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Mathieu-Fibonacci Series

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Abstract

Using reciprocal sums of powers of Fibonacci and Lucas numbers, we present series representations for generalized Mathieu series via Lambert-type series. An application of the Laplace-type integral for Dirichlet series yields some integral representations of reciprocal sums of the Fibonacci numbers and generalized Mathieu series. Finally, we analyze the asymptotic behavior of Mathieu-Fibonacci series by the Mellin transform method.

1 Introduction

The Mathieu series is a functional series, introduced by Émile Léonard Mathieu (1835–1890), for the purpose of his research on the elasticity of solid bodies. This series, its generalizations and their alternating variants have been studied by many authors. The recent book by Tomovski et al. [22] presents many of these results, including integral representations, inequalities and asymptotic expansions. Srivastava and Tomovski [20] introduced the generalized Mathieu series

$$S_\mu^{(\alpha,\beta)}(r;\mathbf{a}) = S_\mu^{(\alpha,\beta)}(r;(a_n)_{n=1}^\infty) = \sum_{n=1}^\infty \frac{2a_n^\beta}{(a_n^\alpha + r^2)^\mu}, \quad r,\alpha,\beta,\mu \in \mathbb{R}^+, \tag{1}$$

where it is tacitly assumed that the monotone–increasing, divergent sequence of positive real numbers

$$\mathbf{a} = (a_n)_{n=1}^\infty \qquad (\lim_{n\to\infty} a_n = \infty)$$

is chosen such that the infinite series (1) converges, that is, the auxiliary series $\sum_{n=1}^\infty \frac{1}{a_n^{\mu\alpha-\beta}}$ is convergent. We will also find it useful to consider the function $a_x = a(x),\ x \in \mathbb{N}$, and its inverse $a^{-1}(x)$. The alternating variant $\tilde{S}_\mu^{(\alpha,\beta)}(r;\mathbf{a})$ of (1) is defined by [21]

$$\tilde{S}_\mu^{(\alpha,\beta)}(r;\mathbf{a}) = \tilde{S}_\mu^{(\alpha,\beta)}(r;(a_n)_{n=1}^\infty) = \sum_{n=1}^\infty (-1)^{n-1} \frac{2a_n^\beta}{(a_n^\alpha + r^2)^\mu}, \quad r,\alpha,\beta,\mu \in \mathbb{R}^+, \tag{2}$$

if the auxiliary alternating series $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{a_n^{\mu\alpha-\beta}}$ is convergent. From (1) and (2), Srivastava and Tomovski [20] found the series representations

$$S_\mu^{(\alpha,\beta)}(r;\mathbf{a}) = 2\sum_{m=0}^\infty \binom{\mu+m-1}{m}(-r^2)^m \sum_{n=1}^\infty \frac{1}{a_n^{(\mu+m)\alpha-\beta}}, \quad 0<r<1, \tag{3}$$

and

$$\tilde{S}_\mu^{(\alpha,\beta)}(r;\mathbf{a}) = 2\sum_{m=0}^\infty \binom{\mu+m-1}{m}(-r^2)^m \sum_{n=1}^\infty \frac{(-1)^{n-1}}{a_n^{(\mu+m)\alpha-\beta}}, \quad 0<r<1.$$

Integral and series representations for various selections of the sequence $(a_n)_{n=1}^\infty$ were derived recently, for example, for $(n^\gamma)_{n=1}^\infty$, $((n!)^\gamma)_{n=1}^\infty$, $((\log n!)^\gamma)_{n=1}^\infty$, etc. [22]. One of the goals of the present article is to give series representations of generalized Mathieu series associated with Fibonacci numbers and related sequences. We apply some results on series of reciprocal Fibonacci numbers, a subject with a long history [2, 6, 13, 17, 24]. Moreover, we show how our generalized Mathieu series can be analyzed asymptotically for $r \to \infty$. This gives rise to an interesting numerical phenomenon: Fourier series with extremely small coefficients occur in the expansions, which cause oscillations that are numerically very hard to see by computing the generalized Mathieu series.

