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The Vertical Recursive Relation of Riordan Arrays and Its Matrix Representation

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Abstract

It is known that the entries of a Riordan array satisfy horizontal recursive relations represented by the A- and Z-sequences. In this paper, we study a vertical recursive relation approach to Riordan arrays. This vertical recursive approach gives a way to represent the entries of a Riordan array (g, f) in terms of a recursive linear combination of the coefficients of g. We also give a matrix representation of the vertical recursive relation. The set of all those matrices forms a group, called the quasi-Riordan group. We present extensions of the horizontal recursive relation and the vertical recursive relation in terms of c- and C-Riordan arrays, with illustrations by using the rook triangle and the Laguerre triangle. These extensions represent a way to study nonlinear recursive relations of the entries of some triangular matrices from linear recursive relations of the entries of Riordan arrays. In addition, the matrix representation of the vertical recursive relation of Riordan arrays provides transforms between lower order and higher order finite Riordan arrays, where the mth order Riordan array is defined by $(g, f)_m = (d_{n,k})_{m\geq n,k\geq 0}$. Furthermore, the vertical relation approach to Riordan arrays provides a unified approach to construct identities.

1 Introduction

Riordan matrices are infinite, lower triangular matrices defined by the generating function of their columns. With matrix multiplication, they form a group, called the *Riordan group* (see Shapiro, Getu, Woan and Woodson [29]).

More formally, let us consider the set of formal power series ring $\mathcal{F} = K[\![t]\!]$, where $\mathbb{K}$ is the field of $\mathbb{R}$ or $\mathbb{C}$. The *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{K}$), is the minimum number $r \in \mathbb{N}_0$ such that $f_r \neq 0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the positive integer set. We let $\mathcal{F}_r$ denote the set of formal power series of order r. Let $g(t) \in \mathcal{F}_0$ and $f(t) \in \mathcal{F}_1$; the pair $(g,\ f)$ defines the *(proper) Riordan matrix* $D = (d_{n,k})_{n,k\in\mathbb{N}_0} = (g, f)$ where

$$d_{n,k} = [t^n]g(t)f(t)^k \tag{1}$$

or, in other words, having gf^k as the generating function of the kth column of (g, f). The *first fundamental theorem of Riordan matrices* concerns the action of the proper Riordan matrices on the formal power series presented by

$$(g(t), f(t))h(t) = g(t)(h \circ f)(t),$$

which can be abbreviated as $(g, f)h = gh(f)$. Thus we immediately see that the usual row-by-column product of two Riordan matrices is also a Riordan matrix:

$$(g_1,\ f_1)(g_2,\ f_2) = (g_1 g_2(f_1),\ f_2(f_1)). \tag{2}$$

The Riordan matrix $I = (1,\ t)$ is the identity matrix because its entries are $d_{n,k} = [t^n]t^k = \delta_{n,k}$.

Let $(g\,(t)\,,\ f(t))$ be a Riordan matrix. Then its inverse is

$$(g\,(t)\,,\ f(t))^{-1} = \left(\frac{1}{g(\overline{f}(t))}, \overline{f}(t),\right), \tag{3}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$, i.e., $(f \circ \overline{f})(t) = (\overline{f} \circ f)(t) = t$. In this way, the set $\mathcal{R}$ of all proper Riordan matrices forms a group (see [29]) called the *Riordan group.*

Here is a list of six important subgroups of the Riordan group (see [29, 31, 1, 30]):

- the *Appell subgroup* $\{(g(z),\ z)\}$.
- the *Lagrange (associated) subgroup* $\{(1,\ f(z))\}$.
- the *k-Bell subgroup* $\{(g(z),\ z(g(z))^k)\}$, where k is a fixed positive integer.
- the *hitting-time subgroup* $\{(zf'(z)/f(z),\ f(z))\}$.
- the *derivative subgroup* $\{(f'(z),\ f(z))\}$.
- the *checkerboard subgroup* $\{(g(z),\ f(z))\}$, where g is an even function and f is an odd function.

The 1-Bell subgroup is referred to as the *Bell subgroup* for short, and the Appell subgroup can be considered as the 0-Bell subgroup if we allow $k = 0$ to be included in the definition of the k-Bell subgroup.

Let G be a group, and let H and N be two subgroups of G with N normal. Then the following statements are equivalent:

- $NH = G$ and $N \cap H = e$.
- Every $g \in G$ can be written uniquely as $g = nh$, where $n \in N$ and $h \in H$.
- Define $\psi : H \to G/N$ by $\psi(h) = Nh$, $h \in H$. Then ψ is an isomorphism.

If these conditions hold, we write $G = N \rtimes H$ and say that G is expressed as a semidirect product of N and H. Since for every $(g, f) \in \mathcal{R}$ we have

$$(g, f) = (g, t)(1, f),$$

where $(g, t) \in \mathcal{A}$ and $(1, f) \in \mathcal{L}$ and $\mathcal{A}$ and $\mathcal{L}$ are the Appell subgroup, the normal subgroup of $\mathcal{R}$, and the Lagrange subgroup, respectively.

An infinite lower triangular matrix $[d_{n,k}]_{n,k\in\mathbb{N}_0}$ is a Riordan matrix if and only if a unique sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}_0$ we have

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots + a_{n-k} d_{n,n}. \tag{4}$$

This is equivalent to

$$f(t) = tA(f(t)) \quad \text{or} \quad t = \bar{f}(t)A(t). \tag{5}$$

Here $A(t)$ is the generating function of the A-sequence. The first formula of (5) is also called the *second fundamental theorem of Riordan matrices*. Moreover, there exists a unique sequence $Z = (z_0, z_1, z_2, \ldots)$ such that every element in column 0 can be expressed as the linear combination

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \cdots + z_n d_{n,n}, \tag{6}$$

or equivalently,

$$g(t) = \frac{1}{1 - tZ(f(t))}, \tag{7}$$

in which and throughout we always assume $g(0) = g_0 = 1$, a usual hypothesis for proper Riordan matrices with normalization. From (7), we may obtain the equation.

$$Z(t) = \frac{g(\bar{f}(t)) - 1}{\bar{f}(t)g(\bar{f}(t))}.$$

A- and Z-sequence characterizations of Riordan matrices were introduced, developed, and/or studied in Merlini, Rogers, Sprugnoli, and Verri [24], Roger [28], Sprugnoli and the author [17], Cheon and Jin [5], Cheon, Luzón, Morón, Prieto Martinez, and Song [7], [13], Jean-Louis and Nkwanta [19], Luzón, Morón, and Prieto-Martinez [21, 22], etc. In [17] the expressions of the A- and Z-sequences of the product depend on the analogous sequences of the two given factors.

