ c_k \circ \cdots \circ c_1(\alpha) \in C_{k+1}$, then $x = f(x')$ with $f = c_{k+1} \in \mathcal{F}$, $x' = c_k \circ \cdots \circ c_1(\alpha) \in C_k \subset C$, the latter by induction assumption. Therefore, $x \in f(C) \subset C$.

For $C \subset C'$, we apply the Algorithm to $x \in C$. The condition in the while loop can be checked because $\mathcal{F}$ is finite and x' must be smaller than x by virtue of (4). The algorithm terminates because x is getting strictly smaller in each iteration of the while loop. The output of $c = c_k \ldots c_1$ then provides the representation of x as an element of C', i.e., $x = c_k \circ \cdots \circ c_1(\alpha)$.

2. ⇒ 3. Let $C = \{c_k \circ \cdots \circ c_1(\alpha) \mid c_\ell \in \mathcal{F},\ \ell \in [k],\ k \in \mathbb{N}_0\}$ and $C' = \bigcap\{\Gamma \subset \mathbb{N} \mid \Gamma \text{ fulfills } \mathrm{SG}(\alpha, \mathcal{F})\}$. For every $\Gamma \subset \mathbb{N}$ which fulfills $\mathrm{SG}(\alpha, \mathcal{F})$ we can prove $C_k \subset \Gamma$ by induction as before. So $C_k \subset C'$ and consequently $C \subset C'$. Obviously, C fulfills $\mathrm{SG}(\alpha, \mathcal{F})$, so that $C' \subset C$.

Algorithm

Procedure $C \subset C'$
Parameter x**:** element of C
Parameter c**:** string of elements of $\mathcal{F}$
  input x
  $c \leftarrow \epsilon$ (empty word)
  while $\exists\, f \in \mathcal{F}\ \exists\, x' \in C : x = f(x')$
    $x \leftarrow x',\ c \leftarrow cf$
  end while
  output c

3. $\Rightarrow$ 4. $C = \bigcap\{\Gamma \subset \mathbb{N} \mid \Gamma \text{ fulfills } \mathrm{SG}(\alpha, \mathcal{F})\}$ fulfills $\mathrm{SG}(\alpha, \mathcal{F})$. If $\Gamma \subset \mathbb{N}$ fulfills $\mathrm{SG}(\alpha, \mathcal{F})$, then $C \subset \Gamma$.

4. $\Rightarrow$ 1. Let C be the smallest subset of $\mathbb{N}$ that fulfills $\mathrm{SG}(\alpha, \mathcal{F})$ and assume that $x \in C \setminus (\{\alpha\} \cup \bigcup\{f(C) \mid f \in \mathcal{F}\})$. Let $C' := C \setminus \{x\}$. Then $\alpha \in C'$ and if $f \in \mathcal{F}$ and $x' \in C'$, then $f(x') \neq x$, i.e., $f(x') \in C'$. So C' fulfills $\mathrm{SG}(\alpha, \mathcal{F})$, but is smaller than C, a contradiction. $\square$

2 Sierpiński graphs

For $p \in \mathbb{N}$, $p \geq 2$, and $n \in \mathbb{N}_0$ we define Sierpiński graph S_p^n with $V(S_p^n) = P^n$ and edge set as in (1). Let $m \in \mathbb{N}$. An *m-key vertex* in S_p^n is an $s \in P^n$ with $\delta(s, (p-1)^n) = m \cdot \delta(s, 0^n)$. If s is an m-key vertex, the value $\delta(s, 0^n)$ is called an *m-key distance*. If there is no doubt about the m we just write "key vertex (distance)". The set of m-key vertices in S_p^n is denoted by ${}_m\Psi_{p,n}$, occasionally without the indices m or p. A special case is $n = 0$, where the only vertex ϵ is a key vertex for every m, i.e., ${}_m\Psi_{p,0} = \{\epsilon\}$, and 0 is the only key distance. For $n \in \mathbb{N}$, key distances are always positive.

In the discussion of the case $p = 2$, Mersenne numbers $M_n = 2^n - 1$ play a central role. The following is probably well-known:

Lemma 2. *Every odd $k > 1$ divides some M_κ with $\lceil \ln(k+1)/\ln(2) \rceil \leq \kappa < k$. In particular, every odd number is a proper divisor of some Mersenne number.*

Proof. It suffices to prove the first statement because Mersenne number M_κ is a proper divisor of $M_{2\kappa} = (2^\kappa + 1)M_\kappa$ for $\kappa \in \mathbb{N}$.

The set of residues modulo k of powers of 2 has size at most $k - 1$ because $k > 1$ is odd and therefore the remainder 0 is impossible. So by the pigeonhole principle there must be $0 \leq i < j \leq k - 1$ such that $\frac{2^j}{k} - \frac{2^i}{k} \in \mathbb{N}$, whence $k \mid 2^i(2^{j-i} - 1)$. Again because k is odd we get $k \mid 2^\kappa - 1$ with $1 \leq \kappa := j - i \leq k - 1$. $\square$

As we have seen before, S_2^n, $n \in \mathbb{N}$, is a path graph on 2^n vertices which can be labeled by binary strings $s \in \{0,1\}^n$, leading from 0^n to 1^n in natural order of their values as binary numbers. An m-key vertex s must therefore satisfy $(m+1)\delta = M_n$, where the m-key distance δ is $\delta(s, 0^n) = (s)_2$ and must be odd. We get

$$1 \le \delta = \frac{M_n}{m+1} \le \frac{M_n}{2},\ n \ge 2.$$

So we find m-key vertices if and only if $m+1 > 1$ is a divisor of M_n and δ is a proper divisor of M_n. From Lemma 2 we see that there are m-key vertices iff m is even and that every odd δ is a key distance. We call $\delta = 1$ *trivial*, which leads to an $(M_n - 1)$-key distance with trivial key vertex $0^{n-1}1$. Note that

M_n is prime if and only if no non-trivial key vertex exists in S_2^n.

So for odd m and $n \in \mathbb{N}$ we have ${}_m\Psi_{2,n} = \emptyset$. For $m = 2$ we get ${}_2\Psi_{2,n} = \left\{(01)^{n/2}\right\}$, if n is even and ${}_2\Psi_{2,n} = \emptyset$, if n is odd. This reflects the famous formula $M_n \bmod 3 = n \bmod 2$ (cf. [8, p. 100]). For $n = 2\nu$, $\nu \in \mathbb{N}$, the (positive) 2-key distances form the sequence A002450 of odd Lichtenberg numbers $\ell_{2\nu-1} = \frac{1}{3}(2^{2\nu} - 1) = \delta\left((01)^{\nu}, 0^{2\nu}\right)$. (For the *Lichtenberg sequence* (A000975), see [5] and [13]). For $m = 4$ we note that $5 \,|\, M_n \Leftrightarrow 4 \,|\, n$, as can be seen by looking at the residues modulo 5 of powers of 2, so that there are 4-key vertices if and only if $n = 4\nu$, $\nu \in \mathbb{N}_0$, namely ${}_4\Psi_{2,4\nu} = \{(0011)^{\nu}\}$. The sequence of 4-key distances is A182512$(\nu) = \frac{1}{5}(2^{4\nu} - 1) = 0, 3, 51, 819, 13107, \ldots$.

2.1 The case $m = 1$

As a warm-up for general p we ask whether for some *key vertices* $s \in P^n$, $n \in \mathbb{N}$, the distances to two extreme vertices, 0^n and $(p-1)^n$ say, are equal. From (3) we see that this cannot happen for $p = 2$. For $p \ge 3$ we have from (2):

$$\begin{aligned}
\delta(s, (p-1)^n) = \delta(s, 0^n) \quad &\Leftrightarrow \quad \sum_{d=1}^{n} (s_d \ne p-1) \cdot 2^{d-1} = \sum_{d=1}^{n} (s_d \ne 0) \cdot 2^{d-1} \\
&\Leftrightarrow \quad \forall\, d \in [n] : s_d \ne p-1 \Leftrightarrow s_d \ne 0 \\
&\Leftrightarrow \quad \forall\, d \in [n] : s_d \in [p-2] \\
&\Leftrightarrow \quad s \in [p-2]^n.
\end{aligned}$$

Theorem 3. *For $p \in \mathbb{N}$, $p \ge 2$, and $n \in \mathbb{N}_0$ we have ${}_1\Psi_{p,n} = [p-2]^n$.*

So there are $(p-2)^n$ key vertices and the corresponding *key distance* $\delta(s, 0^n)$ is always $2^n - 1$. In particular, for $p = 3$ there is only one key vertex at distance $2^n - 1$ from both 0^n and 2^n, namely extreme vertex 1^n.

2.2 The case $m = 2$

The *Fibonacci sequence* turns up in $|_2\Psi_{3,n}| = F_{n-1}$, which is also the number of 2-key distances occurring for S_p^n; see Proposition 6 below. (This is formally compatible for $n = 0$, if we put $F_{-1} = 1$.) In order to generalize this result we define $F_{q,n}$ for $q, n \in \mathbb{N}_0$ by

$$F_{q,0} = 0, \tag{5}$$

$$F_{q,1} = 1, \tag{6}$$

$$F_{q,n+2} = q \cdot F_{q,n+1} + F_{q,n}. \tag{7}$$

(F_q is the *Lucas sequence of the first kind* $U(P, Q)$ for the parameters $P = q$ and $Q = -1$; see [12, formula (10)]. The numbers $F_{q,n}$ are sometimes called *q-Fibonacci numbers*, as, e.g., in [3].) Again, for formal reasons, we put $F_{q,-1} = 1$, compatible with (5), (6), and (7) for $n = -1$. Special cases are

$$\begin{aligned} F_{0,n} &= n \bmod 2, \\ F_{1,n} &= F_n, \\ F_{2,n} &= P_n, \end{aligned}$$

where F_n are the *Fibonacci numbers* (A000045) and P_n are the *Pell numbers* (A000129), respectively. Let $Q_\pm := \frac{1}{2}\left(q \pm \sqrt{4+q^2}\right)$; then

$$\begin{aligned} F_{q,n} &= \frac{1}{\sqrt{4+q^2}}\left((F_{q,1} - Q_-F_{q,0})Q_+^n - (F_{q,1} - Q_+F_{q,0})Q_-^n\right) \qquad (8) \\ &= \frac{Q_+^n - Q_-^n}{Q_+ - Q_-} \end{aligned}$$

is the solution of (7), the latter if (5) and (6) are fulfilled. For $q \in \mathbb{N}$ the ratios $F_{q,n+1}/F_{q,n}$ tend to Q_+ as $n \to \infty$. These irrational numbers have recently been called *metallic means*; see, e.g., [4, p. 2]. Since this expression is used inconsistently in literature, we prefer to refer to them as *precious metal means* as, e.g., the golden ($q = 1$, $Q_+ = \frac{1}{2}\left(1 + \sqrt{5}\right)$), silver ($q = 2$, $Q_+ = 1 + \sqrt{2}$) and bronze ($q = 3$, $Q_+ = \frac{1}{2}\left(3 + \sqrt{13}\right)$) ratio. They have the constant infinite continued fraction representation $[q; \overline{q}]$.