2 Series of reciprocal Fibonacci and Lucas numbers

Before stating our first result, we recall some basic facts about Fibonacci and Lucas numbers (see Grimaldi [11] for more information). Both of these number sequences satisfy the second-order recurrence relation $G_{n+1} = G_n + G_{n-1}$, but they have different initial terms. Fibonacci numbers F_n start with $F_0 = 0$ and $F_1 = 1$; Lucas numbers L_n have the initial values $L_0 = 2$ and $L_1 = 1$. With $\varphi = \frac{1+\sqrt{5}}{2}$, the explicit Binet forms of Fibonacci and Lucas numbers are given by

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \frac{(-1)^n}{\varphi^n}\right), \tag{4}$$
$$L_n = \varphi^n + \frac{(-1)^n}{\varphi^n}.$$

Zucker [24] proved that for any integer $s \geq 1$ the reciprocal sums

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^s}, \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^s}, \qquad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^{2s}}, \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^{2s}},$$

among others, can be expressed as rational functions of Jacobi's theta functions with rational coefficients. Let $q = \frac{1}{\varphi^2}$, where $\varphi^2 = \frac{3+\sqrt{5}}{2}$. Zucker used the fact that the above series can be expressed as Lambert-type series, for any complex s with $\mathrm{Re}(s) > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^s} = (\sqrt{5})^s \sum_{n=1}^{\infty} \frac{q^{ns}}{(1-q^{2n})^s}, \tag{5}$$
$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^s} = \sum_{n=1}^{\infty} \frac{q^{ns}}{(1+q^{2n})^s},$$
$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^{2s}} = 5^s \sum_{n=1}^{\infty} \frac{q^{(2n-1)s}}{(1+q^{2n-1})^{2s}},$$
$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^{2s}} = \sum_{n=1}^{\infty} \frac{q^{(2n-1)s}}{(1-q^{2n-1})^{2s}}.$$

For $s = 1$, the Lambert-type series for (5) was already given by Landau [13].

3 Series representations

In this section we will give series representation for $S_\mu^{(\alpha,\beta)}(r; (F_{2n})_{n=1}^\infty)$, $S_\mu^{(\alpha,\beta)}(r; (L_{2n})_{n=1}^\infty)$, $S_\mu^{(\alpha,\beta)}(r; (F_{2n-1}^2)_{n=1}^\infty)$, and $S_\mu^{(\alpha,\beta)}(r; (L_{2n-1}^2)_{n=1}^\infty)$, by using formula (3). We have

$$S_\mu^{(\alpha,\beta)}(r; (F_{2n})_{n=1}^\infty) = 2\sum_{m=0}^{\infty} \binom{\mu+m-1}{m} (-r^2)^m \sum_{n=1}^{\infty} \frac{1}{F_{2n}^{(\mu+m)\alpha-\beta}}, \tag{6}$$

where μ, α, β are so chosen that $\mu\alpha - \beta > 0$. Then, by (5), we get

$$S_\mu^{(\alpha,\beta)}(r; (F_{2n})_{n=1}^\infty) = 2\sum_{m=0}^{\infty} \binom{\mu+m-1}{m} (-r^2)^m (\sqrt{5})^{(\mu+m)\alpha-\beta} \sum_{n=1}^{\infty} \frac{q^{n((\mu+m)\alpha-\beta)}}{(1-q^{2n})^{(\mu+m)\alpha-\beta}}. \tag{7}$$

Analogously,

$$S_{\mu}^{(\alpha,\beta)}(r;(L_{2n})_{n=1}^{\infty}) = 2\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2)^m\sum_{n=1}^{\infty}\frac{1}{L_{2n}^{(\mu+m)\alpha-\beta}}$$
$$= 2\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2)^m\sum_{n=1}^{\infty}\frac{q^{n((\mu+m)\alpha-\beta)}}{(1+q^{2n})^{(\mu+m)\alpha-\beta}}, \tag{8}$$

and

$$S_{\mu}^{(\alpha,\beta)}(r;(F_{2n-1}^2)_{n=1}^{\infty}) = 2\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2)^m 5^{(\mu+m)\alpha-\beta}\sum_{n=1}^{\infty}\frac{q^{(2n-1)((\mu+m)\alpha-\beta)}}{(1+q^{2n-1})^{2((\mu+m)\alpha-\beta)}},$$

as well as

$$S_{\mu}^{(\alpha,\beta)}(r;(L_{2n-1}^2)_{n=1}^{\infty}) = 2\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2)^m\sum_{n=1}^{\infty}\frac{q^{(2n-1)((\mu+m)\alpha-\beta)}}{(1-q^{2n-1})^{2((\mu+m)\alpha-\beta)}}.$$

Example 1. Setting $\mu = 2, \alpha = 1, \beta = 1$, and $r^2 = \frac{1}{\sqrt{5}}$ in (7), we get

$$S_2^{(1,1)}\left(\frac{1}{\sqrt{5}};(F_{2n})_{n=1}^{\infty}\right) = 2\sqrt{5}\sum_{m=1}^{\infty} m(-1)^{m-1}L_m(q),$$

where

$$L_m(q) = \sum_{n=1}^{\infty}\frac{q^{mn}}{(1-q^{2n})^m}$$

is a sequence of Lambert series. For example, $L_1(q) = L(q) - L(q^2)$, where

$$L(q) = \sum_{n=1}^{\infty}\frac{q^n}{1-q^n} = \frac{\Psi_q(1)+\log(1-q)}{\log q}. \tag{9}$$