The Catalan numbers, $C_n = \binom{2n}{n}/(n+1)$, are a sequence of integers that occur in many counting situations. The books *Enumerative Combinatorics*, Volume 2 [33] and the more recent *Catalan Numbers* by Richard Stanley [34] give a wealth of information about them.

The generating function of the Catalan numbers is denoted by $C(z)$, which can be written as

$$C(t)=\sum_{n=0}^{\infty}C_nt^n=\frac{1-\sqrt{1-4t}}{2t},$$

which is equivalent to the functional equation $C(t)=1+tC(t)^2$. It can be shown that [11]

$$C(t)^k=\sum_{n=0}^{\infty}\frac{k}{2n+k}\binom{2n+k}{n}t^n.$$

A *Dyck path* of length $2n$ is a path in the plane lattice $\mathbb{Z}\times\mathbb{Z}$ from the origin $(0,0)$ to $(2n,0)$, made up with steps $(1,1)$ and $(1,-1)$. The other requirement is that a path can never go below the x -axis. We refer to n as the *semilength* of the path. It is well known that the number of Dyck paths of semilength n is the n-th Catalan number C_n. A partial Dyck path is a Dyck path without requiring that the end point be on the x-axis. Hence we have that

$$C(n,k):=[t^n]C(t)^k=\frac{k}{2n+k}\binom{2n+k}{n} \tag{8}$$

is the number of the partial Dyck paths from $(0,0)$ to $(2n+k,k)$.

For a positive integer m, an m-Dyck path is a path from the origin to $(mn,0)$ using the steps $(1,1)$ and $(1,1-m)$ and again not going below the x-axis. We refer to mn as the length of the path. A partial m-Dyck path is defined as an m-partial Dyck path. It is well known that the number of m-Dyck paths of length mn is (see, for example, [11])

$$F_m(n,1)=\frac{1}{mn+1}\binom{mn+1}{n},$$

the Fuss-Catalan numbers. For $m=2$, the Fuss-Catalan numbers are the Catalan numbers $F_2(n,1)$. More generally, the Fuss-Catalan numbers are

$$F_m(n,r):=\frac{r}{mn+r}\binom{mn+r}{n}, \tag{9}$$

which are named after N. I. Fuss and E. C. Catalan (see [9, 11, 20, 25, 26, 12, 15, 18]). The Fuss-Catalan numbers have many combinatorial applications (see, for example, Shapiro and the author [18]).

The generating function $F_m(t)$ for the Fuss-Catalan numbers, $\{F_m(n,1)\}_{n\geq 0}$ is called the generalized binomial series in [11], and it satisfies the function equation $F_m(t)=1+tF_m(t)^m$. Hence from Lambert's formula for the Taylor expansion of the powers of $F_m(t)$ (see [11]), we have that

$$F_m^r\equiv F_m(t)^r=\sum_{n\geq 0}\frac{r}{mn+r}\binom{mn+r}{n}t^n \tag{10}$$

for all $r \in \mathbb{R}$. The key case(10) leads to the following formula for $F_m(t)$:

$$F_m(t) = 1 + tF_m^m(t). \tag{11}$$

Actually,

$$\begin{aligned}1 + tF_m^m(t) &= 1 + \sum_{n\geq 0} \frac{m}{mn+m}\binom{mn+m}{n} t^{n+1} \\ &= 1 + \sum_{n\geq 1} \frac{m}{mn}\binom{mn}{n-1} t^n \\ &= \sum_{n\geq 0} \frac{1}{mn+1}\binom{mn+1}{n} t^n = F_m(t).\end{aligned}$$

For the cases $m = 1$ and 2, we have $F_1 = 1/(1-t)$ and $F_2 = C(z)$, respectively. When $m = 3$, the Fuss-Catalan numbers $(F_3)_n$ form the sequence A001764 (see [32]),

$$1, 1, 3, 12, 55, 273, 1428, \ldots,$$

the so-called ternary numbers. The ternary numbers count the number of 3-Dyck paths or ternary paths. The generating function of the ternary numbers is denoted by $T(t) = \sum_{n=0}^{\infty} T_n t^n$ with $T_n = \frac{1}{3n+1}\binom{3n+1}{n}$, and is given equivalently by the equation

$$T(t) = 1 + tT(t)^3.$$

2 A vertical recursive relation view of Riordan arrays

In a Riordan array $(g, f) = (d_{n,k})_{n,k\geq 0}$, the first column (0th column), with its generating function $g(t)$, usually possesses an interesting combinatorial interpretation or represents an important sequence, while the other columns might be considered as the compositions of the first column in the view of the following recurrence relation:

$$d_{n,k} = [t^n]gf^k = \sum_{j=1}^{n} f_j [t^{n-j}]gf^{k-1} = \sum_{j=1}^{n-k+1} f_j d_{n-j,k-1} \tag{12}$$

for $n, k \geq 1$, where $f(t) = \sum_{j\geq 1} f_j t^j$. For instance,

$$\begin{aligned}&d_{1,1} = f_1 d_{0,0}, \\ &d_{2.1} = f_2 d_{0,0} + f_1 d_{1,0},\ d_{2,2} = f_1 d_{1,1}, \\ &d_{3,1} = f_3 d_{0,0} + f_2 d_{1,0} + f_1 d_{2.0},\ d_{3,2} = f_2 d_{2,1} + f_1 d_{1,1},\ d_{3,3} = f_1 d_{2,2}, \ldots\end{aligned}$$

Example 1. For the Pascal triangle $(1/(1-t), t/(1-t))$, from (12) we obtain the well-known recursive relation

$$\binom{n}{k} = \sum_{j=1}^{n-k+1} \binom{n-j}{k-1}. \tag{13}$$

Recurrence relation (12) provides a resource for constructing identities and an algorithm for computing powers and multiplications of formal power series. The latter approach may simplify the corresponding computation process by using Faà di Bruno's formula.