Our first main result now reads

Theorem 4. *For $p \in \mathbb{N}$, $p \geq 2$, and $n \in \mathbb{N}_0$ we have $|_2\Psi_{p,n}| = F_{p-2,n-1}$.*

Proof. Let $\overline{s} = s_n \ldots s_1 \in P^n$ and $s = s_{n+1}\overline{s} \in \Psi_{n+1}$. Then $s_{n+1} = 0$, because $(p-1)^{n+1}$ is the closest extreme vertex to vertex $(p-1)\overline{s}$ in S_p^{n+1} and if $s_{n+1} \in [p-2]$, then

$$\delta(s, (p-1)^{n+1}) = \delta(\overline{s}, (p-1)^n) + 2^n < 2^{n+1}$$

and

$$\delta(s, 0^{n+1}) = \delta(\overline{s}, 0^n) + 2^n \geq 2^n.$$

Let $\Phi_n := \{s \in P^n \mid 2^n + \delta(s,(p-1)^n) = 2 \cdot \delta(s,0^n)\}$, i.e., $\Psi_{n+1} = 0\Phi_n$. We show that $|\Phi_n|$ fulfills the recurrence (5), (6), (7) for $q = p-2$. If $s \in \Phi_0$, then $1 = 0$, whence $|\Phi_0| = 0$, i.e., (5) holds. So let $n \in \mathbb{N}$. Then

$$\begin{aligned} s \in \Phi_n \;\; &\Leftrightarrow \;\; 2^n + \sum_{d=1}^{n}(s_d \neq p-1) \cdot 2^{d-1} = 2 \cdot \sum_{d=1}^{n}(s_d \neq 0) \cdot 2^{d-1} \\ &\Leftrightarrow \;\; 2^n + \sum_{d=1}^{n-1}(s_{d+1} \neq p-1) \cdot 2^d + (s_1 \neq p-1) = (s_n \neq 0) \cdot 2^n + \sum_{d=1}^{n-1}(s_d \neq 0) \cdot 2^d \\ &\Leftrightarrow \;\; s_1 = p-1,\ \forall\, d \in [n-1] :\ s_d = 0 \Leftrightarrow s_{d+1} = p-1,\ s_n \neq 0. \end{aligned} \tag{9}$$

For $n = 1$ we have $s \in \Phi_1$ iff $s = p-1$, so $|\Phi_1| = 1$, i.e., (6) is satisfied. Together with (9) (cf. also the standard drawings of S_p^n, e.g., in [8, Chapter 4]) we can deduce

$$\forall\, n \in \mathbb{N}_0 : \Phi_{n+2} = [p-2]\Phi_{n+1} \,\dot\cup\, (p-1)0\Phi_n.$$

Therefore $|\Phi_n|$ also satisfies (7) for $q = p-2$. □

Remark 5. 1. 2-key vertices s lie at $\frac{2}{3} = \left(0.\overline{10}\,\right)_2$ on the only optimal path from $(p-1)^n$ to 0^n which passes s.

2. For $p = 2$ it follows immediately from (9) that $\Psi_n = \emptyset$, if n is odd, and that otherwise $s = (01)^{n/2}$ is the only element of Ψ_n, as we have seen before.

Sierpiński graph S_2^n and R^n, the state graph of the Chinese Rings (see [8, Chapter 2]), being isomorphic, we see that if the number of rings is odd, there is no state at $\frac{2}{3}$ distance between the extreme states 0^n and 10^{n-1}, while for an even positive number of rings there is exactly one, which is the state 1^n. □

The approach taken in [11] was slightly different. We looked at the binary representation of the key distance $\delta(s,0^{n+1})$ and observed [11, Lemma 3.2(2)] that the last bit is 1 and that the representation does not contain a square 00 [11, Lemma 3.2(3)] (this would, e.g., contradict the distance formula (2), because there would be a 0 at the same place in the binary representations of $\delta(s,0^{n+1})$ and $\delta(s,(p-1)^{n+1})$; for $p = 2$ there are no squares 11 either because there are only two types of bits). Conversely, every binary number with these properties represents some $\delta(s,0^{n+1})$. To achieve this, one can construct a bijection between P^n and the set of those binary matrices $b = (b_{j,d-1})_{j \in P, d \in [n]} \in \{0,1\}^{p \times n}$ which satisfies

$$\forall\, d \in [n] : \sum_{i=0}^{p-1} b_{i,d-1} = p-1; \tag{10}$$

in fact, $b_{j,d-1} = (s_d \neq j)$ for $s \in P^n$. This can be based on the fact that the set of those binary matrices which satisfy (10) has size p^n (as can easily be seen by induction on n).

Note that this bijection shows that $p-1$ rows of the matrix suffice to recover s, because the missing row can be reconstructed by virtue of (10). Moreover, from this representation one can immediately deduce [8, Corollary 4.7].

Let us add our observations that $s_{n+1} = 0$ for key vertices $s = s_{n+1}\overline{s}$ in S_p^{n+1} [11, Lemma 3.2(1)] and that therefore the first and last bits of $\delta(\overline{s}, 0^n)$ are 1. The quest for key distances in S_p^{n+1} can then be reduced, for $p \geq 3$, to the problem of finding, for $n \geq 2$, the value of $|B_{n-2}|$ for the sets B_ℓ defined as the sets of bit strings of length $\ell \in \mathbb{N}_0$ which do not contain the substring 00. A counting like this can be found in [2, Section 1.2]. Quite obviously, B_0 just contains the empty word, and $B_1 = \{0, 1\}$. As before, we get

$$B_{\ell+2} = \{1t \mid t \in B_{\ell+1}\} \,\dot{\cup}\, \{01t \mid t \in B_\ell\},$$

whence $|B_\ell| = F_{\ell+2}$.

The elements of the union of the B_ℓ, $\ell \in \mathbb{N}$, considered as decimal numbers, form the sequence a given by

$$a_0 = 0,\ \forall\, n \in \mathbb{N},\ n \geq 2\ \forall k \in [F_n]_0 : a_{F_{n+1}-1+k} = a_{F_{n-1}-1+k} + 2^{n-2};$$

this is, apart from the offset, the sequence A003754 of the OEIS, i.e., $a_n =$ A003754$(n+1)$ for $n \in \mathbb{N}_0$.

The distances occurring in $C_n := \{\delta(s, 0^n) \mid s \in \Phi_n\}$ are none for $n = 0$, $(1)_2$ for $n = 1$, and $(1\beta 1)_2$ with β running through B_{n-2} for $n \geq 2$ or, in other words, $C_n = 2^{n-1} + (C_{n-1} \cup C_{n-2})$. Hence these distances are all different so that $|C_n| = F_n$. We arrive at

Proposition 6. *The number of 2-key distances in S_p^n, $p \geq 3$, is F_{n-1}.*

The sequence c obtained from $\bigcup\limits_{n \in \mathbb{N}_0} C_n$, ordered by size, is given by $c_n = 2a_{n-1} + 1$ for $n \in \mathbb{N}$, i.e.,

$$c_0 = 0,\ \forall\, n \in \mathbb{N}\ \forall k \in [F_n]_0 : c_{F_{n+1}+k} = c_{F_{n-1}+k} + 2^{n-1}.$$

It is A247648 $= 2\cdot$A003754$+1$ and starts $(0,)1, 3, 5, 7, 11, 13, 15, 21, 23, 27, \ldots$; see [11, p. 77]. The sequence forms the self-generating set obtained from $\alpha = 1$ and $\mathcal{F} = \{k \mapsto 2k+1, k \mapsto 4k+1\}$ in Lemma 1. In particular, the sequence c includes the odd Lichtenberg numbers, i.e., the positive 2-key distances for $p = 2$, which are generated by $k \mapsto 4k+1$ with seed 1.

The sets Φ_n contain, for $p \in \mathbb{N}$, $p \geq 3$, and $n \in \mathbb{N}$, the vertices $s \in [p-2]^{n-1}(p-1)$ with maximal distance $\delta(s, 0^n) = 2^n - 1$. Similarly, the vertices $s = ((p-1)0)^{(n-1)/2}\,(p-1)$, if n is odd, and $s \in ((p-1)0)^{(n-2)/2}\,[p-2](p-1)$, if n is even, have minimal distance $\delta(s, 0^n) = J_{n+1}$ (*Jacobsthal numbers* (A001045); cf. [5]). For $p = 3$ it is possible to prove that all elements of Φ_n are those which lie on the straight line joining maximal distance with minimal distance vertices in the standard triangular drawing of S_3^n. If the side length of the triangle is chosen to be 1, this *magic line* is the same for all n [11, Theorem 3.3] and leads to a fractal if intersected with the Sierpiński triangle (of side length 1), see [11, Section 4]. When drawn as tetrahedra with side length 1, the graphs S_4^n contain an analogue *magic triangle* accommodating all 2-key vertices and leading to another fascinating fractal structure, the *Pell fractal* (cf. [11, Section 5]).

2.3 The case $m = 3$

A 3-key vertex s of S_p^n must satisfy

$$\sum_{d=1}^{n}(s_d \neq p-1)\cdot 2^{d-1} = 3\sum_{d=1}^{n}(s_d \neq 0)\cdot 2^{d-1}. \tag{11}$$

If $s_n \neq 0$, then RHS $\geq 3\cdot 2^{n-1} > M_n \geq$ LHS; therefore $s_n = 0$ and (11) can be replaced by

$$2^{n-1} + \sum_{d=1}^{n-1}(s_d \neq p-1)\cdot 2^{d-1} = 3\sum_{d=1}^{n-1}(s_d \neq 0)\cdot 2^{d-1}. \tag{12}$$

For $n = 1$ this leads to a contradiction, whence ${}_3\Psi_{p,1} = \emptyset$. So let $n \geq 2$ and assume that $s_{n-1} = 0$. Then RHS $\leq 3\cdot M_{n-2} < 3\cdot 2^{n-2} = 2^{n-1} + 2^{n-2} \leq$ LHS, a contradiction. Similarly, if $s_{n-1} = p-1$, then LHS $\leq 2^{n-1} + M_{n-1} < 3\cdot 2^{n-2} \leq$ RHS, another contradiction. Therefore, $s_{n-1} \in [p-2]$ and (12) reduces to

$$\sum_{d=1}^{n-2}(s_d \neq p-1)\cdot 2^{d-1} = 3\sum_{d=1}^{n-2}(s_d \neq 0)\cdot 2^{d-1}. \tag{13}$$

For $n = 2$ we are done with ${}_3\Psi_{p,2} = 0[p-2]$. For $n \geq 3$ we notice that (13) is the same as (11), but with n replaced by $n-2$. It follows that

$${}_3\Psi_{p,n} = 0[p-2]\,{}_3\Psi_{p,n-2}$$

with $\delta(s,(p-1)^n) = M_n$ for $s \in {}_3\Psi_{p,n}$. We can summarize the case $m = 3$ in the following theorem.