Here $\Psi_q(z)$ is the q-digamma function, defined as

$$\Psi_q(z) = \frac{1}{\Gamma_q(z)}\frac{\partial\Gamma_q(z)}{\partial z},$$

where $\Gamma_q(z)$ is the q-gamma function. We refer to Andrews at al. [3, Section 10.3]; the well-known identity (9) follows immediately from the definition of the q-gamma function. The second member of the sequence $L_m(q)$ is

$$L_2(q) = \sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} = \sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} = \sum_{n=1}^{\infty}\xi_1(n)q^{2n},$$

where $\xi_1(n) = \sum_{d|n} d$ is the sum-of-divisors function (see Agarwal [1]). Similarly,

$$S_2^{(1,1)}\left(\frac{1}{5}; (F_{2n-1}^2)_{n=1}^{\infty}\right) = 10\sum_{m=1}^{\infty} m(-1)^{m-1}\tilde{L}_m(q),$$

where

$$\tilde{L}_m(q) = \sum_{n=1}^{\infty} \frac{q^{(2n-1)m}}{(1+q^{2n-1})^{2m}}$$

is a sequence of Lambert-type series. In the following, we use the notation of Whittaker and Watson [23] for Jacobi theta functions. The identities we use are found in Whittaker and Watson [23, pp. 471, 489] or in Borwein and Borwein [5, Section 3.7]. For example,

$$\tilde{L}_1(q) = \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1+q^{2n-1})^2} = -\frac{\vartheta_3''(0,q)}{8\vartheta_3(0,q)},$$

where

$$\vartheta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz$$

and the derivative is taken with respect to z.

Example 2. Setting $\mu = 2, \alpha = 1, \beta = 1$, and $r = 1$ in (8), we get

$$S_2^{(1,1)}(1; (L_{2n})_{n=1}^{\infty}) = 2\sum_{m=1}^{\infty} m(-1)^m \hat{L}_m(q),$$

where $\hat{L}_m(q)$ is a variant of Lambert series:

$$\hat{L}_m(q) = \sum_{n=1}^{\infty} \frac{q^{mn}}{(1+q^{2n})^m}.$$

For example,

$$\hat{L}_1(q) = \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} = \frac{1}{4}(\vartheta_3(0,q)^2 - 1),$$

$$\hat{L}_2(q) = \sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2} = -\frac{1}{8}\left(1 + \frac{\vartheta_2''(0,q)}{\vartheta_2(0,q)}\right),$$

etc., where

$$\vartheta_2(z,q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(2n+1)z.$$

Similarly,

$$S_2^{(1,1)}(1;(L_{2n-1}^2)_{n=1}^{\infty}) = 2\sum_{n=1}^{\infty} m(-1)^{m-1}L_m^*(q),$$

where

$$L_m^*(q) = \sum_{n=1}^{\infty} \frac{q^{(2n-1)m}}{(1-q^{2n-1})^{2m}}.$$

For example,

$$L_1^*(q) = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} = \frac{\vartheta_4''(0,q)}{8\vartheta_4(0,q)},$$

where

$$\vartheta_4(z,q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz.$$

The second term can be expressed via $L_1^*(q)$, namely

$$\begin{aligned}
L_2^*(q) &= \sum_{n=1}^{\infty} q^{4n-2}\sum_{k=0}^{\infty}\binom{k+3}{3}q^{(2n-1)k} \\
&= \sum_{k=0}^{\infty}\binom{k+3}{3}\frac{q^{k+2}}{1-q^{2k+4}} = \frac{1}{6}\sum_{n=1}^{\infty}\frac{(n^3-n)q^n}{1-q^{2n}} \\
&= \frac{1}{6}\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^{2n}} - \frac{1}{6}L_1^*(q) = -\frac{1}{192}\frac{d^3}{dz^3}\frac{\vartheta_4'(z,q)}{\vartheta_4(z,q)}\Big|_{z=0} - \frac{1}{48}\frac{\vartheta_4''(0,q)}{\vartheta_4(0,q)}.
\end{aligned}$$

In the last step, we evaluated the third derivative of the identity (see p. 489 in [23])

$$\sum_{n=1}^{\infty}\frac{4\sin(2nz)q^n}{1-q^{2n}} = \frac{\vartheta_4'(z,q)}{\vartheta_4(z,q)}$$

at zero.

Ling [14, 15, 16] used Weierstrassian elliptic functions to obtain closed forms for $L_m(q)$, $\tilde{L}_m(q)$, $\hat{L}_m(q)$, and $L_m^*(q)$.