Theorem 2. *Let $(g, f) = (d_{n,k})_{n,k\geq 0}$ be a Riordan array, and let $C(n, k)$ and $F_m(n, r)$ be defined by (8) and (9), respectively. Then we obtain the recurrence relation (12). In particular, for $g = F_m$ we have*

$$\begin{aligned}&\frac{k+1}{m(n-k)+k+1}\binom{m(n-k)+k+1}{n-k}\\ &= \sum_{j=0}^{n-k} \frac{k}{(mj+1)(m(n-j-k)+k)}\binom{mj+1}{j}\binom{m(n-j-k)+k}{n-k-j}.\end{aligned} \tag{14}$$

If $m = 2$, then $g = F_2 = C$ and (14) becomes

$$\begin{aligned}&\frac{k+1}{2n-k+1}\binom{2n-k+1}{n-k}\\ &= \sum_{j=0}^{n-k} \frac{k}{(2j+1)(2(n-j-k)+k)}\binom{2j+1}{j}\binom{2(n-j-k)+k}{n-k-j},\end{aligned} \tag{15}$$

or equivalently,

$$C(n-k, k+1) = \sum_{j=0}^{n-k} C(j, 1)C(n-j-k, k), \tag{16}$$

where $C(m, \ell)$ are defined by (8), and $C(j, 1) = [t^j]C(t)$, the jth Catalan number.

Theorem 3. *Let $(g, f) = (d_{n,k})_{n,k\geq 0}$ be a Riordan array. Then its entries $d_{n,k}$, $n, k \geq 0$, can be represented recursively by*

$$\begin{aligned}d_{n,k} = &\sum_{i_k=1}^{n-(k-1)} f_{i_k} \sum_{i_{k-1}=1}^{n-(k-2)-i_k} f_{i_{k-1}} \sum_{i_{k-2}=1}^{n-(k-3)-i_k-i_{k-1}} f_{i_{k-2}} \cdots\\ &\sum_{i_1=1}^{n-i_k-i_{k-1}-i_2} f_{i_1} g_{n-i_1-i_2-\cdots-i_k}.\end{aligned} \tag{17}$$

Let A and B be $m \times m$ and $n \times n$ matrices, respectively. Then we define the direct sum of A and B by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}_{(m+n)\times(m+n)}. \tag{18}$$

Definition 4. Let $g \in \mathcal{F}_0$ with $g(0) = 1$ and $f \in \mathcal{F}_1$. We let $[g, f]$ denote the following array, called a quasi-Riordan array.

$$[g, f] := (g, f, tf, t^2 f, \ldots), \tag{19}$$

where g, f, tf, $t^2 f \cdots$ are the generating functions of the 0th, 1st, 2nd, 3rd, $\cdots$, columns of the matrix $[g, f]$, respectively. It is clear that $[g, f]$ can be written as

$$[g, f] = \begin{pmatrix} g(0) & 0 \\ (g - g(0))/t & (f, t) \end{pmatrix}, \tag{20}$$

where $(f, t) = (f, tf, t^2 f, t^3 f, \ldots)$. Particularly, if $f = tg$, then the quasi-Riordan array $[g, tg] = (g, t)$, an Appell-type Riordan array.

Note that $[g, f]$ defined by (19) is not the Riordan-Krylov matrix defined in [6], which relationship is worth being investigated. Paul Barry pointed out a connection between quasi-Riordan arrays and almost-Riordan arrays (cf. Barry [2] and Barry and Pantelidis [3]) in a personal communication, which is being studied by Barry, Pantelidis, and the author.

Theorem 5. *Let $(g, f) = (d_{n,k})_{n,k\geq 0}$ be a Riordan array, and let $([1] \oplus (g, f))$ and $[g, f]$ be defined by (17) and (19), respectively. Then (g, f) has the expression*

$$(g, f) = [g, f]([1] \oplus (g, f)). \tag{21}$$

Proof. We write the formal power series g and f in the Riordan array (g, f) as $g = \sum_{n\geq 0} g_n t^n$ and $f = \sum_{n\geq 1} f_n t^n$. Then

$$\begin{aligned}
&\begin{pmatrix} g(0) & 0 \\ (g - g(0))/t & (f, t) \end{pmatrix} ([1] \oplus (g, f)) \\
&= \begin{pmatrix} d_{0,0} & & & & \\ d_{1,0} & f_1 & & & \\ d_{2,0} & f_2 & f_1 & & \\ d_{3,0} & f_3 & f_2 & f_1 & \\ d_{4,0} & f_4 & f_3 & f_2 & f_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \\ 0 & d_{0,0} & & & \\ 0 & d_{1,0} & d_{1,1} & & \\ 0 & d_{2,0} & d_{2,1} & d_{2,2} & \\ 0 & d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} d_{0,0} & & & & \\ d_{1,0} & f_1 d_{0,0} & & & \\ d_{2,0} & f_2 d_{0,0} + f_1 d_{1,0} & f_1 d_{1,1} & & \\ d_{3,0} & f_3 d_{0,0} + f_2 d_{1,0} + f_1 d_{2,0} & f_2 d_{1,1} + f_1 d_{2,1} & f_1 d_{2,2} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} d_{0,0} & & & & & \\ d_{1,0} & d_{1,1} & & & & \\ d_{2,0} & d_{2,1} & d_{2,2} & & & \\ d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} & & \\ d_{4,0} & d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{22}$$

where the last step follows (12), which completes the proof of (19). □

For an integer $r \geq 0$ we let $h|_r := \sum_{n=0}^{r} h_n t^n$ denote the r-th truncations of a power series $h = \sum_{n\geq 0} h_n t^n$.

Corollary 6. *Let $(g, f) = (d_{n,k})_{n,k\geq 0}$ be a Riordan array with $g = \sum_{n\geq 0} g_n t^n$ and $f = \sum_{n\geq 1} f_n t^n$, and let $(g, f)_n$ be the nth order leading principle submatrix of (g, f). Then we have the recursive relation of $(g, f)_n$ in the following form:*

$$(g, f)_n = [g, f]_n([1] \oplus (g, f)_{n-1}), \tag{23}$$

where $[g, f]_n$ is the nth order leading principle submatrix of the quasi-Riordan array $[g, f]$ defined by (17), namely

$$[g, f]_n = \begin{pmatrix} g(0) & 0 \\ ((g - g(0))/t)|_{n-1} & (f, t)_{n-1} \end{pmatrix}, \tag{24}$$

where the $n - 1$st truncation of $(g - g(0)/t)|_{n-1}$ is $(g - g(0)/t)|_{n-1} = \sum_{k=1}^{n-1} g_k t^{k-1}$, and $(f, t)_{n-1} = (f, tf, t^2 f, \ldots, t^{n-1} f)$. We call $[g, f]_n$ the recursive matrix of the Riordan array (g, f).