Theorem 7. *The set of 3-key vertices in S_p^n is empty for odd n and otherwise ${}_3\Psi_{p,n} = (0[p-2])^{n/2}$ with $|{}_3\Psi_{p,n}| = (p-2)^{n/2}$. The sequence of positive 3-key distances is $\frac{1}{3}M_{2k} = \ell_{2k-1} = 1, 5, 21, 85, \ldots$ for $k \in \mathbb{N}$; these are the odd Lichtenberg numbers, A002450. It is the self-generating sequence for seed 1 and generating function set $\{k \mapsto 4k+1\}$.*

2.4 The case $m = 4$

For this case we need some preparation. For $q \in \mathbb{N}_0$ let the sequences $(FF_{q,n})_{n\in\mathbb{N}_0}$ be defined by

$$FF_{q,0} = FF_{q,1} = FF_{q,2} = 0, \tag{14}$$
$$FF_{q,3} = 1, \tag{15}$$
$$FF_{q,n+4} = q\,(FF_{q,n+3} + FF_{q,n+1}) + FF_{q,n}. \tag{16}$$

As before and consistent with (14), (15) and (16), we put $FF_{q,-1} = 1$. For $q = 0$, the sequence is $FF_{0,n} = (n \bmod 4 = 3)$. If $q = 1$, we write FF_n for $FF_{1,n}$; then the sequence

FF_{3+n} is A006498. The sequence of differences $\overline{FF}$ is, apart from the shift of the offset, A070550. For the sequence of partial sums ΣFF, cf. the somewhat obscure entry A097083 of the OEIS. The sequences $FF_{2,n}$ and $FF_{3,n}$ are, but for the offsets, A089928 and A089931, respectively. The relation between the sequences $FF_{q,n}$ and $F_{q,n}$ is the following.

Proposition 8. *For all $q, k \in \mathbb{N}_0$: $FF_{q,2k} = F_{q,k-1}F_{q,k}$, $FF_{q,2k+1} = F_{q,k}^2$.*

Proof. Induction on k, where the cases $k = 0$ and $k = 1$ are obvious. For $k \in \mathbb{N}$ we get:

$$\begin{aligned} FF_{q,2(k+1)} = FF_{q,2(k-1)+4} &= q \cdot FF_{q,2(k-1)+3} + q \cdot FF_{q,2(k-1)+1} + FF_{q,2(k-1)} \\ &= q \cdot F_{q,k}^2 + q \cdot F_{q,k-1}^2 + F_{q,k-2}F_{q,k-1} \\ &= q \cdot F_{q,k}^2 + F_{q,k-1}(q \cdot F_{q,k-1} + F_{q,k-2}) \\ &= q \cdot F_{q,k}^2 + F_{q,k-1}F_{q,k} = F_{q,k}F_{q,k+1} \end{aligned}$$

and

$$\begin{aligned} FF_{q,2(k+1)+1} = FF_{q,2k-1+4} &= q \cdot FF_{q,2k+2} + q \cdot FF_{q,2k} + FF_{q,2k-1} \\ &= q \cdot FF_{q,2(k+1)} + q \cdot FF_{q,2k} + FF_{q,2(k-1)+1} \\ &= q \cdot F_{q,k}F_{q,k+1} + q \cdot F_{q,k-1}F_{q,k} + F_{q,k-1}^2 \\ &= q \cdot F_{q,k}F_{q,k+1} + F_{q,k-1}F_{q,k+1} = F_{q,k+1}^2. \end{aligned}$$

□

In particular, for all $k \in \mathbb{N}_0$ we have

$$\begin{aligned} FF_{1,2k} &= F_k \cdot F_{k-1}, \\ FF_{1,2k+1} &= F_k^2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} FF_{2,2k} &= P_k \cdot P_{k-1}, \\ FF_{2,2k+1} &= P_k^2. \end{aligned}$$

We will now set out to prove

Theorem 9. *For $p \in \mathbb{N}$, $p \geq 2$, and $n \in \mathbb{N}_0$ we have $|{}_4\Psi_{p,n}| = FF_{p-2,n-1}$.*

Proof. For $n = 0$, we have ${}_4\Psi_{p,0} = \{\epsilon\}$, whence $|{}_4\Psi_{p,0}| = 1 = FF_{p-2,-1}$ by our convention. In $S_p^1 \cong K_p$ there is no distance four times a different one, so ${}_4\Psi_{p,1} = \emptyset$ and $|{}_4\Psi_{p,1}| = 0 = FF_{p-2,0}$. For $n \geq 2$ we have that $s \in P^n$ lies in ${}_4\Psi_{p,n}$ iff

$$\begin{aligned} (s_1 \neq p-1) \quad + \quad (s_2 \neq p-1) \cdot 2 \quad + \quad \sum_{d=3}^{n} (s_d \neq p-1) \cdot 2^{d-1} \\ = \sum_{d=3}^{n} (s_{d-2} \neq 0) \cdot 2^{d-1} + (s_{n-1} \neq 0) \cdot 2^n + (s_n \neq 0) \cdot 2^{n+1}. \end{aligned}$$

This, in turn, is only possible if

$$s_1 = p-1 = s_2, \ \forall\, d \in [n-2] : s_d = 0 \Leftrightarrow s_{d+2} = p-1, \ s_{n-1} = 0 = s_n.$$

For $n = 2$ and $n = 3$ this cannot be fulfilled, so ${}_4\Psi_{p,2} = \emptyset = {}_4\Psi_{p,3}$ and consequently $|{}_4\Psi_{p,2}| = 0 = FF_{p-2,1}$ and $|{}_4\Psi_{p,3}| = 0 = FF_{p-2,2}$. For $n \geq 4$ this amounts to $s = 00\overline{s}(p-1)(p-1)$ with $\overline{s} = s_{n-2} \dots s_3 \in P^{n-4}$ fulfilling

$$s_3 \neq p-1 \neq s_4,\ \forall\, d \in [n-4] \setminus [2] : s_d = 0 \Leftrightarrow s_{d+2} = p-1,\ s_{n-3} \neq 0 \neq s_{n-2}. \tag{17}$$

Let $\overline{S}_1 = \overline{S}_2 = \overline{S}_3 = \emptyset$ and for $n \geq 4$ denote the set of $\overline{s}$ fulfilling (17) by $\overline{S}_n$. Then $\overline{S}_4 = \{\epsilon\}$, $\overline{S}_5 = [p-2]$, $\overline{S}_6 = [p-2]^2$, and $\overline{S}_7 = [p-2]^3 \,\dot\cup\, (p-1)[p-2]0$. For $n \geq 8$ we have the following three cases for an $\overline{s} = s_{n-2}s_{n-3} \dots s_3 \in \overline{S}_n$, depending on the number of initial $p-1$ (there cannot be three in a row because of (17)):

1. $0 \neq s_{n-2} \neq p-1$,
2. $s_{n-2} = p-1 \neq s_{n-3} \neq 0$,
3. $s_{n-2} = p-1 = s_{n-3}$.

In case 1, $\overline{s}$ will run through $[p-2]\overline{S}_{n-1}$, because $s_{n-3} \neq 0 \neq s_{n-4}$. In case 2, s_{n-4} has to be 0, and all elements of $(p-1)[p-2]0\overline{S}_{n-3}$ are admissible because $s_{n-5} \neq 0 \neq s_{n-6}$. Finally, in case 3, $s_{n-4} = 0 = s_{n-5}$, and all elements of $(p-1)(p-1)00\overline{S}_{n-4}$ are admissible because $s_{n-6} \neq 0 \neq s_{n-7}$. So we obtain that for $n \geq 8$ (in fact, for $n \geq 5$):

$$\overline{S}_n = [p-2]\overline{S}_{n-1} \cup (p-1)[p-2]0\overline{S}_{n-3} \cup (p-1)(p-1)00\overline{S}_{n-4}, \tag{18}$$

with the unions disjoint. We can conclude that (for $n \in \mathbb{N}$)

$$|\overline{S}_1| = |\overline{S}_2| = |\overline{S}_3| = 0, \tag{19}$$

$$|\overline{S}_4| = 1, \tag{20}$$

$$|\overline{S}_{n+4}| = (p-2)\left(|\overline{S}_{n+3}| + |\overline{S}_{n+1}|\right) + |\overline{S}_n|. \tag{21}$$

Comparison of (19), (20), (21) with (14), (15), (16) yields $|\overline{S}_n| = FF_{p-2,n-1}$ and since $|{}_4\Psi_{p,n}| = |\overline{S}_n|$, the theorem is proved. □

If we ask for $DD_n := \{\delta(s, 0^n) \mid s \in {}_4\Psi_{p,n}\}$, we see that $DD_0 = \{0\}$, $DD_1 = DD_2 = DD_3 = \emptyset$, and for $n \geq 4$ we have

$$DD_n = 3 + \left\{ \sum_{d=3}^{n-2} (s_d \neq 0) \cdot 2^{d-1} \mid \overline{s} = s_{n-2} \dots s_3 \in \overline{S}_n \right\}. \tag{22}$$

All elements of $\overline{S}_n$ have the form $\sigma = \sigma_k \dots \sigma_1$, where $\sigma_\ell \in [p-2] \,\dot\cup\, (p-1)[p-2]0 \,\dot\cup\, \{(p-1)(p-1)00\}$ and $k \in \mathbb{N}_0$ is such that σ has overall length n. It follows that the binary representation of a distance in DD_n has the form $00\beta_k \dots \beta_1 11$ with $\beta_\ell = 1$ if $\sigma_\ell \in [p-2]$, $\beta_\ell = 110$ if $\sigma_\ell \in (p-1)[p-2]0$, and $\beta_\ell = 1100$ if $\sigma_\ell = (p-1)(p-1)00$, respectively. Therefore,

$\max DD_n = M_{n-2}$, if $p > 2$; $\max DD_n = \frac{1}{5}M_n = \min DD_n$, if $p = 2$ and $n \bmod 4 = 0$ (this is A182512; cf. supra); and finally, for $p > 2$,

$$\min DD_n = \begin{cases} (00(1100)^{(n-4)/4}11)_2 = \frac{1}{5}(2^n - 1), & \text{if } n \bmod 4 = 0\,; \\ (00(1100)^{(n-5)/4}111)_2 = \frac{1}{5}(2^n + 3), & \text{if } n \bmod 4 = 1\,; \\ (00(1100)^{(n-6)/4}1111)_2 = \frac{1}{5}(2^n + 11), & \text{if } n \bmod 4 = 2\,; \\ (00(1100)^{(n-7)/4}11011)_2 = \frac{1}{5}(2^n + 7), & \text{if } n \bmod 4 = 3. \end{cases}$$

Asymptotically, for large n, we have $\min DD_n \sim \frac{1}{5}2^n$ and $\max DD_n \sim \frac{1}{4}2^n$. Note further that for $n \geq 6$ every element of DD_n has a binary representation $0011\beta11$ with a bit string β of length $n-6$ and which does *not* contain a substring 000 or 010. From (22) and (18) we also obtain the recurrence relation $DD_{n+4} = 2^{n+1} + (DD_{n+3} \cup (2^n + (DD_{n+1} \cup DD_n)))$ for $n \in \mathbb{N}_0$. The sequence cc resulting from the union over $n \in \mathbb{N}$ of the sets DD_n by order of size is given by $cc(1) = 3$ and $\forall\, n \in \mathbb{N}_0$:

$$\begin{aligned} \forall\, k \in [FF_n + FF_{n+1}]: \quad & cc(\Sigma FF_{n+3} + k) = 3 \cdot 2^{n+1} + cc(\Sigma FF_{n-1} + k), \\ \forall\, k \in [FF_{n+3}]: \quad & cc(\Sigma FF_{n+4} - FF_{n+3} + k) = 2^{n+2} + cc(\Sigma FF_{n+2} + k). \end{aligned}$$

The sequence cc (with offset 1) starts

$$3, 7, 15, 27, 31, 51, 55, 59, 63, 103, 111, 115, 119, 123, 127, \ldots$$

and is A353578 of the OEIS. It can be viewed as the self-generating sequence with seed 3 and generating function set $\{k \mapsto 2k+1, k \mapsto 8k+3, k \mapsto 16k+3\}$ (cf. Lemma 1).

