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The Expected Degree of Noninvertibility of Compositions of Functions

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Abstract

Recently, Defant and Propp defined the degree of noninvertibility of a function $f\colon X \to Y$ between two finite nonempty sets by $\deg(f) = \frac{1}{|X|}\sum_{x\in X}|f^{-1}(f(x))|$. We obtain an exact formula for the expected degree of noninvertibility of the composition of t functions for every $t \in \mathbb{N}$. Subsequently, we use the expected value to quantify a strengthening of a sort of a submultiplicativity property of the degree of noninvertibility. Finally, we generalize an equivalent formulation of the degree of noninvertibility and obtain a combinatorial identity involving the Stirling numbers of the first and second kind.

1 Introduction

Recently, Defant and Propp [1] defined the *degree of noninvertibility* of a function $f\colon X \to Y$ between two finite nonempty sets by

$$\deg(f) = \frac{1}{|X|}\sum_{x\in X}|f^{-1}(f(x))|,$$

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as a measure of how far f is from being injective. For example, if f is k-to-1 for $1 \leq k \leq |X|$, then $\deg(f) = k$. In particular, if f is injective (resp., constant), then $\deg(f) = 1$ (resp., $\deg(f) = |X|$). A major part of their work was devoted to the computation of the degrees of noninvertibility of specific discrete dynamical systems (i.e., functions $f\colon X \to X$), such as the Carolina solitaire map and the bubble sort map, to name a few. Turning their attention to arbitrary functions, they studied, from an extremal point of view, the connection between the degrees of noninvertibility of functions and those of their iterates.

The main objective of this work is the calculation of the expected degree of noninvertibility of a random function, or, more generally, of the composition of t random functions, for every $t \in \mathbb{N}$, thereby providing a probabilistic perspective on this new notion. The expected value is subsequently used to quantify a strengthening of a sort of a submultiplicativity property of the degree of noninvertibility. Additionally, an equivalent formulation of the degree of noninvertibility is generalized, yielding a combinatorial identity involving the Stirling numbers of the first and second kind.

We begin with a detailed presentation of our results. Section 3 contains their proofs. In Section 4 we show how the degrees of noninvertibility of several specific dynamical systems, considered by Defant and Propp [1], compare with the expected value. All the sets in this work are tacitly assumed to be nonempty and of finite sizes.

2 Main results

Let $t \in \mathbb{N}$ to be used throughout this work.

Definition 1. Let $X_1, \ldots, X_{t+1}$ be $t+1$ sets of sizes $n_1, \ldots, n_{t+1}$, respectively. We denote the expected degree of noninvertibility of the composition of t functions between $X_1, \ldots, X_n$ by $\mathcal{D}(X_1, \ldots, X_{t+1})$, i.e.,

$$\mathcal{D}(X_1, \ldots, X_{t+1}) = \frac{1}{\prod_{s=1}^{t} n_{s+1}^{n_s}} \sum_{\substack{f_s\colon X_s \to X_{s+1} \\ 1 \leq s \leq t}} \deg(f_t \circ \cdots \circ f_1).$$

Our first main result is an exact formula for the expected degree of noninvertibility of the composition of t random functions.

Theorem 2. *Let $X_1, \ldots, X_{t+1}$ be $t+1$ sets of sizes $n_1, \ldots, n_{t+1}$, respectively. Then*

$$\mathcal{D}(X_1, \ldots, X_{t+1}) = \frac{\prod_{s=1}^{t+1} n_s - \prod_{s=1}^{t+1} (n_s - 1)}{\prod_{s=2}^{t+1} n_s}.$$

In particular, if all the sets are equal, we obtain the following result.

Corollary 3. *Let X be a set of size n. Then*

$$\mathcal{D}(\overbrace{X, \ldots, X}^{t+1 \text{ times}}) = \frac{n^{t+1} - (n-1)^{t+1}}{n^t}.$$

Thus,

$$\lim_{n\to\infty} \mathcal{D}(\overbrace{X,\ldots,X}^{t+1 \text{ times}}) = t+1.$$

Our second main result is concerned with a sort of a submultiplicativity property of the degree of noninvertibility [1, Theorem 3.4], according to which, if X is a set of size n and $f, g\colon X \to X$ are two functions, then

$$\deg(f \circ g) \le \sqrt{n}\sqrt{\deg(f)}\deg(g). \tag{1}$$

This inequality is strengthened in the following theorem. Subsequently, Corollary 3 is used to quantify the improvement.