4 Integral representations of reciprocal Fibonacci series and Mathieu-Fibonacci series

We begin this section with some integral representations of series of reciprocal Fibonacci numbers and their alternating variants. We need the following lemma [12, 19, 21].

Lemma 3. *If $(a_n)_{n=1}^{\infty}$ is a monotone sequence increasing to infinity, then*

$$\sum_{k=1}^{\infty} e^{-a_k s} = s \int_0^{\infty} e^{-st} A(t) dt$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} e^{-a_k s} = s \int_0^{\infty} e^{-st} \tilde{A}(t) dt,$$

where the counting functions $A(t)$ and $\tilde{A}(t)$ are defined by

$$A(t) = \sum_{\substack{k \\ a_k \le t}} 1 = \lfloor a^{-1}(t) \rfloor,$$

the notation $\lfloor \lambda \rfloor$ denotes the integer part of a real number λ, and

$$\tilde{A}(t) = \sum_{\substack{k \\ a_k \le t}} (-1)^{k-1} = \frac{1 - (-1)^{\lfloor a^{-1}(t) \rfloor}}{2} = \sin^2\left(\frac{\pi}{2} \lfloor a^{-1}(t) \rfloor\right).$$

Let us find all natural numbers k for which $\log F_{2k} \le t$. From $\frac{1}{\sqrt{5}}(\varphi^{2k} - \frac{1}{\varphi^{2k}}) \le e^t$ we have $\varphi^{4k} - e^t\sqrt{5}\varphi^{2k} - 1 \le 0$. The quadratic equation associated with this inequality has two solutions $\varphi^{2k} = \frac{\sqrt{5} \pm \sqrt{5e^{2t}+4}}{2}$. The first one $\frac{\sqrt{5} - \sqrt{5e^{2t}+4}}{2}$ is negative, so

$$\frac{3+\sqrt{5}}{2} \le \varphi^{2k} \le \frac{\sqrt{5} + \sqrt{5e^{2t}+4}}{2} := u(t).$$

Hence, the possible values of k are all natural numbers of the segment $[1, \lfloor \log_{\varphi} \sqrt{u(t)} \rfloor]$. Since $\log F_{2k}$ is monotone increasing to infinity and

$$e^{-s \log F_{2k}} = \frac{1}{F_{2k}^s},$$

we get

$$A(t) = \sum_{\substack{k \\ \log F_{2k} \le t}} 1 = \sum_{k=1}^{\lfloor \log_{\varphi} \sqrt{u(t)} \rfloor} 1 = \lfloor \log_{\varphi} \sqrt{u(t)} \rfloor$$

and

$$\tilde{A}(t) = \sum_{\substack{k \\ \log F_{2k} \le t}} (-1)^{k-1} = \sum_{k=1}^{\lfloor \log_{\varphi} \sqrt{u(t)} \rfloor} (-1)^{k-1} = \sin^2\left(\frac{\pi}{2} \lfloor \log_{\varphi} \sqrt{u(t)} \rfloor\right).$$

Then, Lemma 3 yields

$$\sum_{k=1}^{\infty} \frac{1}{F_{2k}^s} = s \int_0^{\infty} e^{-st} \lfloor \log_{\varphi} \sqrt{u(t)} \rfloor dt \tag{10}$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{F_{2k}^{s}}=s\int_0^{\infty}e^{-st}\sin^2\left(\frac{\pi}{2}\lfloor\log_\varphi\sqrt{u(t)}\rfloor\right)dt.$$

The natural numbers k satisfying $\log F_{2k-1}\le t$ are those of the segment $\left[1,\lfloor\frac{\log_\varphi v(t)+1}{2}\rfloor\right]$, where $v(t)=\frac{\sqrt{5}e^t+\sqrt{5e^{2t}-4}}{2}$. Hence, again by Lemma 3,

$$\sum_{k=1}^{\infty}\frac{1}{F_{2k-1}^{2s}}=2s\int_0^{\infty}e^{-2st}\left\lfloor\frac{\log_\varphi v(t)+1}{2}\right\rfloor dt$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{F_{2k-1}^{2s}}=2s\int_0^{\infty}e^{-2st}\sin^2\left(\frac{\pi}{2}\left\lfloor\frac{\log_\varphi v(t)+1}{2}\right\rfloor\right)dt.$$