As an example, we consider the case $n = 8$. Theorem 9 and Proposition 8 assert that there are $FF_{p-2,7} = F_{p-2,3}^2$ 4-key vertices. For $p = 2$ this is $(3 \bmod 2)^2 = 1$, for $p = 3$ this is $F_3^2 = 2^2 = 4$, while for $p = 4$ this is $P_3^2 = 5^2 = 25$. The 4-key vertices come in four forms:

$$\begin{aligned} &0^2[p-2]^4(p-1)^2, \\ &0^2[p-2](p-1)[p-2]0(p-1)^2, \\ &0^2(p-1)[p-2]0[p-2](p-1)^2, \text{ and} \\ &0^2(p-1)^2 0^2(p-1)^2. \end{aligned}$$

The numbers of key vertices of these types are $(p-2)^4$, $(p-2)^2$, $(p-2)^2$, and 1, totaling 1 for $p = 2$, 4 for $p = 3$, and 25 for $p = 4$, as expected. The corresponding key distances are $(00111111)_2 = 63 = cc_9$, $(00111011)_2 = 59 = cc_8$, $(00110111)_2 = 55 = cc_7$, and $(00110011)_2 = 51 = cc_6$. Figure 1 illustrates the four 4-key vertices 00111122, 00121022, 00210122, and 00220022 when $p = 3$.

3 Sierpiński triangle graphs

The approximation of the Sierpiński triangle by a sequence of graphs is even more direct when we consider *Sierpiński triangle graphs* $\widehat{S}^n$. They are embedded as the case $p = 3$ in

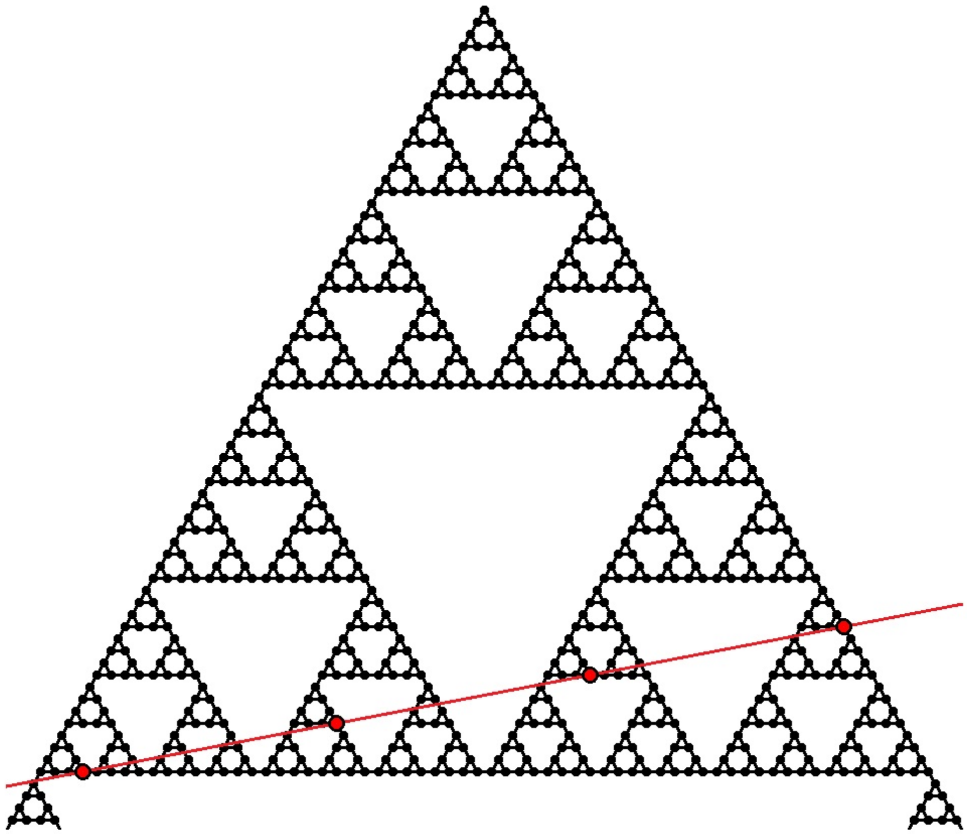

Figure 1: 4-key vertices in S_3^8 (subgraph $00S_3^6$ shown)

the class $\widehat{S}_p^n$ with vertex sets

$$V(\widehat{S}_p^n) = \widehat{P} \cup \left\{ s_\nu \dots s_2 s_1 \mid s_\nu \dots s_2 \in P^{\nu-1},\ \nu \in [n],\ s_1 = \widehat{ij},\ \{i,j\} \in \tbinom{P}{2} \right\},$$

where $p \in \mathbb{N}$, $p \geq 2$, and $\widehat{P}$ stands for the set of *primitive vertices* $\widehat{k}$, $k \in P = [p]_0$; in particular, $\widehat{S}_p^0$ is the complete graph on $\widehat{P}$. All non-primitive vertices $s_\nu \dots s_2\widehat{ij}$ in $\widehat{S}_p^n$ come about by contracting the edge between vertices $s_\nu \dots s_2 ij^{n-\nu+1}$ and $s_\nu \dots s_2 ji^{n-\nu+1}$ in S_p^{n+1}; note that $\widehat{ij} = \widehat{ji}$. The primitive vertex $\widehat{k}$ corresponds to extreme vertex k^{n+1}, and all non-contracted edges of S_p^{n+1} are preserved in $\widehat{S}_p^n$. For a direct definition of the edge set of $\widehat{S}_p^n$, see [7, Definition 3]. The Sierpiński triangle graph $\widehat{S}_p^{1+n}$ can be obtained recursively by taking p copies $k\widehat{S}_p^n$ in which a $k \in P$ has been concatenated to the left of the vertices of $\widehat{S}_p^n$ and finally writing $\widehat{k}$ for $k\widehat{k}$ and identifying $k\widehat{\ell}$ and $\ell\widehat{k}$ for $\ell \in P$, $k \neq \ell$, resulting in *critical vertex* $\widehat{k\ell}$. Consequently, $\widehat{S}_p^n$ is connected; the canonical distance function is denoted by $\delta^{(n)}$. In the case of $p = 2$, we obtain a $\widehat{0}, \widehat{1}$-path of length 2^n with the only critical vertex $0\widehat{1}$. For $p = 3$ we write $\widehat{S}^n := \widehat{S}_3^n$; see Figure 2.

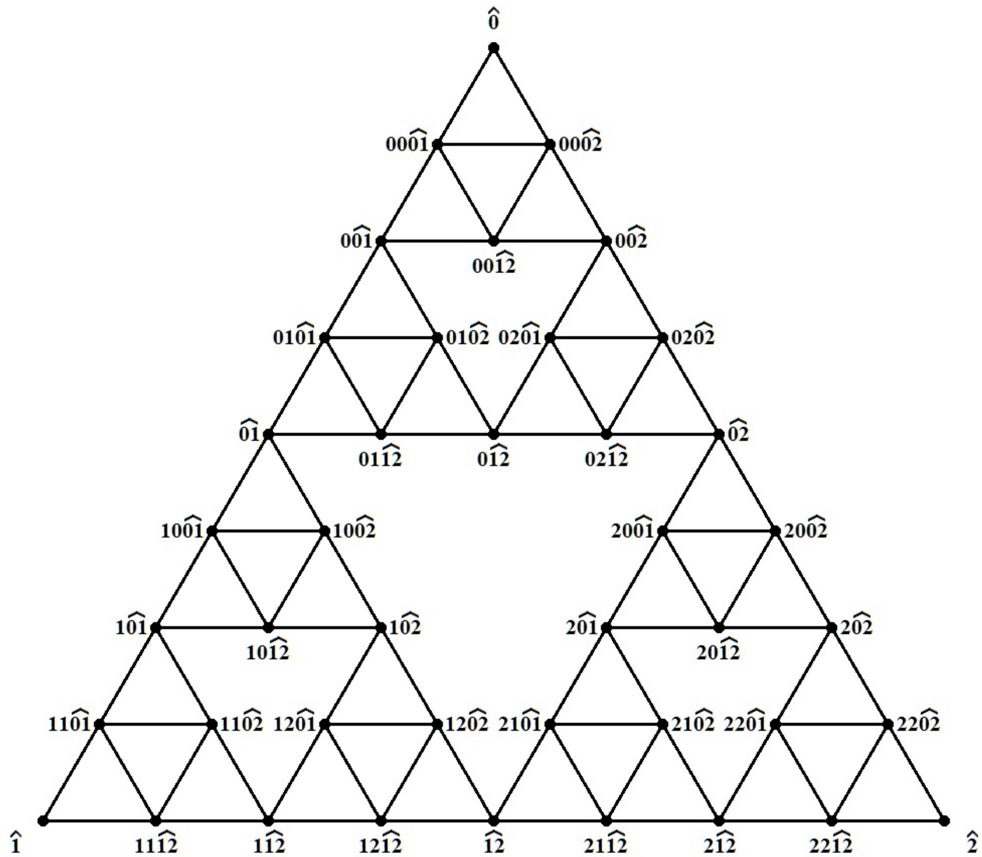

Figure 2: Drawing of the Sierpiński triangle graph $\widehat{S}^3$

For our purpose the distance of a vertex to a primitive vertex is of utmost importance. We have (cf. [7, Equations (3) to (5)]):

$$\delta^{(n)}(\widehat{k}, \widehat{\ell}) = 2^n \cdot (k \neq \ell) \tag{23}$$

and

$$\begin{aligned}
\delta^{(n)}(s_\nu \dots s_2 i\widehat{j}, \widehat{\ell}) &= 2^{n-\nu} \delta^{(\nu)}(s_\nu \dots s_2 i\widehat{j}, \widehat{\ell}) \\
&= 2^{n-\nu} \left(1 + (i \neq \ell \neq j) + \sum_{d=1}^{\nu-1} (s_{d+1} \neq \ell) \cdot 2^d \right) .
\end{aligned} \tag{24}$$

As before, we are interested in *m-key vertices* s for which, without loss of generality, the distance to $\widehat{p-1}$ is m times the distance to $\widehat{0}$. Primitive vertices $\widehat{k}$ are m-key vertices, iff

$m = 1$ and $k \in [p-2]$; the key distance is 2^n. We write ${}_1\widehat{\Phi}_{p,0} = \widehat{[p-2]}$ and ${}_m\widehat{\Phi}_{p,0} = \emptyset$ for $m > 1$. Moreover, by (24) it suffices to look at the case $\nu = n \in \mathbb{N}$, i.e., we consider the sets ${}_m\widehat{\Phi}_{p,n}$ given by

$$\left\{ s\widehat{ij} \mid s = s_n \dots s_2 \in P^{n-1}, \{i,j\} \in \tbinom{P}{2};\ \delta^{(n)}(s\widehat{ij}, \widehat{p-1}) = m \cdot \delta^{(n)}(s\widehat{ij}, \widehat{0}) \right\}.$$

The set of m-key vertices in $\widehat{S}_p^n$ is then $\widehat{\Psi}_n = \bigcup_{\nu=0}^{n} {}_m\widehat{\Phi}_{p,\nu}$ and its size is $|\widehat{\Psi}_n| = \sum_{\nu=0}^{n} |{}_m\widehat{\Phi}_{p,\nu}|$.