Theorem 4. *Let X, Y, and Z be three sets and let $g\colon X \to Y$ and $f\colon Y \to Z$ be two functions. Then*

$$\deg(f \circ g) \le \max_{z\in Z} |f^{-1}(z)| \deg(g). \tag{2}$$

That Theorem 4 is a strengthening of (1), in the case that $X = Y = Z$, follows from the following Lemma.

Lemma 5. *Let X be a set of size n and let Y be an additional set. Let $f\colon X \to Y$ be a function. Then*

$$\max_{y\in Y} |f^{-1}(y)| \le \sqrt{n}\sqrt{\deg(f)}.$$

To quantitatively compare between (1) and (2) (still assuming $X = Y = Z$), we notice that, by Corollary 3, the order of the expectation of $\sqrt{n}\sqrt{\deg(f)}$ is $\Theta(\sqrt{n})$. On the other hand, the order of the expectation of $\max_{x\in X} |f^{-1}(x)|$ is $\Theta\left(\frac{\log(n)}{\log(\log(n))}\right)$, a result due to Gonnet [2] (see also the references in A208250 in the On-Line Encyclopedia of Integer Sequences (OEIS) [4]).

To motivate our last main result, notice that, if $f\colon X \to Y$ is a function between two sets X and Y, then

$$\deg(f) = \frac{1}{|X|}\sum_{y\in Y} |f^{-1}(y)|^2.$$

This identity, which is easy to prove and which we shall freely use throughout this work, opens the door for a generalization: For $p \in \mathbb{N}$, let

$$\deg(f, p) = \frac{1}{|X|}\sum_{y\in Y} |f^{-1}(y)|^p.$$

We obtain an exact formula for the expected value of $\deg(f, p)$. It involves the Stirling numbers of the first and second kind, denoted by $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$, respectively (e.g., [3, pp. 243–253]).

Theorem 6. *Let $p \in \mathbb{N}$ and let X and Y be two sets of sizes n and m, respectively. Then*

$$\frac{1}{m^n} \sum_{f\colon X \to Y} \deg(f,p) = \frac{1}{m^{p-1}} \sum_{k=1}^{p} \left\{ {p \atop k} \right\} \left(\sum_{j=1}^{k} (-1)^{k-j} \left[{k \atop j} \right] n^{j-1} \right) m^{p-k}.$$

3 The proofs

Before we begin, let us introduce some notation to reduce clutter. We denote the set of nonnegative integers by $\mathbb{N}_0$. Let $m, n \in \mathbb{N}$. Bold face letters stand for vectors, i.e., $\mathbf{k} = (k_1, \ldots, k_n)$. We define $O(m,n) = \{\mathbf{k} \in \mathbb{N}_0^n : \sum_{i=1}^n k_i = m\}$. If $\mathbf{k}, \boldsymbol{\ell} \in \mathbb{N}_0^n$, we let $\mathbf{k}^{\boldsymbol{\ell}}$ stand for $\prod_{i=1}^n k_i^{\ell_i}$. We denote by $\mathrm{mult}(\mathbf{k})$ the multinomial coefficient corresponding to $\mathbf{k} \in O(m,n)$, i.e., $\mathrm{mult}(\mathbf{k}) = \binom{m}{k_1,\ldots,k_n}$. Finally, if $p \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}_0^n$, then the p-norm of $\mathbf{k}$ is denoted by $||\mathbf{k}||_p$, i.e., $||\mathbf{k}||_p^p = \sum_{i=1}^n k_i^p$. We set $||\mathbf{k}||_0^0 = n$.

The proof of Theorem 2 relies on the following lemma.

Lemma 7. *Let $X_1, \ldots, X_{t+1}$ be $t+1$ sets of sizes $n_1, \ldots, n_{t+1}$, respectively. Then*

$$\mathcal{D}(X_1, \ldots, X_{t+1}) = \frac{1}{n_1 \prod_{s=1}^t n_{s+1}^{n_s}} \sum_{\substack{\mathbf{k}^{(s)} \in O(n_s, n_{t+1}) \\ 1 \le s \le t}} \left(\prod_{r=1}^t \mathrm{mult}(\mathbf{k}^{(r)}) (\mathbf{k}^{(r+1)})^{\mathbf{k}^{(r)}} \right) ||\mathbf{k}^{(1)}||_2^2,$$

where $\mathbf{k}^{(t+1)} = (\overbrace{1, \ldots, 1}^{n_{t+1} \text{ times}})$.