Analogous integral representations for series of reciprocal Lucas numbers can be obtained in a similar way. We can now derive further integral representations for generalized Mathieu series. From (6) and (10), we obtain

$$S_\mu^{(\alpha,\beta)}(r;(F_{2n})_{n=1}^{\infty})=2\int_0^{\infty}e^{-(\mu\alpha-\beta)t}\lfloor\log_\varphi\sqrt{u(t)}\rfloor\Sigma(t)dt,$$

where

$$\begin{aligned}\Sigma(t)&=\sum_{m=0}^{\infty}[(\mu+m)\alpha-\beta]\binom{\mu+m-1}{m}(-r^2e^{-\alpha t})^m\\&=(\mu\alpha-\beta)\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2e^{-\alpha t})^m+\alpha\sum_{m=0}^{\infty}m\binom{\mu+m-1}{m}(-r^2e^{-\alpha t})^m\\&=\frac{\mu\alpha-\beta}{(1+r^2e^{-\alpha t})^{\mu}}+\alpha\mu(-r^2e^{-\alpha t})\sum_{m=1}^{\infty}\binom{\mu+m-1}{m-1}(-r^2e^{-\alpha t})^{m-1}\\&=\frac{\mu\alpha-\beta}{(1+r^2e^{-\alpha t})^{\mu}}-\frac{\mu\alpha r^2e^{-\alpha t}}{(1+r^2e^{-\alpha t})^{\mu+1}}=\frac{\mu\alpha-\beta(1+r^2e^{-\alpha t})}{(1+r^2e^{-\alpha t})^{\mu+1}}.\end{aligned}$$

Hence,

$$S_\mu^{(\alpha,\beta)}(r;(F_{2n})_{n=1}^{\infty})=2\int_0^{\infty}e^{-(\mu\alpha-\beta)t}\frac{\mu\alpha-\beta(1+r^2e^{-\alpha t})}{(1+r^2e^{-\alpha t})^{\mu+1}}\lfloor\log_\varphi\sqrt{u(t)}\rfloor dt$$
$$(r,\alpha,\beta,\mu>0,\mu\alpha-\beta>0).$$

Analogously, under the same conditions on the parameters, we get the following integral representations:

$$\tilde{S}_\mu^{(\alpha,\beta)}(r;(F_{2n})_{n=1}^{\infty})=2\int_0^{\infty}e^{-(\mu\alpha-\beta)t}\frac{\mu\alpha-\beta(1+r^2e^{-\alpha t})}{(1+r^2e^{-\alpha t})^{\mu+1}}\sin^2\left(\frac{\pi}{2}\lfloor\log_\varphi\sqrt{u(t)}\rfloor\right)dt,$$

$$S_\mu^{(\alpha,\beta)}(r;(F_{2n-1}^2)_{n=1}^{\infty})=2\int_0^{\infty}e^{-(\mu\alpha-\beta)t}\frac{\mu\alpha-\beta(1+r^2e^{-\alpha t})}{(1+r^2e^{-\alpha t})^{\mu+1}}\left\lfloor\frac{\log_\varphi v(t)+1}{2}\right\rfloor dt,$$

and

$$\tilde{S}_\mu^{(\alpha,\beta)}(r;(F_{2n-1}^2)_{n=1}^\infty) = 2\int_0^\infty e^{-(\mu\alpha-\beta)t}\frac{\mu\alpha-\beta(1+r^2e^{-\alpha t})}{(1+r^2e^{-\alpha t})^{\mu+1}}\sin^2\left(\frac{\pi}{2}\left\lfloor\frac{\log_\varphi v(t)+1}{2}\right\rfloor\right)dt.$$

5 Asymptotic expansions

We consider here the generalized Mathieu series (1), with $\alpha\mu > \beta$, $0 < \alpha \le 3$, and $\mathbf{a}$ the sequence of Fibonacci numbers (F_n). We focus on this choice of $\mathbf{a}$, because the associated Dirichlet series

$$D_{\mathrm{Fib}}(s) := \sum_{n=1}^\infty F_n^{-s}, \tag{11}$$

which occurs in the Mellin transform of this Mathieu-Fibonacci series, has been studied before. Several related generalized Mathieu series, involving F_{2n} or Lucas numbers, can be treated very similarly. Egami [6] and Navas [17] both proved, independently, that the series (11), which converges for $\mathrm{Re}(s) > 0$, has a meromorphic continuation to the whole complex plane, with a lattice of simple poles

$$-2k + \frac{(2n+k)\pi i}{\log\varphi}, \quad k \in \mathbb{N}_0,\ n \in \mathbb{Z}. \tag{12}$$

This follows from applying the binomial series to (4), which yields the expression

$$D_{\mathrm{Fib}}(s) = 5^{s/2}\sum_{k=0}^\infty \binom{-s}{k}\frac{1}{\varphi^{s+2k}+(-1)^{k+1}}.$$