As we already know, $\widehat{S}_2^n$ is a path graph on $2^n + 1$ vertices whose leaves are the primitive vertices $\widehat{0}$ and $\widehat{1}$. A δ is an m-key distance iff $(m+1)\delta = 2^n$, i.e., if

$$1 \le \delta = \frac{2^n}{m+1} \le 2^{n-1}.$$

So $m = M_\nu$, $\nu \in [n]$, and $\delta = \delta^{(n)}(s, \widehat{0}) = 2^{n-\nu}$ with m-key vertex $s = 0^{\nu-1}\widehat{01} \in V(\widehat{S}_2^\nu) \subset V(\widehat{S}_2^n)$.

3.1 The case $m = 1$

For $m = 1$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
s\widehat{ij} \in \widehat{\Phi}_n \ &\Leftrightarrow\ 1 + (i \neq p-1 \neq j) + \sum_{d=1}^{n-1} (s_{d+1} \neq p-1) \cdot 2^d \\
&\qquad\qquad = 1 + (i \neq 0 \neq j) + \sum_{d=1}^{n-1} (s_{d+1} \neq 0) \cdot 2^d \\
&\Leftrightarrow\ (i \neq p-1 \neq j) = (i \neq 0 \neq j) \text{ and } \forall\, d \in [n-1] : (s_{d+1} \neq p-1) = (s_{d+1} \neq 0) \\
&\Leftrightarrow\ s \in [p-2]^{n-1} \text{ and } \left(\widehat{ij} = 0\widehat{(p-1)} \text{ or } \{i,j\} \in \tbinom{[p-2]}{2} \right).
\end{aligned}$$

Let $V_p := \left\{ 0\widehat{(p-1)} \right\} \dot\cup \left\{ \widehat{ij} \mid \{i,j\} \in \tbinom{[p-2]}{2} \right\}$ and $f_p := |V_p| = 1 + \tbinom{p-2}{2}$. Then we have shown:

Theorem 10. *For all $p \in \mathbb{N}$, $p \ge 2$, and all $n \in \mathbb{N}_0$:*

$${}_1\widehat{\Psi}_{p,n} = \widehat{[p-2]} \,\dot\cup \bigcup_{\nu=1}^{n} [p-2]^{\nu-1} V_p;$$

$$\left| {}_1\widehat{\Psi}_{3,n} \right| = 1 + n;\ \left| {}_1\widehat{\Psi}_{p,n} \right| = p - 2 + f_p \frac{(p-2)^n - 1}{p-3},\ \text{ if } p \neq 3.$$

In particular, ${}_1\widehat{\Psi}_{2,n} = \emptyset$ if $n = 0$, ${}_1\widehat{\Psi}_{2,n} = \{\widehat{01}\}$ otherwise;

$${}_1\widehat{\Psi}_{3,n} = \{\widehat{1}\} \,\dot\cup \{1^\mu \widehat{02} \mid \mu \in [n]_0\};\ |{}_1\widehat{\Psi}_{4,n}| = 2^{n+1}.$$

When we ask for 1-key distances, we can enter the 1-key vertices from Theorem 10 into the distance formulas (23) and (24). The case $p = 2$ can contribute only one value, and only for $n \neq 0$, namely 2^{n-1}. For $p \geq 3$ we get 2^n and $2^n - 2^{n-\nu}$, $\nu \in [n]$. These sets only overlap at powers of 2, so that the sequence of all 1-key distances is given by $\binom{n}{2} + \nu \mapsto 2^n - 2^{n-\nu}$ for $n \in \mathbb{N}$ and $\nu \in [n]$. These are the numbers whose binary representation is $(1^{n-\mu}0^\mu)_2$ with $\mu \in [n]_0$. They form, apart from the offset, sequence A023758 of the OEIS.

Figure 3 illustrates the six key vertices in $\widehat{S}_3^5 = \widehat{S}^5$ that are equidistant from primitive vertices $\widehat{0}$ and $\widehat{2}$. From left to right, these vertices are $\widehat{1}$ at distance 32, $1^{4-\mu}\widehat{0}\widehat{2}$ for μ from 0 to 4 at distances $(1^{5-\mu}0^\mu)_2$, i.e., 31, 30, 28, 24, and 16, respectively.

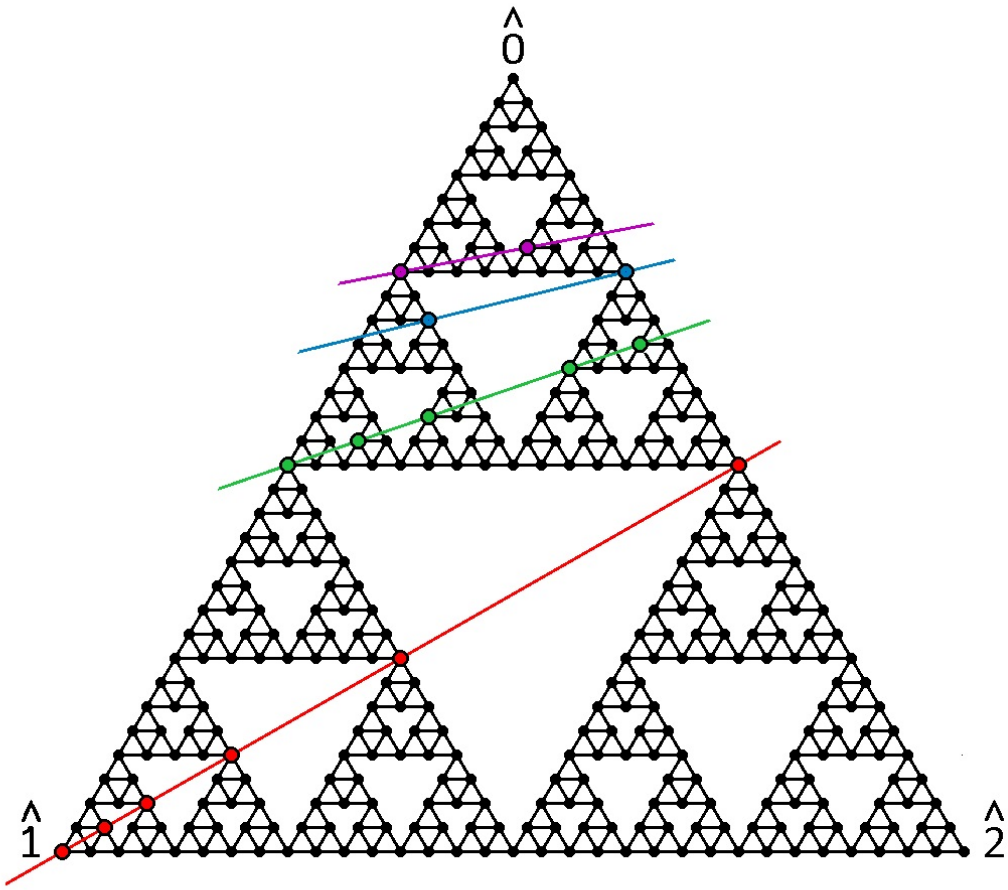

Figure 3: m-key vertices in $\widehat{S}^5$ for $m = 1$ (red), 2 (green), 3 (blue), and 4 (violet)

3.2 The case $m = 2$

For $q, n \in \mathbb{N}_0$ let us define $\widetilde{F}_{q,n}$ by

$$\begin{aligned} \widetilde{F}_{q,0} &= q, \\ \widetilde{F}_{q,1} &= \tbinom{q}{2}, \\ \widetilde{F}_{q,n+2} &= q \cdot \widetilde{F}_{q,n+1} + \widetilde{F}_{q,n}. \end{aligned} \tag{25}$$

We notice that $\widetilde{F}_{0,n} = 0$, $\widetilde{F}_{1,n+1} = F_n$, and $\widetilde{F}_{2,n+2} = 4 \cdot P_{n+1} + P_n =$ A048654$(n+1)$.

Theorem 11. *For $p \in \mathbb{N}$, $p \geq 2$, ${}_2\widehat{\Phi}_{p,0} = \emptyset$ and for $n \in \mathbb{N}$ we have $|{}_2\widehat{\Phi}_{p,n}| = \widetilde{F}_{p-2,n-1}$.*

Proof. Since 3 does not divide 2^n, there are no 2-key vertices in $\widehat{S}_2^n$; so we may assume that $p \geq 3$.

We know already that a 2-key vertex cannot be primitive, i.e., ${}_2\widehat{\Phi}_{p,0} = \emptyset$. From (24) we deduce for $n \in \mathbb{N}_0$:

$$\begin{aligned} s\widehat{ij} \in {}_2\widehat{\Phi}_{p,n+1} \;\Leftrightarrow\; & (i \neq p-1 \neq j) + \sum_{d=1}^{n} (s_{d+1} \neq p-1) \cdot 2^d \\ & = 1 + 2(i \neq 0 \neq j) + \sum_{d=1}^{n} (s_{d+1} \neq 0) \cdot 2^{d+1}. \end{aligned}$$

This means that $i \neq p-1 \neq j$ and that for $n = 0$ we have ${}_2\widehat{\Phi}_{p,1} = 0\widehat{[p-2]}$ and consequently $|{}_2\widehat{\Phi}_{p,1}| = p-2 = \widetilde{F}_{p-2,0}$. Moreover, for $n \in \mathbb{N}$ we get $s_{n+1} = 0$ so that we can reduce the problem to finding $|\widehat{\Phi}_n| = |0\widehat{\Phi}_n| = |{}_2\widehat{\Phi}_{p,n+1}|$ for

$$\widehat{\Phi}_n := \left\{ s\widehat{ij} \mid s = s_n \ldots s_2 \in P^{n-1}, \{i,j\} \in \tbinom{P'}{2};\; 2^n + \delta^{(n)}(s\widehat{ij}, \widehat{p-1}) = 2 \cdot \delta^{(n)}(s\widehat{ij}, \widehat{0}) \right\},$$

where $P' := [p-1]_0$. For completeness, we also define

$$\widehat{\Phi}_0 := \left\{ \widehat{k} \mid k \in P,\; 1 + \delta^{(0)}(\widehat{k}, \widehat{p-1}) = 2 \cdot \delta^{(0)}(\widehat{k}, \widehat{0}) \right\} = \widehat{[p-2]},$$

so that $|\widehat{\Phi}_0| = p-2 = \widetilde{F}_{p-2,0}$. Note that $0\widehat{\Phi}_0 = 0\widehat{[p-2]}$ due to the recursive definition of $\widehat{S}_p^{1+n}$. For $n \in \mathbb{N}$ we have

$$s\widehat{ij} \in \widehat{\Phi}_n \Leftrightarrow \sum_{d=1}^{n-1} (s_{d+1} \neq p-1) \cdot 2^d + 2^n = 2(i \neq 0 \neq j) + \sum_{d=1}^{n-1} (s_{d+1} \neq 0) \cdot 2^{d+1}.$$