Proof. Assume that $X_{t+1} = \{x_1, \ldots, x_{n_{t+1}}\}$, and, for every $1 \le s \le t$, let $f_s\colon X_s \to X_{s+1}$. For every $1 \le i \le n_{t+1}$, we define iteratively: $X_i^{(t)} = f_t^{-1}(x_i), k_i^{(t)} = |X_i^{(t)}|$ and, for $1 \le s \le t-1$, we set $X_i^{(s)} = f_s^{-1}(X_i^{(s+1)}), k_i^{(s)} = |X_i^{(s)}|$. Then $(f_t \circ \cdots \circ f_1)^{-1}(x_i) = X_i^{(1)}$. Thus, $|(f_t \circ \cdots \circ f_1)^{-1}(x_i)| = k_i^{(1)}$. Now, for every $1 \le s \le t-1$, there are exactly $\mathrm{mult}(\mathbf{k}^{(s)})(\mathbf{k}^{(s+1)})^{\mathbf{k}^{(s)}}$ functions $g\colon X_s \to X_{s+1}$ such that $|g^{-1}(X_i^{(s+1)})| = k_i^{(s)}, 1 \le i \le n_{t+1}$. □

Proof of Theorem 2. We proceed by induction on t. For the induction step we shall need the following identity, from which we shall also deduce the case $t = 1$: Let $m, n \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}_0^n$. Put $r = \sum_{i=1}^n k_i$. We claim that

$$\sum_{\boldsymbol{\ell} \in O(m,n)} \mathrm{mult}(\boldsymbol{\ell}) \mathbf{k}^{\boldsymbol{\ell}} ||\boldsymbol{\ell}||_2^2 = m(m-1) r^{m-2} ||\mathbf{k}||_2^2 + m r^m. \tag{3}$$

Indeed,

$$\sum_{\boldsymbol{\ell} \in O(m,n)} \mathrm{mult}(\boldsymbol{\ell}) \mathbf{k}^{\boldsymbol{\ell}} ||\boldsymbol{\ell}||_2^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sum_{\boldsymbol{\ell} \in O(m,n)} \operatorname{mult}(\boldsymbol{\ell}) \mathbf{k}^{\boldsymbol{\ell}} \ell_i^2 \\
&= \sum_{i=1}^{n} \left(\sum_{\substack{\boldsymbol{\ell} \in O(m,n) \\ \ell_i = 1}} \operatorname{mult}(\boldsymbol{\ell}) \mathbf{k}^{\boldsymbol{\ell}} \ell_i^2 + \sum_{\substack{\boldsymbol{\ell} \in O(m,n) \\ \ell_i \geq 2}} \operatorname{mult}(\boldsymbol{\ell}) \mathbf{k}^{\boldsymbol{\ell}} \ell_i^2 \right) \\
&= \sum_{i=1}^{n} \left(m k_i (r - k_i)^{m-1} + m \sum_{\substack{\boldsymbol{\ell} \in O(m,n) \\ \ell_i \geq 2}} \binom{m-1}{\ell_1, \ldots, \ell_i - 1, \ldots, \ell_n} \mathbf{k}^{\boldsymbol{\ell}} (\ell_i - 1 + 1) \right) \\
&= \sum_{i=1}^{n} \left(m k_i (r - k_i)^{m-1} + k_i^2 m(m-1) \sum_{\boldsymbol{\ell} \in O(m-2,n)} \operatorname{mult}(\boldsymbol{\ell}) \mathbf{k}^{\boldsymbol{\ell}} + m k_i \sum_{\substack{\boldsymbol{\ell} \in O(m-1,n) \\ \ell_i \geq 1}} \operatorname{mult}(\boldsymbol{\ell}) \mathbf{k}^{\boldsymbol{\ell}} \right) \\
&= m \sum_{i=1}^{n} k_i \left((r - k_i)^{m-1} + k_i (m-1) r^{m-2} + \sum_{\boldsymbol{\ell} \in O(m-1,n)} \operatorname{mult}(\boldsymbol{\ell}) \mathbf{k}^{\boldsymbol{\ell}} - \sum_{\substack{\boldsymbol{\ell} \in O(m-1,n) \\ \ell_i = 0}} \operatorname{mult}(\boldsymbol{\ell}) \mathbf{k}^{\boldsymbol{\ell}} \right) \\
&= m \sum_{i=1}^{n} k_i \left((r - k_i)^{m-1} + k_i (m-1) r^{m-2} + r^{m-1} - (r - k_i)^{m-1} \right) \\
&= m(m-1) r^{m-2} ||\mathbf{k}||_2^2 + m r^m.
\end{aligned}
$$