We will use this continuation to obtain asymptotics of $S_\mu^{(\alpha,\beta)}(r;(F_n)_{n=1}^\infty)$ as $r\to\infty$ by the Mellin transform method. This method has been applied to other generalized Mathieu series in several detailed studies [9, 10, 18], and so we present here only the gist of the approach and the main estimate specific to our problem, without spelling out all the laborious case distinctions for various parameter ranges. In Example 5, we give more details for a special set of parameters, and explain why oscillating factors with very small amplitude appear in the expansions. It is straightforward that the Mellin transform of the Mathieu series under consideration equals [9, (2.10)]

$$\begin{aligned} M(s) &:= \int_0^\infty r^{s-1}S_\mu^{(\alpha,\beta)}(r;(F_n)_{n=1}^\infty)dr \\ &= \Gamma(\mu)^{-1}\Gamma\Big(\mu-\frac{s}{2}\Big)\Gamma\Big(\frac{s}{2}\Big)D_{\mathrm{Fib}}\Big(\alpha\mu-\beta-\frac{\alpha s}{2}\Big), \quad 0<\mathrm{Re}(s)<\sigma, \end{aligned} \tag{13}$$

where

$$\sigma := \min\left(\tfrac{2(\alpha\mu-\beta)}{\alpha}, 2\mu\right).$$

Now $S_\mu^{(\alpha,\beta)}$ can be expressed by the Mellin inversion formula, and an expansion is found by integrating (13) over an appropriate rectangle and collecting residues. This is explained on a similar problem in Flajolet et al. [7, Example 12], to which we refer for details. The estimate for M that is needed to let the height of the rectangle tend to infinity is provided by Lemma 4 below. The outcome is that

$$S_\mu^{(\alpha,\beta)}(r;(F_n)_{n=1}^\infty) = -\sum_{\sigma\le \mathrm{Re}(\xi)<\tilde\sigma} \mathrm{Res}_{s=\xi}\big[M(s)r^{-s}\big] + O(r^{-\tilde\sigma}), \tag{14}$$

where $\tilde\sigma > \sigma$ is arbitrary. This easily yields an expansion in powers of r, possibly with logarithmic factors stemming from double poles. Terms resulting from poles of D_{Fib} feature oscillating factors, due to the vertically aligned poles with equal real part (see (12)). See Example 5 for more on this.

Lemma 4. *For $m \in \mathbb{N}$, we have the estimate*

$$\sum_{k=m}^\infty \left| \binom{-s}{k} \frac{1}{\varphi^{s+2k}+(-1)^{k+1}} \right| = O(\exp(1.04|\mathrm{Im}(s)|)) \tag{15}$$

as $|s|\to\infty$ in the strip $1-2m \le \mathrm{Re}(s) \le \frac{1}{2}$.

Proof. W.l.o.g., it suffices to consider s with $\mathrm{Im}(s) > 0$. We write c for various positive constants, which may depend on m. Since $\mathrm{Re}(s)+2k \ge 1$, we have

$$\begin{aligned} |\varphi^{s+2k}+(-1)^{k+1}| &= \varphi^{\mathrm{Re}(s)+2k}\big|1+(-1)^{k+1}\varphi^{-s-2k}\big| \\ &\ge \varphi^{\mathrm{Re}(s)+2k}\big(1-\varphi^{-\mathrm{Re}(s)-2k}\big) \\ &\ge \varphi^{1-2m+2k}(1-1/\varphi), \end{aligned}$$

and so

$$\frac{1}{|\varphi^{s+2k}+(-1)^{k+1}|} \le c\varphi^{-2k}, \quad k \ge 0. \tag{16}$$

It is well-known that, using the reflection formula of the gamma function, we can rewrite the binomial coefficient as

$$\begin{aligned} \binom{-s}{k} &= \frac{\Gamma(1-s)}{\Gamma(k+1)\Gamma(-s-k+1)} \\ &= \Gamma(1-s)\frac{\sin \pi(s+k)}{\pi}\frac{\Gamma(k+s)}{\Gamma(k+1)} \\ &= (-1)^k\Gamma(1-s)\frac{\sin \pi s}{\pi}\frac{\Gamma(k+s)}{k!}. \end{aligned}$$

We use Stirling's formula in the form

$$|\Gamma(z)| \sim \sqrt{2\pi}e^{-\mathrm{Re}(z)-\mathrm{Im}(z)\arg(z)}|z|^{\mathrm{Re}(z)-1/2}, \quad |z|\to\infty,$$

where $|\arg(z)|$ is bounded away from π. Writing $s = u + iv$, with $u \leq \frac{1}{2}$ bounded and $v \to \infty$, we obtain

$$\begin{aligned}|\Gamma(k+s)| &\leq ce^{-(k+u)-v\arg(k+s)}\big((k+u)^2+v^2\big)^{\frac{k+u}{2}-\frac{1}{4}} \\ &\leq ce^{-k-v\arg(k+s)}\big((k+u)^2+v^2\big)^{\frac{k+u}{2}-\frac{1}{4}}. \qquad (17)\end{aligned}$$