Mao, Mu, and Wang [23] use (24) to give another interesting criterion for the total positivity of Riordan arrays.

In the next section, we will show that the set of all quasi-Riordan arrays forms a group, called the quasi-Riordan group.

3 The quasi-Riordan group

Let $\mathcal{R}_r$ denote the set of all quasi-Riordan arrays defined by (19). In this section, we show that $\mathcal{R}_r$ is a group with respect to regular matrix multiplication.

Proposition 7. *Let $[g, f]$, $[d, h] \in \mathcal{R}_r$, and let $u = \sum_{n\geq 0} u_n t^n \in \mathcal{F}_0$. Then the first fundamental theorem for quasi-Riordan arrays (FFTQRA) is*

$$[g, f]u = gu_0 + \frac{f}{t}(u - u_0), \tag{25}$$

which implies that

$$[g,f][d,h]=\left[g+\frac{f}{t}(d-1),\frac{fh}{t}\right]. \tag{26}$$

Hence, $[1,t]$ *is the identity of* $\mathcal{R}_r$*.*

Proof. The FFTQRA (25) can be proved as follows.

$$\begin{aligned}[g,f]u &= (g,f,tf,t^2f,\cdots)\begin{pmatrix}u_0\\u_1\\u_2\\ \vdots\end{pmatrix}\\ &= gu_0+f\sum_{n\geq 1}u_nt^{n-1}=gu_0+\frac{f}{t}(u-u_0).\end{aligned}$$

By using FFTQRA and noting $d(0)=1$ and $h(0)=0$, we have

$$\begin{aligned}[g,f][d,h] &= (g,f,tf,t^2f,\cdots)(d,h,th,t^2h,\cdots)\\ &= \left(g+\frac{f}{t}(d-1),\frac{f}{t}h,\frac{f}{t}th,\frac{f}{t}t^2h,\cdots\right),\end{aligned}$$

which implies (26).

Substituting $[g,f]=[1,t]$ and $[d,h]=[1,t]$ into (26), we obtain, respectively,

$$\begin{aligned}&[1,t][g,f]=[1+(g-1),f]=[g,f] \quad \text{and}\\ &[g,f][1,t]=\left[g+\frac{f}{t}(1-1),\frac{f}{t}t\right]=[g,f],\end{aligned}$$

which implies $[1,t]$, the identity matrix, is the identity of $\mathcal{R}_r$. □

Theorem 8. *The set of all quasi-Riordan arrays* $\mathcal{R}_r$ *is a group, called the quasi-Riordan group, with respect to the multiplication represented in* (26)*.*

Proof. From (26) of Proposition 7, $\mathcal{R}_r$ is closed with respect to the multiplication. (26) also shows $[1,t]$ is the identity of $\mathcal{R}_r$. For any $[g,f]\in\mathcal{R}_r$, its inverse is

$$[g,f]^{-1}=\left[1+\frac{t}{f}(1-g),\frac{t^2}{f}\right], \tag{27}$$

since

$$\left[1+\frac{t}{f}(1-g),\frac{t^2}{f}\right][g,f]=\left[1+\frac{t}{f}(1-g)+\frac{t^2/f}{t}(g-1),\frac{t^2}{f}\frac{f}{t}\right]=[1,t]. \tag{28}$$

Finally, the associative law is satisfied for any $[g,f]$, $[d,h]$, and $[u,v]\in\mathcal{R}_r$ because

$$([g,f][d,h])[u,v]=\left[g+\frac{f}{t}(d-1),\frac{fh}{t}\right][u,v]$$

$$= \left[g + \frac{f}{t}(d-1) + \frac{fh}{t^2}(u-1), \frac{fhv}{t^2}\right]$$

and

$$\begin{aligned}
[g,f]([d,h])[u,v] &= [g,f]\left[d + \frac{h}{t}(u-1), \frac{hv}{t}\right] \\
&= \left[g + \frac{f}{t}\left(d + \frac{h}{t}(u-1) - 1\right), \frac{fhv}{t^2}\right] \\
&= \left[g + \frac{f}{t}(d-1) + \frac{fh}{t^2}(u-1), \frac{fhv}{t^2}\right]
\end{aligned}$$

shows $([g,f][d,h])[u,v] = [g,f]([d,h][u,v])$. □

Example 9. Consider the quasi-Riordan array $[1/(1-t), t/(1-t)]$, which is an Appell-type Riordan array $(1/(1-t), t)$ with its inverse $(1-t, t)$. From (27) its inverse in $\mathcal{R}_r$ is

$$\begin{aligned}
&\left[\frac{1}{1-t}, \frac{t}{1-t}\right]^{-1} = \left[1 + \frac{t}{f}(1-g), \frac{t^2}{f}\right] \\
&= \left[1 + \frac{t}{t/(1-t)}\left(1 - \frac{1}{1-t}\right), \frac{t^2}{t/(1-t)}\right] = [1-t, t(1-t)] \\
&= (1-t, t) = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & 1 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\end{aligned}$$

Theorem 10. *A quasi-Riordan array $[g,f]$ is a Riordan array if and only if $f = tg$, i.e., when $[g,f]$ is an Appell-type Riordan array. Hence, $\mathcal{A} = \{[g,tg] : g \in \mathcal{F}_0, g(0) = 1\}$ is a subgroup of $\mathcal{R}_r$, which is called the Appell quasi-Riordan subgroup.*

Proof. Let $[g,tg]$, $[d,td] \in \mathcal{R}_r$. Then

$$[g,tg][d,td] = \left[g + \frac{tg}{t}(d-1), \frac{t^2gd}{t}\right] = [gd, tgd].$$