If $n = 1$, this means that $i \neq 0 \neq j$, whence $\widehat{\Phi}_1 = \left\{ \widehat{ij} \mid \{i,j\} \in \tbinom{[p-2]}{2} \right\}$, i.e., $|\widehat{\Phi}_1| = \tbinom{p-2}{2} = \widetilde{F}_{p-2,1}$. For $n \geq 2$ we have $s\widehat{ij} \in \widehat{\Phi}_n$ if and only if

$$s_2 \neq p-1 \Leftrightarrow i \neq 0 \neq j,\; \forall\, d \in [n-1] \setminus \{1\} : s_{d+1} = p-1 \Leftrightarrow s_d = 0,\; s_n \neq 0.$$

As in the proof of Theorem 4 we can deduce from this that

$$\widehat{\Phi}_{n+2} = [p-2]\widehat{\Phi}_{n+1} \,\dot{\cup}\, (p-1)0\widehat{\Phi}_n, \tag{26}$$

so that $|\widehat{\Phi}_{n+2}| = (p-2)|\widehat{\Phi}_{n+1}| + |\widehat{\Phi}_n|$ for $n \in \mathbb{N}_0$, i.e., (25) is satisfied with $q = p-2$. $\square$

As an example, the 2-key vertices in $\widehat{S}_3^5 = \widehat{S}^5$ are $\widehat{01}$, $020\widehat{1}$, $0120\widehat{1}$, $02020\widehat{1}$, and $01120\widehat{1}$ (see Figure 3). To see why there are exactly $5 = F_5$ of them, we have to calculate $\widehat{F}_{q,n} = \sum_{\nu=0}^{n-1} \widetilde{F}_{q,\nu}$. It fulfills

$$\widehat{F}_{q,0} = 0,\ \widehat{F}_{q,1} = q,\ \forall\, n \in \mathbb{N}_0 : \widehat{F}_{q,n+2} = -\tbinom{q}{2} + q\widehat{F}_{q,n+1} + \widehat{F}_{q,n}.$$

This can be solved by putting $G_{q,n} = \widehat{F}_{q,n} - \frac{q-1}{2}$ which then fulfills

$$G_{q,0} = -\tfrac{q-1}{2},\ G_{q,1} = \tfrac{q+1}{2},\ \forall\, n \in \mathbb{N}_0 : G_{q,n+2} = qG_{q,n+1} + G_{q,n}.$$

For $q = 1$ we obtain $\widehat{F}_{1,n} = G_{1,n} = F_n$, and $q = 2$ yields (cf. (8))

$$\begin{aligned} \widehat{F}_{2,n} = G_{2,n} + \tfrac{1}{2} &= \tfrac{1}{4}\left((2\sqrt{2}-1)(1+\sqrt{2})^n - (2\sqrt{2}+1)(1-\sqrt{2})^n + 2\right) \\ &= 0, 2, 3, 7, 16, 38, 91, 219, 528, \ldots\ , \end{aligned}$$

which is A353580 in the OEIS.

To find out about the *2-key distances*, i.e., the distances of 2-key vertices to the primitive vertex $\widehat{0}$, we define, for $\nu, n \in \mathbb{N}_0$:

$$D_\nu = \left\{\delta^{(\nu+1)}(s\widehat{ij}, \widehat{0}) \mid s\widehat{ij} \in {}_2\widehat{\Phi}_{p,\nu+1}\right\} = \left\{\delta^{(\nu)}(s\widehat{ij}, \widehat{0}) \mid s\widehat{ij} \in \widehat{\Phi}_\nu\right\},$$

the latter if $\nu \geq 1$, and

$$B_n = \bigcup_{\nu=0}^{n-1} 2^{n-1-\nu} D_\nu = \bigcup_{\nu=0}^{n-1} 2^\nu D_{n-1-\nu},\ B = \bigcup_{n\in\mathbb{N}} B_n.$$

B_n is the set of distances to $\widehat{0}$ occurring among 2-key vertices in $\widehat{S}_p^n$. It fulfills the recurrence

$$B_0 = \emptyset,\ \forall\, n \in \mathbb{N}_0 : B_{n+1} = 2B_n \cup D_n. \tag{27}$$

For $\nu = 0$ we have ${}_2\widehat{\Phi}_{p,1} = 0\widehat{[p-2]}$ and $\delta^{(1)}(\widehat{0j}, \widehat{0}) = 1$ for $j \in [p-2]$, so that $D_0 = \{1\}$. For $\nu = 1$ we have $\widehat{\Phi}_1 = \left\{\widehat{ij} \mid \{i,j\} \in \binom{[p-2]}{2}\right\}$ and $\delta^{(1)}(\widehat{ij}, \widehat{0}) = 2$, so that $D_1 = \emptyset$, if $p = 3$, and $D_1 = \{2\}$, if $p \geq 4$. Using (26) we get:

$$\forall\, n \in \mathbb{N}_0 : D_{n+2} = 2^{n+1} + (D_{n+1} \cup D_n). \tag{28}$$

Note that for $p = 3$ this is the recurrence of the sets C_n (cf. supra) with the seeds switched and that the elements of D_n are the odd elements of B_{n+1}. Independent of $p \geq 3$ we get

$$B_0 = \emptyset,\ B_1 = \{1\},\ \forall\, n \in \mathbb{N}_0 : B_{n+2} = 2^n + (B_{n+1} \cup B_n). \tag{29}$$

The first two statements are clear, as is $B_2 = \{2\}$ for the base step of an induction proof for the recurrence relation. The induction step is

$$\begin{aligned} B_{n+3} = 2B_{n+2} \cup D_{n+2} &= 2^{n+1} + (2B_{n+1} \cup 2B_n \cup D_{n+1} \cup D_n) \\ &= 2^{n+1} + (B_{n+2} \cup B_{n+1}). \end{aligned}$$

From equations (28) and (29) we immediately get

$$2^{n-1} < D_n \leq 2^n,\ 2^{n-1} < B_{n+1} \leq 2^n;$$

in particular, the sets in the sequence B are disjoint, as are those from the sequence D, whence $|D_n| = F_{n-1}$, if $p = 3$, $|D_n| = F_{n+1}$, if $p \geq 4$, and $|B_n| = F_n$ for $n \in \mathbb{N}_0$. More precisely:

Proposition 12. *For $n \in \mathbb{N}$ we have*
(a) $\max\, B_n = 2^{n-1}$,
(b) $\min\, B_n = A_{n+1}$. *(Arima sequence; see* [5] *and cf.* A005578 *in the* OEIS*. Recall that* $\frac{A_{n+1}}{2^n} \to \frac{1}{3}$ *as $n \to \infty$; cf.* [5, p. 7]*.)*
(c) *If the sequence $b \in \mathbb{N}^{\mathbb{N}}$ is given by*

$$b_1 = 1,\ \forall\, n \in \mathbb{N}_0\ \forall\, k \in [F_{n+2}]_0 : b_{F_{n+3}+k} = b_{F_{n+1}+k} + 2^n,$$

then $B = b(\mathbb{N})$. (This corresponds to sequence A052499 *of the* OEIS*: $b_n =$* A052499*$(n-1)$.)*

Proof. Statement (a) follows by induction from (29). Similarly, the recurrence for $\min\, B_n$ in (b) is

$$\min\, B_1 = 1,\ \min\, B_2 = 2,\ \forall\, n \in \mathbb{N} : \min\, B_{n+2} = 2^n + \min\, B_n,$$

a recurrence also fulfilled by the Arima numbers A_{n+1}; cf. [5, p. 7].

For (c) we can show by induction and making use of (29) that

$$\forall\, n \in \mathbb{N}_0 : B_n = \{b_k \mid k \in [F_{n+2}]_0 \setminus [F_{n+1}]_0\}\,.$$

As $\mathbb{N} = \dot{\bigcup\limits_{n \in \mathbb{N}_{(0)}}} [F_{n+2}]_0 \setminus [F_{n+1}]_0$, the elements of sequence b exhaust the whole set B. □

Remark 13. The maximum distance from $\widehat{0}$ among 2-key vertices in $\widehat{S}_p^n$, $n \in \mathbb{N}$, is attained for $s = \widehat{0j}$, $j \in [p-2]$, and $s \in 0[p-2]^\nu \widehat{ij}$, $\{i.j\} \in \binom{[p-2]}{2}$, $\nu \in [n-1]_0$. The minimum is taken in vertices $s = (0(p-1))^{\lfloor (n-1)/2 \rfloor}\, \widehat{0j}$, $j \in [p-2]$ and in addition, if n is even, in vertices $s = (0(p-1))^{(n-2)/2}\, 0\widehat{ij}$, $\{i, j\} \in \binom{[p-2]}{2}$. □

If we compare (29) with the recurrence for the sequence c, we see that $2b_n = c_n + 1$, i.e., $2 \cdot$ A052499$(n-1) = 2 \cdot$ A003754$(n) + 2$, whence A052499$(n-1) =$ A003754$(n) + 1$ for $n \in \mathbb{N}$ (cf. [1, Corollary 1]).

The recurrence in (29) shows that the sequence B does not depend on p, so we may assume that $p = 3$, i.e., $D_1 = \emptyset$. Then another consequence of equations (28) and (29) is the following.

Proposition 14. *Let $n \in \mathbb{N}_0$. Then $D_{n+1} = 4B_n - 1$ and $B_{n+2} = 2B_{n+1} \dot{\cup} (4B_n - 1)$.*

Proof. For $n = 0$ we have $D_1 = \emptyset = 4B_0 - 1$. For $n = 1$ we get $D_2 = \{3\} = 4\{1\} - 1 = 4B_1 - 1$. Now for $n \in \mathbb{N}_0$:

$$\begin{aligned} D_{n+3} &= 2^{n+2} + D_{n+2} \cup D_{n+1} \\ &= 2^{n+2} + (4B_{n+1} - 1) \cup (4B_n - 1) \\ &= 2^{n+2} + 4(B_{n+1} \cup B_n) - 1 \\ &= 4(2^n + B_{n+1} \cup B_n) - 1 \\ &= 4B_{n+2} - 1. \end{aligned}$$

The second statement then follows by (27). The union is disjoint for parity reasons. □

From Proposition 14 it follows that $B = \{1\} \cup 2B \cup (4B - 1)$ (disjoint unions), so that B fulfills the definition given in [1, p. 2] and which is assumed to characterize the sequence A052499, albeit with offset 0, in the OEIS. It is, however, not stated in literature, why the set $B \subset \mathbb{N}$ should be determined uniquely by the above condition. It is an example of a self-generating set; cf. Lemma 1.