Now we split the sum in (15) into $k \leq v$ and $k > v$. If $k \leq v$, then

$$\arg(k+s) = \arctan\frac{v}{u+k} \geq \arctan\frac{v}{u+v} \geq \frac{\pi}{4} - \varepsilon,$$

where $\varepsilon > 0$ is arbitrary. Together with (17), this implies

$$\begin{aligned}|\Gamma(k+s)| &\leq ce^{-k-v(\pi/4-\varepsilon)}\big((v+u)^2+v^2\big)^{\frac{k+u}{2}-\frac{1}{4}} \\ &\leq cv^k e^{-k-v(\pi/4-\varepsilon)}3^{k/2},\end{aligned}$$

hence

$$\begin{aligned}\sum_{m\leq k\leq v}\left|\binom{-s}{k}\frac{1}{\varphi^{s+2k}+(-1)^{k+1}}\right| &\leq c|\Gamma(1-s)\sin\pi s|\sum_{m\leq k\leq v}\frac{|\Gamma(k+s)|}{k!\varphi^{2k}} \\ &\leq c|\Gamma(1-s)\sin\pi s|e^{-v(\pi/4-\varepsilon)}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{v\sqrt{3}}{e\varphi^2}\Big)^k \\ &= c|\Gamma(1-s)\sin\pi s|\exp\Big(-v(\pi/4-\varepsilon)+\frac{v\sqrt{3}}{e\varphi^2}\Big) \\ &\leq c|\Gamma(1-s)\sin\pi s|e^{-0.54v}, \qquad (18)\end{aligned}$$

for ε small enough. We will see below that this last expression dominates our estimate for $\sum_{k>v}$, which we now develop. By (17), for $k > v$ and any $\varepsilon > 0$,

$$\begin{aligned}|\Gamma(k+s)| &\leq ce^{-k}\big((k+u)^2+k^2\big)^{\frac{k+u}{2}-\frac{1}{4}} \\ &\leq ce^{-k}k^{k+u-1/2}(2+\varepsilon)^{k/2}.\end{aligned}$$

This part of our sum thus satisfies, by using Stirling's formula for $k!$ in the third line,

$$\begin{aligned}\sum_{k>v}\left|\binom{-s}{k}\frac{1}{\varphi^{s+2k}+(-1)^{k+1}}\right| &\leq c|\Gamma(1-s)\sin\pi s|\sum_{k>v}\frac{|\Gamma(k+s)|}{k!\varphi^{2k}} \\ &\leq c|\Gamma(1-s)\sin\pi s|\sum_{k>v}\frac{e^{-k}k^{k+u-1/2}(2+\varepsilon)^{k/2}}{k!\varphi^{2k}} \\ &\leq c|\Gamma(1-s)\sin\pi s|\sum_{k>v}\frac{e^{-k}k^{k+u-1/2}(2+\varepsilon)^{k/2}}{k^{1/2}(k/e)^k\varphi^{2k}} \\ &= c|\Gamma(1-s)\sin\pi s|\sum_{k>v}k^{u-1}\left(\frac{\sqrt{2+\varepsilon}}{\varphi^2}\right)^k.\end{aligned}$$

It is easy to see that the last sum satisfies, for arbitrary $\varepsilon, \delta > 0$ and v large enough,

$$\sum_{k>v} k^{u-1}\left(\frac{\sqrt{2+\varepsilon}}{\varphi^2}\right)^k \le \sum_{k>v}(1+\delta)^k\left(\frac{\sqrt{2+\varepsilon}}{\varphi^2}\right)^k \le c\exp\left(v\log\frac{\sqrt{2+\varepsilon}(1+\delta)}{\varphi^2}\right).$$

This implies

$$\sum_{k>v} k^{u-1}\left(\frac{\sqrt{2+\varepsilon}}{\varphi^2}\right)^k = O(e^{-0.6v}),$$

for $\varepsilon > 0$ small enough, which is negligible compared to the last factor in (18). By Stirling's formula,

$$\Gamma(1-s) = O\left(\exp\left(-(\tfrac{\pi}{2}-\varepsilon)|\mathrm{Im}(s)|\right)\right). \tag{19}$$

The result follows from (18), (19) and

$$\sin \pi s = O\big(\exp\left(\pi|\mathrm{Im}(s)|\right)\big),$$

since

$$-\frac{\pi}{2} + \pi - 0.54 < 1.04. \qquad \square$$

By this lemma, (13), Stirling's formula (cf. (19)), and our assumption that $0 < \alpha \le 3$, the Mellin transform (13) decays exponentially along vertical lines, uniformly for $\mathrm{Re}(s)$ bounded. This implies (14); again, see Flajolet et al. [7] for details. The constraint $\alpha \le 3$ can presumably be removed by improving the estimates in Lemma 4.