$\mathcal{A}$ is closed under multiplication. In addition, the inverse of $[g,f]$

$$[g,tg]^{-1} = \left[1 + \frac{t}{tg}(1-g), \frac{t^2}{tg}\right] = \left[\frac{1}{g}, \frac{t}{g}\right]$$

is also in $\mathcal{A}$. □

Theorem 11. *Let $\mathcal{R}_r$ be the quasi-Riordan group. Then every conjugate of an element $[g, f] \in \mathcal{R}_r$ is in the set $\mathcal{R}(f)_r := \{[d, f] : d \in \mathcal{F}_0, d(0) = 1\}$. Hence, $\mathcal{A} = \{[g, tg] : g \in \mathcal{F}_0, g(0) = 1\}$ is a normal subgroup of $\mathcal{R}_r$.*

Proof. Let $[d, h] \in \mathcal{R}_r$. Then for an arbitrarily fixed $[g, f] \in \mathcal{R}_r$, we have

$$
\begin{aligned}
[d,h][g,f][d,h]^{-1} &= [d,h][g,f]\left[1+\frac{t}{h}(1-d), \frac{t^2}{h}\right] \\
&= \left[d+\frac{h}{t}(g-1)-\frac{f}{t}(d-1), f\right] \in \mathcal{R}(f)_r.
\end{aligned}
$$

In particular, if $f = tg$ in $[g, f]$, then

$$
[d,h][g,tg][d,h]^{-1} = \left[d+\frac{h}{t}(g-1)-g(d-1), tg\right],
$$

which implies $\mathcal{A} = \{[g, tg] : g \in \mathcal{F}_0, g(0) = 1\}$ is a normal subgroup of $\mathcal{R}_r$. □

4 A vertical recursive relation view of (c)- and (C)-Riordan arrays

In this section, we extend the vertical recursive relation to (c) Riordan arrays. We start from the following definition of the (c)-class of Riordan arrays.

Definition 12. Let $(g, f) = (d_{n,k})_{n,k\geq 0}$ be a Riordan array, and let $c = (c_0, c_1, c_2, \ldots)$ be a sequence satisfying $c_0 = 1$ and $c_k \neq 0$ for all $k = 1, 2, \ldots$. We define

$$
(g,f)_c = \left(\frac{c_n}{c_k} d_{n,k}\right)_{n,k\geq 0}. \tag{29}
$$

Then the set

$$
\{\mathcal{R}\}_c := \{(g,f)_c : (g,f) = (d_{n,k})_{n,k\geq 0} \in \mathcal{R}\} \tag{30}
$$

is called the (c)-class of $\mathcal{R}$ or the set of the (c)-Riordan arrays. Since we may change c_n and c_k respectively to $1/c_n$ and $1/c_k$,

$$
\left(\frac{c_k}{c_n} d_{n,k}\right)_{n,k\geq 0} = \left(\frac{1/c_n}{1/c_k} d_{n,k}\right)_{n,k\geq 0}
$$

is in the $(1/c)$-class of $\mathcal{R}$, where $(1/c) = (1/c_0, 1/c_1, 1/c_2, \ldots)$.

Let $(g, f) = (d_{n,k})_{n,k\geq 0}$ be a Riordan array, and let $C = (c_{n,k})_{n,k\geq 0}$ be a lower triangular matrix satisfying $c_{n,0} = 1$ and $c_{n,k} \neq 0$ for all $0 \leq k \leq n$ and $c_{n,k} = 0$ for all $k > n$. We define

$$(g,f)_C = \left(\frac{c_{n,n}}{c_{n,k}} d_{n,k}\right)_{n,k\geq 0}. \tag{31}$$

The set

$$\{\mathcal{R}\}_C := \{(g,f)_C : (g,f) = (d_{n,k})_{n,k\geq 0} \in \mathcal{R}\} \tag{32}$$

is called the (C)-class of $\mathcal{R}$, or the set of the (C)-Riordan arrays. Similarly, $\left(\frac{c_{n,k}}{c_{n,n}} d_{n,k}\right)_{n,k\geq 0}$ is in the $(1/C)$-class of $\mathcal{R}$, where $(1/C) = (1/c_{n,k})_{n,k\geq 0}$.

Wang and Wang [35] defined the (c)-Riordan arrays by using the generalized formal power series. Gould and the author [10] claimed that for a sequence (c) the (c)-class of $\mathcal{R}$, or the set of the (c)-Riordan arrays $\{\mathcal{R}\}_c$, forms a group. More precisely, we have the following theorem.

Theorem 13. *[10] Let $c = (c_0, c_1, c_2, \ldots)$ be a sequence satisfying $c_0 = 1$ and $c_k \neq 0$ for all $k = 1, 2, \ldots,$ and let $\{\mathcal{R}\}_c$ be the (c)-class defined by (30). Then $\{\mathcal{R}\}_c$ forms a group with respect to the regular matrix multiplication. We let $\mathcal{R}_c$ denote this group and called the (c)-Riordan group with respect to the sequence (c).*

Proof. If $(g_1, f_1)_c, (g_2, f_2)_c \in \{(g,f)\}_c$, and define $(g_1, f_1) = (d_{n,k})_{n,k\geq 0}$, $(g_2, f_2) = (e_{n,k})_{n,k\geq 0}$, and $(g_1, f_1)(g_2, f_2) = (g_1 g_2(f_1), f_2(f_1)) = (h_{n,k})_{n,k\geq 0}$, then $(d_{n,k})_{n,k\geq 0}(e_{n,k})_{n,k\geq 0} = (h_{n,k})_{n,k\geq 0}$, and

$$\begin{aligned}
(g_1, f_1)_c (g_2, f_2)_c &= \left(\sum_{j=0}^{\min n,k} \left(\frac{c_n}{c_j} d_{n,j}\right)\left(\frac{c_j}{c_k} e_{j,k}\right)\right)_{n,k\geq 0} \\
&= \left(\frac{c_n}{c_k} \sum_{j=0}^{\min n,k} (d_{n,j} e_{j,k})\right)_{n,k} \\
&= \left(\frac{c_n}{c_k} h_{n,k}\right)_{n,k\geq 0} = (g_1 g_2(f_1), f_2(f_1))_c.
\end{aligned}$$

Hence, we may find $(1.t)_c$ is the identity of $\mathcal{R}_c$ and the inverse of an element $(g,f)_c \in \mathcal{R}_c$ is $(1/g(\overline{f}), \overline{f})$, where $\overline{f}$ is the compositional inverse of f. Finally, we have

$$\begin{aligned}
((g,f)_c (d,h)_c)\,(u,v)_c &= (gd(f)u(h(v)), v(h(f))) \\
&= (g,f)_c\,((d,h)_c (u,v)_c)\,,
\end{aligned}$$

completing the proof. $\square$

More material on (c)-Riordan arrays can be found in [14, 30].