3.3 The case $m = 3$

Primitive vertices cannot be 3-key vertices in $\widehat{S}_p^n$, which are therefore the elements of $\widehat{\Psi}_n := \bigcup_{\nu=1}^{n} {}_3\widehat{\Phi}_{p,\nu}$, where for $n \in \mathbb{N}$:

$${}_3\widehat{\Phi}_{p,n} = \left\{ s\widehat{ij} \mid s = s_n \ldots s_2 \in P^{n-1},\ \{i.j\} \in \tbinom{P}{2};\ \delta^{(n)}(s\widehat{ij}, \widehat{p-1}) = 3 \cdot \delta^{(n)}(s\widehat{ij}, \widehat{0}) \right\}.$$

A vertex $s\widehat{ij}$ lies in ${}_3\widehat{\Phi}_{p,n}$, iff

$$(i \neq p-1 \neq j) + \sum_{d=2}^{n} (s_d \neq p-1) \cdot 2^{d-1} = 2 + 3(i \neq 0 \neq j) + 3 \sum_{d=2}^{n} (s_d \neq 0) \cdot 2^{d-1}. \qquad (30)$$

If $n = 1$, then LHS $\leq 1 < 2 \leq$ RHS, so ${}_3\widehat{\Phi}_{p,1} = \emptyset = \widehat{\Psi}_1$. So let $n \geq 2$ and assume that $s_n \neq 0$. Then RHS $\geq 2 + 3 \cdot 2^{n-1} > 2^n - 1 \geq$ LHS, a contradiction. Therefore, $s_n = 0$ and

(30) becomes

$$(i \neq p-1 \neq j) + \sum_{d=2}^{n-1} (s_d \neq p-1) \cdot 2^{d-1} + 2^{n-1} = 2 + 3(i \neq 0 \neq j) + 3 \sum_{d=2}^{n-1} (s_d \neq 0) \cdot 2^{d-1}. \quad (31)$$

If $n = 2$, then necessarily $(i \neq p-1 \neq j) = 0 = (i \neq 0 \neq j)$, whence ${}_3\widehat{\Phi}_{p,2} = \{00\widehat{(p-1)}\} = \widehat{\Psi}_2$. Let $n \geq 3$ and assume that $s_{n-1} = 0$. Then RHS $\leq 2 + 3M_{n-2} = 2^{n-1} + 2^{n-2} - 1 < 2^{n-1} + 2^{n-2} \leq$ LHS, a contradiction. Similarly, if $s_{n-1} = p-1$, then LHS $\leq M_{n-2} + 2^{n-1} = 3 \cdot 2^{n-2} - 1 < 2 + 3 \cdot 2^{n-2} \leq$ RHS; again a contradiction. It follows that $s_{n-1} \in [p-2]$ and (31) becomes

$$(i \neq p-1 \neq j) + \sum_{d=2}^{n-2} (s_d \neq p-1) \cdot 2^{d-1} = 2 + 3(i \neq 0 \neq j) + 3 \sum_{d=2}^{n-2} (s_d \neq 0) \cdot 2^{d-1}. \quad (32)$$

We notice that (32) is the same as (30), but with n replaced by $n-2$. It follows that ${}_3\widehat{\Phi}_{p,n} = \emptyset$, if n is odd, and ${}_3\widehat{\Phi}_{p,n} = (0[p-2])^{(n-2)/2}00\widehat{(p-1)}$, if n is even. In the latter case, $\delta^{(n)}(s\widehat{ij}, \widehat{p-1}) = M_n$ for $s\widehat{ij} \in {}_3\widehat{\Phi}_{p,n}$.

We can summarize the case $m = 3$ in the following theorem.

Theorem 15. *The set of* 3*-key vertices in* $\widehat{S}_p^n$ *is*

$$\widehat{\Psi}_n = \bigcup_{\mu=0}^{\lfloor n/2 \rfloor - 1} (0[p-2])^\mu 00\widehat{(p-1)}$$

with

$$|\widehat{\Psi}_n| = \sum_{\mu=0}^{\lfloor n/2 \rfloor - 1} (p-2)^\mu = \begin{cases} \lfloor n/2 \rfloor, & \text{if } p = 3\,; \\ \dfrac{(p-2)^{\lfloor n/2 \rfloor} - 1}{p-3}, & \text{if } p \neq 3\,. \end{cases}$$

The set of 3-key distances from $\widehat{S}_p^n$ *is* $\widehat{B}_n := \left\{\frac{1}{3} 2^{n-\nu} M_\nu \mid \nu \in [n] \text{ even}\right\}$ *with* $|\widehat{B}_n| = \lfloor n/2 \rfloor$ *(A004526).*

Remark 16. 1. In our test case $\widehat{S}_3^5$ we have key vertices $00\widehat{2}$ and $0100\widehat{2}$ with key distances 8 and 10, respectively (see Figure 3).

2. Note that $\widehat{B}_0 = \emptyset = \widehat{B}_1$ and that for $n \geq 2$ we have $\min \widehat{B}_n = 2^{n-2}$ and $\max \widehat{B}_n = \ell_{n-1}$, the Lichtenberg numbers (A000975). As $\ell_{n-1} < 2^{n-1}$, the sets $\widehat{B}_n$ are disjoint. The elements of $\widehat{B}_n$ can be written as $\frac{1}{3} 2^{n-\nu} M_\nu = \frac{1}{3}(2^n - 2^{n-\nu}) = 2^{n-\nu} \ell_{\nu-1}$ for even $\nu \in [n]$. The set of all 3-key distances is

$$\widehat{B} := \bigcup_{n=0}^{\infty} \widehat{B}_n = \{2^i \ell_{2j+1} \mid i, j \in \mathbb{N}_0\} = \{(1(01)^j 0^i)_2 \mid i, j \in \mathbb{N}_0\}. \quad (33)$$

This set can be written as a sequence $\widehat{b} \in \mathbb{N}^{\mathbb{N}}$ in an interesting way. If we define $\widetilde{\Delta}_0 = 0 = \widetilde{\Delta}_1$ and $\widetilde{\Delta}_{N+2} = \widetilde{\Delta}_N + N + 1$ for $N \in \mathbb{N}_0$, i.e.,

$$\widetilde{\Delta}_N = \sum_{n=0}^{N} \lfloor n/2 \rfloor = \lfloor N^2/4 \rfloor = \lfloor N/2 \rfloor \cdot \lceil N/2 \rceil = \tfrac{1}{4}(N^2 - N \bmod 2)$$

(see the many entries for A002620 in the OEIS and note that $\widetilde{\Delta}_{N+1} + \widetilde{\Delta}_N = \binom{N+1}{2} = \Delta_N$), every $n \in \mathbb{N}$ can be written uniquely as $n = \widetilde{\Delta}_{N-1} + \rho$ with $N = \lceil 2\sqrt{n}\, \rceil \geq 2$ and a $\rho \in [\lfloor N/2 \rfloor]$. Then $\widehat{B} = \widehat{b}(\mathbb{N})$ for the sequence $\widehat{b}$ given by

$$\widehat{b}(\widetilde{\Delta}_{N-1} + \rho) = \tfrac{1}{3}(2^N - 2^{N-2\rho}) = 2^{N-2\rho}\ell_{2\rho-1} = \left(1(01)^{\rho-1}0^{N-2\rho}\right)_2,$$

i.e., with $i = N - 2\rho$ and $j = \rho - 1$ in (33). (This sequence $\widehat{b}$ is A181666.) The bijection

$$\mathbb{N} \ni \widetilde{\Delta}_{N-1} + \rho \leftrightarrow (N - 2\rho, \rho - 1) \in \mathbb{N}_0^2$$

is quite remarkable.

$\widehat{B}$ is also the self-generating set (cf. Lemma 1) with seed 1 and engendered by the two generating functions given by $\mathbb{N} \ni k \mapsto 2k$ and $f(2^i(2h+1)) = 2^i(8h+5)$ for $i, h \in \mathbb{N}_0$; note that $f(2^i \ell_{2j+1}) = 2^i \ell_{2(j+1)+1}$, whence $f(\widehat{B}) = \widehat{B} \setminus \{2^i \mid i \in \mathbb{N}_0\}$. $\square$

3.4 The case $m = 4$

Again, primitive vertices cannot be 4-key vertices in $\widehat{S}_p^n$, which are therefore the elements of $\widehat{\Psi}_n := \bigcup_{\nu=1}^{n} {}_4\widehat{\Phi}_{p,\nu}$, where for $n \in \mathbb{N}$:

$${}_4\widehat{\Phi}_{p,n} = \left\{ s\widehat{ij} \mid s = s_n \dots s_2 \in P^{n-1},\ \{i.j\} \in \tbinom{P}{2};\ \delta^{(n)}(s\widehat{ij}, \widehat{p-1}) = 4 \cdot \delta^{(n)}(s\widehat{ij}, \widehat{0}) \right\}.$$

A vertex $s\widehat{ij}$ lies in ${}_4\widehat{\Phi}_{p,n}$, iff

$$(i \neq p-1 \neq j) + \sum_{d=2}^{n} (s_d \neq p-1) \cdot 2^{d-1} = 3 + 4(i \neq 0 \neq j) + \sum_{d=2}^{n} (s_d \neq 0) \cdot 2^{d+1}.$$

As the RHS is odd, we must have $i \neq p-1 \neq j$ and

$$\sum_{d=2}^{n} (s_d \neq p-1) \cdot 2^{d-1} = 2 + 4(i \neq 0 \neq j) + \sum_{d=2}^{n} (s_d \neq 0) \cdot 2^{d+1}.$$

The case $n = 1$ cannot be satisfied, so that ${}_4\widehat{\Phi}_{p,1} = \emptyset$ and $|{}_4\widehat{\Phi}_{p,1}| = 0$. Let $n \geq 2$. Then $s_2 \neq p-1$, whence

$$\sum_{d=3}^{n} (s_d \neq p-1) \cdot 2^{d-1} = 4(i \neq 0 \neq j) + \sum_{d=2}^{n} (s_d \neq 0) \cdot 2^{d+1}. \tag{34}$$

For $n = 2$ we necessarily have $i = 0$ and $j \in [p-2]$ and $s_2 = 0$, so that ${}_4\widehat{\Phi}_{p,2} = 00\widehat{[p-2]}$ and $|{}_4\widehat{\Phi}_{p,2}| = p-2$; key distance is $\delta^{(2)}(00\widehat{j}, \widehat{0}) = 1$. For $n \geq 3$ we get $s_{n-1} = 0 = s_n$, which for $n = 3$ means $\{i,j\} \in \binom{[p-2]}{2}$, $s_2 = 0 = s_3$, whence ${}_4\widehat{\Phi}_{p,3} = \left\{00\widehat{ij} \mid \{i,j\} \in \binom{[p-2]}{2}\right\}$ and $|{}_4\widehat{\Phi}_{p,3}| = \binom{p-2}{2}$; key distance is $\delta^{(3)}(00\widehat{ij}, \widehat{0}) = 2$. For $n = 4$ we get $i \neq 0 \neq j$, $s_2 \in [p-2]$, and $s_3 = 0 = s_4$, i.e., ${}_4\widehat{\Phi}_{p,4} = \left\{00s_2\widehat{ij} \mid s_2 \in [p-2],\ \{i,j\} \in \binom{[p-2]}{2}\right\}$ and $|{}_4\widehat{\Phi}_{p,4}| = (p-2)\binom{p-2}{2}$; key distance is $\delta^{(4)}(00s_2\widehat{ij}, \widehat{0}) = 4$. For $n \geq 5$ we deduce from (34) that, in addition to the conditions already fixed, $s_3 = p-1 \Leftrightarrow \widehat{ij} \in 0\widehat{[p-2]}$, $\forall\, d \in [n-2] \setminus [3] : s_d = p-1 \Leftrightarrow s_{d-2} = 0$ and $s_{n-3} \neq 0 \neq s_{n-2}$. This leads to the following recurrence relation for $n \in \mathbb{N}_0$.