Example 5. We derive first order asymptotics for $r \to \infty$ of the Mathieu-Fibonacci series

$$S_2^{(1,1)}(r;(F_n)_{n=1}^\infty) = \sum_{n=1}^\infty \frac{2F_n}{(F_n+r^2)^2},$$

i.e., $\alpha = 1$, $\beta = 1$, $\mu = 2$. Numerical evaluations suggest that it is of order $O(r^{-2})$, which is true, and that $r^2 S_2^{(1,1)}(r;(F_n))$ converges to $2/\log\varphi$. The latter is wrong, however; we will see that there is an oscillating factor with very small fluctuations. By (13), the Mellin transform is

$$M(s) = \Gamma\Big(2-\frac{s}{2}\Big)\Gamma\Big(\frac{s}{2}\Big)D_{\mathrm{Fib}}\Big(1-\frac{s}{2}\Big), \quad 0 < \mathrm{Re}(s) < 2. \tag{20}$$

The factor $\Gamma(2-s/2)$ has poles at $s = 4, 6, 8, \ldots$, while $\Gamma(s/2)$ has no poles in the right half-plane. By (12), the poles of the third factor are located at

$$4k+2-\frac{2(2n+k)\pi i}{\log\varphi}, \quad k \in \mathbb{N}_0,\ n \in \mathbb{Z}. \tag{21}$$

We just present first order asymptotics here, resulting from the simple poles of $D_{\mathrm{Fib}}(1-s/2)$ with $\mathrm{Re}(s)=2$, i.e., $k=0$ in (21). It is straightforward to push the expansion further, if desired. There are double poles, the first one at $s=10$, which induce logarithmic factors for some of the higher order terms. From (14) and (20), we find

$$\begin{aligned} S_2^{(1,1)}(r;(F_n)) &= -\sum_{n\in\mathbb{Z}} \mathrm{Res}_{s=2-\frac{4n\pi i}{\log\varphi}} \big[M(s)r^{-s}\big] + O(r^{-4}) \\ &= -\sum_{n\in\mathbb{Z}} r^{-2+\frac{4n\pi i}{\log\varphi}} \Gamma\Big(1+\frac{2n\pi i}{\log\varphi}\Big)\Gamma\Big(1-\frac{2n\pi i}{\log\varphi}\Big) \\ &\qquad \times \mathrm{Res}_{s=2-\frac{4n\pi i}{\log\varphi}} D_{\mathrm{Fib}}\Big(1-\frac{s}{2}\Big) + O(r^{-4}). \end{aligned} \tag{22}$$

The explicit formula [17, Proposition 1] for the residues of D_{Fib} yields

$$\mathrm{Res}_{s=2-\frac{4n\pi i}{\log\varphi}} D_{\mathrm{Fib}}\Big(1-\frac{s}{2}\Big) = -2\,\mathrm{Res}_{z=\frac{2n\pi i}{\log\varphi}} D_{\mathrm{Fib}}(z) = -\frac{2\cdot 5^{n\pi i/\log\varphi}}{\log\varphi}.$$

Inserting this into (22) gives the desired result:

$$S_2^{(1,1)}(r;(F_n)) = \frac{2H\big(\log r + \frac14 \log 5\big)}{r^2\log\varphi} + O(r^{-4}), \quad r\to\infty,$$

where the periodic function H is defined by

$$H(x) := 1 + \frac{2}{\log\varphi}\sum_{n=1}^{\infty} \Big|\Gamma\Big(1+\frac{2n\pi i}{\log\varphi}\Big)\Big|^2 \cos\Big(\frac{4n\pi x}{\log\varphi}\Big), \quad x\in\mathbb{R}.$$

Due to the exponential decrease of the gamma function along vertical lines, the coefficients of this Fourier series are extremely small: of order 10^{-16} for $n=1$, and 10^{-34} for $n=2$. This explains why, numerically, our Mathieu series seems to behave like $2/(r^2\log\varphi)$. The phenomenon is not restricted to this concrete example, as it is due to the gamma factors in (13). This kind of "fake asymptotics", stemming from very small Fourier coefficients, has been observed before in other asymptotic problems [4, 8].

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