Example 14. Let $(g,f)=(1/(1-t),t/(1-t))$. Then $(1/(1-\ell t),t/(1-\ell t))=(1/(1-t),t/(1-t))_c$ with $c=(c_n)_{n\geq 0}=(\ell^n)_{n\geq 0}$, where $\ell\in\mathbb{K}$. The (c)-Riordan array $(1/(1-\ell t),t/(1-\ell t))_c$ with $c=(c_n)_{n\geq 0}=(\ell^n)_{n\geq 0}$ begins

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \ell & 1 & 0 & 0 & 0 & 0 & 0 \\ \ell^2 & \ell^2 & 1 & 0 & 0 & 0 & 0 \\ \ell^3 & \ell^2+\ell^3 & \ell^2+\ell & 1 & 0 & 0 & 0 \\ \ell^4 & 2\ell^3+\ell^4 & 2\ell^2+2\ell^3 & \ell^2+2\ell & 1 & 0 & 0 \\ \ell^5 & 3\ell^4+\ell^5 & 4\ell^3+3\ell^4 & 4\ell^2+3\ell^3 & \ell^2+3\ell & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 15. Let $(g,f)=(1/(1-t),t/(1-t))=(\binom{n}{k})_{n,k\geq 0}$, and let $c=(n!)_{n\geq 0}$. Then

$$(r_{n,k})_{n,k\geq 0}=\left(\frac{n!}{k!}\binom{n}{k}\right)_{n,k\geq 0}=\left((n-k)!\left(\binom{n}{k}\right)^2\right)\in\{(1/(1-t),t/(1-t))\}_c, \tag{33}$$

where $c=(c_n)_{n\geq 0}=(n!)_{n\geq 0}$. The (c)-Riordan array $(r_{n,k})_{n,k\geq 0}$ begins

$$(r_{n,k})_{n,k\geq 0}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 \\ 6 & 18 & 9 & 1 & 0 & 0 & 0 \\ 24 & 96 & 72 & 16 & 1 & 0 & 0 \\ 120 & 600 & 600 & 200 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The matrix $(r_{n,k})_{n,k\geq 0}$ is called a rook matrix. The polynomial $r_n(x)=\sum_{k=0}^n r(n,k)x^{n-k}$ is called the rook polynomial of nth order (cf. Fielder [8] and Riordan [27, §7.2, 7.3, and 7.4]).

In general, consider a board that represents a full or a part chess board. Let m be the number of squares present in the board. Two pawns or rooks placed on a board are said to be in non capturing positions if they are not in same row or same column. For $2\leq k\leq m$, let r_k denote the number of ways in which k rooks can be placed on a board such that no two rooks capture each other. Then the polynomial $1+r_1x+r_2x^2+\cdots+r_mx^m$ is called the (general) rook polynomial for the board considered. If the board is denoted by C, then the corresponding polynomial is denoted by $r(C,x)$. The rook polynomials are defined for $m\geq 2$. If $m=1$, then the board contains only one square so that $r_k=0$ for $k\geq 2$, and $r(C,x)=1+x$. Hence, the coefficient $r_{n,k}$ of the roof polynomial $r_n(x)=\sum_{k=0}^n r(n,k)x^{n-k}$ is referred to as the number of ways in which k rooks can be placed on a $n\times n$ full chess board such that no two rooks capture each other. The rook polynomials for 1×1, 2×2, 3×3, 4×4, and 5×5 full boards are, respectively,

$$r_1(x)=x+1,$$

$$r_2(x) = 2x^2 + 4x^1 + 1,$$
$$r_3(x) = 6x^3 + 18x^2 + 9x + 1,$$
$$r_4(x) = 24x^4 + 96x^3 + 72x^2 + 16x + 1,$$
$$r_5(x) = 120x^5 + 600x^4 + 600x^3 + 200x^2 + 5x + 1.$$

In a given board C, suppose we choose a particular square and let $(*)$ denote the square. Let D denote the board obtained from C by deleting the row and column containing the square $(*)$, and let E be the board obtained from C by deleting only the square $(*)$. Then the rook polynomial for the board C is given by $r(C,x) = xr(D,x) + r(E,x)$. This is known as expansion formula for $r(C,x)$.

We let C_{n+1} denote the $(n+1)\times(n+1)$ full board. We suppose $n\times n$ full board C_n is located at the left upper corner of C_{n+1}, and let E_n denote the board obtained from the board C_{n+1} by deleting only the square at the right lower corner of C_{n+1}. Then, we have

$$\begin{aligned} r(E_n,x) &= r(C_{n+1},x) - xr(C_n,x) = r_{n+1}(x) - xr_n(x) \\ &= \sum_{k=0}^{n+1} \frac{(n+1)!}{k!}\binom{n+1}{k} x^{n+1-k} - x\sum_{k=0}^{n} \frac{n!}{k!}\binom{n}{k} x^{n-k} \\ &= \sum_{k=0}^{n+1} \frac{n!}{k!}\left((n+1)\binom{n+1}{k} - \binom{n}{k}\right) x^{n+1-k} \\ &= \sum_{k=0}^{n} \frac{n!}{k!}\binom{n}{k}\left(\frac{(n+1)^2}{n-k+1} - 1\right) x^{n+1-k} + 1 \\ &= \sum_{k=0}^{n} \frac{n!}{k!}\frac{n^2+n+k}{n-k+1}\binom{n}{k} x^{n+1-k} + 1 \\ &= \sum_{k=0}^{n} \frac{n^2+n+k}{n-k+1}(n-k)!\left(\binom{n}{k}\right)^2 x^{n+1-k} + 1. \end{aligned} \tag{34}$$

We define $r(E_n,x) = \sum_{k=0}^{n+1} E_{n,k}x^{n+1-k}$ and call it the remainder polynomial, where

$$E_{n,k} = \frac{n^2+n+k}{n-k+1}(n-k)!\left(\binom{n}{k}\right)^2 \tag{35}$$

for $0 \le k \le n$ and $E_{n,n+1} = 1$. Then, the lower triangle matrix $(E_{n,k})_{n,k\ge 0}$, called the remainder triangle, begins

$$(E_{n,k})_{n,k\ge 0} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & 14 & 8 & 1 & 0 & 0 & 0 \\ 18 & 78 & 63 & 15 & 1 & 0 & 0 \\ 96 & 504 & 528 & 184 & 4 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Here is a recursive relation related to the rook triangle $(r_{n,k})_{n,k\geq 0}$ and the triangle $(E_{n,k})_{n,k\geq 0}$.