$${}_4\widehat{\Phi}_{p,4+n} = 00[p-2]{}_4\widehat{\Phi}''_{p,3+n} \,\dot\cup\, 00(p-1)[p-2]{}_4\widehat{\Phi}'_{p,1+n} \,\dot\cup\, 00(p-1)(p-1){}_4\widehat{\Phi}_{p,n},$$

where each prime indicates the deletion of a leading 0; e.g., ${}_4\widehat{\Phi}'_{p,2} = 0\widehat{[p-2]}$. This means that

$$|{}_4\widehat{\Phi}_{p,4+n}| = (p-2)|{}_4\widehat{\Phi}_{p,3+n}| + (p-2)|{}_4\widehat{\Phi}_{p,1+n}| + |{}_4\widehat{\Phi}_{p,n}|.$$

If for $q \in \mathbb{N}_0$ we define the sequences $(\widetilde{FF}_{q,n})_{n\in\mathbb{N}_0}$ by

$$\widetilde{FF}_{q,0} = 0 = \widetilde{FF}_{q,1},\ \widetilde{FF}_{q,2} = q,\ \widetilde{FF}_{q,3} = \tbinom{q}{2},$$

$$\widetilde{FF}_{q,n+4} = q(\widetilde{FF}_{q,n+3} + \widetilde{FF}_{q,n+1}) + \widetilde{FF}_{q,n},$$

we get

Theorem 17. *If $p \in \mathbb{N}$, $p \geq 2$, and $n \in \mathbb{N}_0$, then $|{}_4\widehat{\Phi}_{p,n}| = \widetilde{FF}_{p-2,n}$.*

For $q = 0$ we have $\widetilde{FF}_{0,n} = 0$, which reflects the fact that 5 does not divide 2^n. For $q = 1$ the sequence is $\widetilde{FF}_{1,n} = \overline{FF}_{n+1}$. The sequence of partial sums is $|{}_4\widehat{\Psi}_{3,n}| = FF_{n+1}$. In our standard example, the graph $\widehat{S}_3^5$, we therefore have two 4-key vertices, namely $00\widehat{01}$ and $0021\widehat{01}$ with 4-key distances 8 and 7, respectively (see Figure 3). The sequence

$$\widetilde{FF}_{2,n} = \tfrac{1}{8}\left((5-3\sqrt{2})(1+\sqrt{2})^n + (5+3\sqrt{2})(1-\sqrt{2})^n + x_n\right),$$

where $x_n = -10,\ 2,\ 10,\ -2$, if $n \bmod 4 = 0,\ 1,\ 2,\ 3$, respectively, starts

$$0, 0, 2, 1, 2, 8, 20, 45, 108, 264, 638, 1537, \ldots;$$

this is A353581 in the OEIS. Its partial sums form sequence A353582, namely

$$\begin{aligned} |{}_4\widehat{\Psi}_{4,n}| &= \tfrac{1}{16}\left((4-\sqrt{2})(1+\sqrt{2})^n + (4+\sqrt{2})(1-\sqrt{2})^n + y_n\right) \\ &= 0, 0, 2, 3, 5, 13, 33, 78, 186, 450, 1088, 2625, \ldots \end{aligned}$$

with $y_n = -8,\ -4,\ 16,\ 12$, if $n \bmod 4 = 0,\ 1,\ 2,\ 3$, respectively.

For the sets of 4-key distances in $\widehat{S}_p^n$, $p \geq 3$, we get the recurrence

$$\widehat{DD}_0 = \emptyset = \widehat{DD}_1,\ \widehat{DD}_2 = \{1\},\ \widehat{DD}_3 = \{2\},$$

$$\widehat{DD}_{n+4} = 2^{n+1} + \left(\widehat{DD}_{n+3} \cup \left(2^n + \left(\widehat{DD}_{n+1} \cup \widehat{DD}_n\right)\right)\right).$$

For $n \geq 2$ we have $\max \widehat{DD}_n = 2^{n-2}$ and

$$\min \widehat{DD}_n = \begin{cases} \frac{1}{5}(2^n + 4), & \text{if } n \bmod 4 = 0\,; \\ \frac{1}{5}(2^n + 3), & \text{if } n \bmod 4 = 1\,; \\ \frac{1}{5}(2^n + 1), & \text{if } n \bmod 4 = 2\,; \\ \frac{1}{5}(2^n + 2), & \text{if } n \bmod 4 = 3\,. \end{cases}$$

Asymptotically, for large n, we have $\min \widehat{DD}_n \sim \frac{1}{5}2^n$ and $\max \widehat{DD}_n \sim \frac{1}{4}2^n$.

The sequence $\widehat{cc}$ obtained from the union over $n \in \mathbb{N}$ of the sets $\widehat{DD}_n$ by order of size is given by $\widehat{cc}(1) = 1$ and $\forall\, n \in \mathbb{N}_0$:

$$\begin{aligned} \forall\, k \in [FF_n + FF_{n+1}] : \quad & \widehat{cc}(\Sigma FF_{n+3} + k) = 3 \cdot 2^{n-1} + \widehat{cc}(\Sigma FF_{n-1} + k), \\ \forall\, k \in [FF_{n+3}] : \quad & \widehat{cc}(\Sigma FF_{n+4} - FF_{n+3} + k) = 2^n + \widehat{cc}(\Sigma FF_{n+2} + k). \end{aligned}$$

The sequence $\widehat{cc}$ (with offset 1) starts

$$1, 2, 4, 7, 8, 13, 14, 15, 16, 26, 28, 29, 30, 31, 32, \ldots$$

and is A353579 in the OEIS. It can be viewed as the self-generating sequence with seed 1 and generating function set $\{k \mapsto 2^n + k, k \mapsto 3 \cdot 2^{n+1} + k, k \mapsto 3 \cdot 2^{n+2} + k\}$, where n is the smallest non-negative integer such that $k \leq 2^n$ (cf. Lemma 1).

4 Outlook

For fixed m and p, the string sets of m-key vertices, ${}_m\Psi_{p,n}$ for Sierpiński graphs S_p^n and ${}_m\widehat{\Psi}_{p,n}$ for Sierpiński triangle graphs $\widehat{S}_p^n$, are often, perhaps always, regular languages, denoted by regular expressions. For example, the language of non-empty strings in ${}_2\Psi_{3,n}$ can be represented by the regular expression

$$0(1 \vee 20)^\star 2,$$

illustrating equation (9) in the proof of Theorem 4. This regular expression denotes the language of all strings that begin with the character 0 and end with the character 2, with zero or more substrings, each either 1 or 20, in between; the star character stands for the star closure, or Kleene closure, of a language. If we wish to include the empty string, which is the only key vertex when $n = 0$, we can use the more compact but perhaps less intuitive regular expression

$$(01^{\star}2)^{\star}.$$

From this, all distance properties can be deduced via the formulas (2) and (23), (24), respectively. The counting sequences $|_{2^k}\Psi_{p,n}|$ for m-key vertices when $m = 2^k$ appear to have interesting forms, extending the formulas for $k \in \{0, 1, 2\}$ presented here. Moreover, it will be interesting to investigate the fractal structures engendered by the underlying sets of key vertices.

5 Acknowledgments

We thank Sandi Klavžar (Ljubljana) for a valuable discussion. We are grateful for the very thoughtful remarks and suggestions made by an anonymous referee.

References

[1] J.-P. Allouche, J. Shallit, and G. Skordev, Self-generating sets, integers with missing blocks, and substitutions, *Discrete Math.* **292** (2005), 1–15.

[2] A. T. Benjamin and J. J. Quinn, *Proofs That Really Count*, The Mathematical Association of America, 2003.

[3] S. Falcón and Á. Plaza, On the Fibonacci k-numbers, *Chaos Solitons Fractals* **32** (2007), 1615–1624.

[4] T. Grubb and C. Wolird, Cyclotomic points and algebraic properties of polygon diagonals, *Integers* **21** (2021), A40.

[5] A. M. Hinz, The Lichtenberg sequence, *Fibonacci Quart.* **55** (2017), 2–12.

[6] A. M. Hinz and C. Holz auf der Heide, An efficient algorithm to determine all shortest paths in Sierpiński graphs, *Discrete Appl. Math.* **177** (2014), 111–120.

[7] A. M. Hinz, C. Holz auf der Heide, and S. S. Zemljič, Metric properties of Sierpiński triangle graphs, *Discrete Appl. Math.*, in press.

[8] A. M. Hinz, S. Klavžar, and C. Petr, *The Tower of Hanoi—Myths and Maths*, Second Edition, Springer, 2018.

[9] A. M. Hinz, S. Klavžar, and S. S. Zemljič, A survey and classification of Sierpiński-type graphs, *Discrete Appl. Math.* **217** (2017), 565–600.

[10] A. M. Hinz and A. Schief, The average distance on the Sierpiński gasket, *Probab. Theory Related Fields* **87** (1990), 129–138.

[11] A. M. Hinz and P. K. Stockmeyer, Discovering Fibonacci numbers, Fibonacci words, and a Fibonacci fractal in the Tower of Hanoi, in K. Dilcher, ed., *Proc. 18th Int'l Conf. on Fibonacci Numbers and Their Applications*, in *Fibonacci Quart.* **57** (5) (2019), 72–83.

[12] E. Lucas, Théorie des fonctions numériques simplement périodiques, *Amer. J. Math.* **1** (1878), 184–240, 289–321.

[13] P. K. Stockmeyer, An exploration of sequence A000975, in C. Ballot, P. G. Anderson, and T. Komatsu, eds., *Proc. 17th Int'l Conf. on Fibonacci Numbers and Their Applications*, in *Fibonacci Quart.* **55** (5) (2017), 174–185.

2020 *Mathematics Subject Classification*: Primary 05C12; Secondary 11B39, 68R15.
Keywords: self-generating set, Sierpiński graph, Sierpiński triangle graph, distance in graphs, integer sequence, Fibonacci number, Pell number.

(Concerned with sequences A000045, A000129, A000225, A000975, A001045, A002450, A002620, A003754, A004526, A005578, A006498, A023758, A048654, A052499, A070550, A089928, A089931, A097083, A181666, A182512, A247648, A353578, A353579, A353580, A353581, and A353582.)

Received February 9 2022; revised versions received May 13 2022; June 9 2022. Published in *Journal of Integer Sequences*, June 10 2022.

Return to Journal of Integer Sequences home page.

sequence	OEIS®	initial entries for $n =$																	
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
F	A000045	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597
$\overline{FF}$	(A070550)	0	0	0	1	0	0	1	2	2	3	6	10	15	24	40	65	104	168
FF	(A006498)	0	0	0	1	1	1	2	4	6	9	15	25	40	64	104	169	273	441
ΣFF	(A097083)?	0	0	0	1	2	3	5	9	15	24	39	64	104	168	272	441	714	1155
a	(A003754)	0	1	2	3	5	6	7	10	11	13	14	15	21	22	23	26	27	29
b	(A052499)		1	2	3	4	6	7	8	11	12	14	15	16	22	23	24	27	28
$\widehat{b}$	A181666		1	2	4	5	8	10	16	20	21	32	40	42	64	80	84	85	128
c	A247648	0	1	3	5	7	11	13	15	21	23	27	29	31	43	45	47	53	55
cc	A353578		3	7	15	27	31	51	55	59	63	103	111	115	119	123	127	207	219
$\widehat{cc}$	A353579		1	2	4	7	8	13	14	15	16	26	28	29	30	31	32	52	55

Table 1: Some integer sequences addressed in the text (An OEIS entry in brackets means that the offset is shifted.)