Proposition 16. *Let $(r_{n,k})_{n,k\geq 0}$ and $(E_{n,k})_{n,k\geq 0}$ be the rook triangles and the remainder triangle defined in Example 15, and let $r_n(x)=\sum_{k=0}^{n} r(n,k)x^{n-k}$ and $r(E_n,x)=\sum_{k=0}^{n} E_{n,k}x^{n-k}$ be the rook polynomial and the remainder polynomial, respectively. Then, there exists the recursive relation*

$$r_{n+1}(x)=xr_n(x)+r(E_n,x) \tag{36}$$

for $n\geq 0$, which can be expressed as a matrix form

$$\overline{(r_{n,k})}=(r_{n,k})+(E_{n,k}), \tag{37}$$

where $n\geq 0$ and $\overline{(r_{n,k})}$ is obtained from $(r_{n,k})$ by deleting its first row. Furthermore, we have the expansion formula for $r_{n+1}(x)$ as

$$r_{n+1}(x)=\sum_{k=0}^{n} x^{n-k}r(E_k,x)+x^{n+1}. \tag{38}$$

Proof. Equation (36) comes from the expansion formula. Substituting $r_n(x)=\sum_{k=0}^{n} r(n,k)x^{n-k}$ and $r(E_n,x)=\sum_{k=0}^{n} E_{n,k}x^{n-k}$ into (36) and comparing the coefficients of the same powers of x on the both sides yields

$$r_{n+1,k}=r_{n,k}+E_{n,k},$$

which can be combined as the matrix form (37) for $n,k\geq 0$. From (36) we have

$$\begin{aligned} r_{n+1}(x)&=xr_n(x)+r(E_n,x)\\ xr_n(x)&=x^2r_{n-1}(x)+xr(E_{n-1},x)\\ x^2r_{n-1}(x)&=x^3r_{n-2}(x)+x^2r(E_{n-2},x)\\ &\ \vdots\\ x^nr_1(x)&=x^{n+1}r_0(x)+x^nr(E_0,x). \end{aligned}$$

Adding up the above system and cancelling the same terms from the both sides yields (38). □

Example 17. Let $(g,f)=(1/(1-t),t/(1-t))=(\binom{n}{k})_{n,k\geq 0}$. Then

$$\begin{aligned} (L_{n,k})_{n,k\geq 0}&=\left(\frac{(-1)^n/(n)_n}{(-1)^k/(n)_k}\binom{n}{k}\right)_{n,k\geq 0}\\ &=\left(\frac{(-1)^{n-k}}{(n-k)!}\binom{n}{k}\right)\in\{(1/(1-t),t/(1-t))\}_C, \end{aligned} \tag{39}$$

where $C=(c_{n,k})_{n,k\geq 0}=((-1)^n/(n)_k)_{n,k\geq 0}$. The (C)-Riordan array $(L_{n,k})_{n,k\geq 0}$ begins

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -2 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{3}{2} & -3 & 1 & 0 & 0 & 0 \\ \frac{1}{24} & -\frac{2}{3} & 3 & -4 & 1 & 0 & 0 \\ -\frac{1}{120} & \frac{5}{24} & -\frac{5}{3} & 5 & -5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The matrix $(L_{n,k})_{n,k\geq 0}$ is called the Laguerre matrix. The polynomial $L_n(n,k)=\sum_{k=0}^n L_{n,k}x^{n-k}$ is called the Laguerre polynomial of order n (cf., for example, Hsu, Shiue, and the author [16] and Riordan [27, pp. 164–170]).

Comparing (33) and (39) and noting that

$$r_{n,k}=\frac{n!}{k!}\binom{n}{k},$$

we immediately obtain

$$r_{n,n-k}=\frac{n!}{(n-k)!}\binom{n}{k}=\frac{n!}{(n-k)!}(-1)^{n-k}(n-k)!L_{n,k}=(-1)^{n-k}n!L_{n,k}, \tag{40}$$

which implies

$$\sum_{k=0}^{n} r_{n,n-k}x^{n-k}=n!\sum_{k=0}^{n} L_{n,k}(-x)^{n-k}=n!L_n(-x) \tag{41}$$

for $0\leq k\leq n$. The left-hand side of (41) can be written as

$$\sum_{k=0}^{n} r_{n,n-k}x^{n-k}=\sum_{k=0}^{n} r_{n,k}x^k=x^n\sum_{k=0}^{n} r_{n,k}\left(\frac{1}{x}\right)^{n-k}=x^n r_n\left(\frac{1}{x}\right).$$

Hence, we have the well-known formula

$$r_n(x)=n!x^nL_n\left(-\frac{1}{x}\right)$$

for $n\geq 0$.

Definition 18. We may extend the rook triangle and the rook polynomials to a general case by defining the following generalized rook triangle and generalized rook polynomials:

$$(\hat{r}_{n,k})_{n,k\geq 0}=\left(\frac{n!}{k!}d_{n,k}\right)_{n,k\geq 0}=\left((n-k)!\binom{n}{k}d_{n,k}\right)\in\{(g(t),f(t)\}_c, \tag{42}$$

where $c = (c_n)_{n\geq 0} = (n!)_{n\geq 0}$ and $(g, f) = (d_{n,k})_{n,k\geq 0}$. In addition, the polynomial

$$\hat{r}_n(x) = \sum_{k=0}^{n} \hat{r}_{n,k} x^{n-k} \tag{43}$$

is referred to as the *generalized rook polynomial* of degree n.

We may also extend the Laguerre triangle and the Laguerre polynomials to a general case by defining the following generalized Riordan type Laguerre triangle and generalized Riordan type Laguerre polynomials:

$$\begin{aligned}(\hat{L}_{n,k})_{n,k\geq 0} &= \left(\frac{(-1)^n/(n)_n}{(-1)^k/(n)_k} d_{n,k}\right)_{n,k\geq 0} \\ &= \left(\frac{(-1)^{n-k}}{(n-k)!}\binom{n}{k} d_{n,k}\right) \in \{(g(t), f(t)\}_{\hat{C}}, \end{aligned}\tag{44}$$

where $\hat{C} = (c_{n,k})_{n,k\geq 0} = (n!)_{n\geq 0}$ and $(g, f) = (d_{n,k})_{n,k\geq 0}$. In addition, the polynomial

$$\hat{L}_n(x) = \sum_{k=0}^{n} \hat{L}_{n,k} x^{n-k} \tag{45}$$

is referred to as the generalized Laguerre polynomial of degree n.

Similar to the relationship between the rook triangle and the Laguerre triangle and the relationship between the rook polynomials and the Laguerre polynomials, we may extend the relationships to the general cases as follows.

Theorem 19. *Let the generalized rook triangle and the rook polynomials and the generalized Riordan type Laguerre triangle and the generalized Riordan type Laguerre polynomials be defined in Definition 18. Then we have*

$$\hat{r}_{n,n-k} = (-1)^{n-k} n! \hat{L}_{n,k} \tag{46}$$

for $0 \leq k \leq n$ *and*

$$\hat{r}_n(x) = n! x^n \hat{L}_n\left(-\frac{1}{x}\right)$$

for $n \geq 0$.

From Examples 15 and 47, we have a relationship between

Theorem 20. *Let* $(g, f) = (d_{n,k})_{n,k\geq 0}$ *be a Riordan array, and let* $(g, f)_c$ *and* $(g, f)_{\hat{C}}$ *be defined in Definition 12. Then* $(g, f)_c = (d^{(c)}_{n,k})_{n,k\geq 0}$ *and* $(g, f)_{\hat{C}} = d^{(\hat{C})}_{n,k\geq 0}$ *satisfy the horizontal recursive relations, which are extensions of* A*- and* Z*-sequence representation (4) and (6) of* (g, f) *to* $(g, f)_c$ *and* $(g, f)_{\hat{C}}$*, respectively:*

$$d^{(c)}_{n+1,k+1} = \frac{c_{n+1}}{c_n c_{k+1}} \sum_{j=0}^{n-k} a_j c_{k+j} d^{(c)}_{n,k+j} \tag{47}$$

for $k \geq 0$ and

$$d^{(c)}_{n+1,0} = \frac{c_{n+1}}{c_n} \sum_{j=0}^{n} z_j c_j d^{(c)}_{n,j} \tag{48}$$

as well as

$$d^{(\hat{C})}_{n+1,k+1} = \frac{c_{n+1,n+1}}{c_{n,n} c_{n+1,k+1}} \sum_{j=0}^{n-k} a_j c_{n,k+j} d^{(\hat{C})}_{n,k+j} \tag{49}$$

for $k \geq 0$ and

$$d^{(\hat{C})}_{n+1,0} = \frac{c_{n+1,n+1}}{c_{n,n}} \sum_{j=0}^{n} z_j c_{n,j} d^{(\hat{C})}_{n,j}. \tag{50}$$

Theorem 21. *Let $(g, f) = (d_{n,k})_{n,k\geq 0}$ be a Riordan array, and let $(g, f)_c$ and $(g, f)_{\hat{C}}$ be defined in Definition 12. Then $(g, f)_c = (d^{(c)}_{n,k})_{n,k\geq 0}$ and $(g, f)_{\hat{C}} = d^{(\hat{C})}_{n,k\geq 0}$ satisfy the vertical recursive relations, which are extensions of vertical recursive relation (12) of the entries of (g, f) to $(g, f)_c$ and $(g, f)_{\hat{C}}$, namely,*

$$d^{(c)}_{n,k} = \frac{c_n}{c_k} \sum_{j=1}^{n-k+1} f_j \frac{c_{k-1}}{c_{n-j}} d^{(c)}_{n-j,k-1} \tag{51}$$

and

$$d^{(\hat{C})}_{n,k} = \frac{c_{n,n}}{c_{n,k}} \sum_{j=1}^{n-k+1} f_j \frac{c_{n-j,k-1}}{c_{n-j,n-j}} d^{(\hat{C})}_{n-j,k-1}, \tag{52}$$

respectively.

Example 22. It is known that the A- and Z- sequences of $(1/(1-t), t/(1-t))$ are $(1, 1, 0, \ldots)$ and $(1, 0, 0, \ldots)$, respectively. Then for the rook triangle represented in Example 15 with $d_{n,k} = \binom{n}{k}$, from (47) and (48), we obtain the horizontal recursive relations for the entries of the rook triangle,

$$r_{n,k} = n r_{n-1,k} + \frac{n}{k} r_{n-1,k-1} \tag{53}$$

for $k \geq 1$ and

$$r_{n,0} = n r_{n-1,0}. \tag{54}$$

From (51) and noting (13) we obtain the vertical recursive relation for the entries of the rook triangle

$$r_{n,k} = \sum_{j=1}^{n-k+1} \frac{(n)_j}{k} r_{n-j,k-1} \tag{55}$$

for $k \geq 1$.

From (33), one may obtain an identity equivalent to (53); namely,

$$\binom{n}{k}^2 = \frac{n}{n-k} \binom{n-1}{k}^2 + \frac{n}{k} \binom{n-1}{k-1}^2.$$

Similarly, from (55), we get

$$\binom{n}{k}^2 = \sum_{j=1}^{n-k+1} \frac{(n)_j}{k(n-k)_{j-1}} \binom{n-j}{k-1}^2.$$

Example 23. Similarly, for the Laguerre triangle represented in Example 17 with $d_{n,k} = \binom{n}{k}$, from (47) and (48), we obtain the horizontal recursive relations for the entries of the Laguerre triangle; namely,

$$L_{n,k} = L_{n-1,k-1} - \frac{1}{n-k} L_{n-1,k} \tag{56}$$

for $k \geq 1$ and

$$L_{n,0} = -\frac{1}{n} L_{n-1,0}. \tag{57}$$

From (51) and noting (13) we obtain the vertical recursive relation for the entries of the Laguerre triangle; namely,

$$L_{n,k} = \frac{1}{(n-k)!} \sum_{j=1}^{n-k+1} (-1)^{j-1}(n-k-j+1)! L_{n-j,k-1} \tag{58}$$

for $k \geq 1$. It can be seen that (58) is equivalent to (13).

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