Let $t = 1$. Then, using Lemma 7,

$$
\begin{aligned}
\mathcal{D}(X_1, X_2) &= \frac{1}{n_1 n_2^{n_1}} \sum_{\mathbf{k}^{(1)} \in O(n_1, n_2)} \operatorname{mult}(\mathbf{k}^{(1)}) ||\mathbf{k}^{(1)}||_2^2 \\
&\overset{(3)}{=} \frac{n_1 (n_1 - 1) n_2^{n_1 - 2} n_2 + n_1 n_2^{n_1}}{n_1 n_2^{n_1}} \\
&= \frac{n_1 + n_2 - 1}{n_2} \\
&= \frac{n_1 n_2 - (n_1 - 1)(n_2 - 1)}{n_2}.
\end{aligned}
$$

Suppose that the claim holds for $t \in \mathbb{N}$. Using Lemma 7 and the induction hypothesis, we have

$$
\begin{aligned}
&\mathcal{D}(X_1, \ldots, X_{t+2}) \\
&= \frac{1}{n_1 \prod_{s=1}^{t+1} n_{s+1}^{n_s}} \sum_{\substack{\mathbf{k}^{(s)} \in O(n_s, n_{t+2}) \\ 1 \leq s \leq t+1}} \left(\prod_{r=1}^{t+1} \operatorname{mult}(\mathbf{k}^{(r)}) (\mathbf{k}^{(r+1)})^{\mathbf{k}^{(r)}} \right) ||\mathbf{k}^{(1)}||_2^2 \\
&= \frac{1}{n_1 \prod_{s=1}^{t+1} n_{s+1}^{n_s}} \sum_{\substack{\mathbf{k}^{(s)} \in O(n_s, n_{t+2}) \\ 2 \leq s \leq t+1}} \left(\prod_{r=2}^{t+1} \operatorname{mult}(\mathbf{k}^{(r)}) (\mathbf{k}^{(r+1)})^{\mathbf{k}^{(r)}} \right) \sum_{\mathbf{k}^{(1)} \in O(n_1, n_{t+2})} \operatorname{mult}(\mathbf{k}^{(1)}) (\mathbf{k}^{(2)})^{\mathbf{k}^{(1)}} ||\mathbf{k}^{(1)}||_2^2
\end{aligned}
$$

$$\overset{(3)}{=} \frac{1}{n_1 \prod_{s=1}^{t+1} n_{s+1}^{n_s}} \sum_{\substack{\mathbf{k}^{(s)} \in O(n_s, n_{t+2}) \\ 2 \le s \le t+1}} \left(\prod_{r=2}^{t+1} \mathrm{mult}(\mathbf{k}^{(r)}) (\mathbf{k}^{(r+1)})^{\mathbf{k}^{(r)}} \right) \left(n_1 (n_1 - 1) n_2^{n_1 - 2} ||\mathbf{k}^{(2)}||_2^2 + n_1 n_2^{n_1} \right)$$

$$= \frac{n_1 n_2^{n_1} \prod_{s=2}^{t+1} n_{s+1}^{n_s}}{n_1 \prod_{s=1}^{t+1} n_{s+1}^{n_s}} + \frac{n_1 - 1}{n_2^2 \prod_{s=2}^{t+1} n_{s+1}^{n_s}} \sum_{\substack{\mathbf{k}^{(s)} \in O(n_s, n_{t+2}) \\ 2 \le s \le t+1}} \left(\prod_{r=2}^{t+1} \mathrm{mult}(\mathbf{k}^{(r)}) (\mathbf{k}^{(r+1)})^{\mathbf{k}^{(r)}} \right) ||\mathbf{k}^{(2)}||_2^2$$

$$= 1 + (n_1 - 1) \frac{\prod_{s=2}^{t+2} n_s - \prod_{s=2}^{t+2} (n_s - 1)}{n_2 \prod_{s=3}^{t+2} n_s}$$

$$= \frac{\prod_{s=1}^{t+2} n_s - \prod_{s=1}^{t+2} (n_s - 1)}{\prod_{s=2}^{t+2} n_s}. \qquad \square$$

Proof of Corollary 3. We have

$$\mathcal{D}(\overbrace{X, \ldots, X}^{t+1 \text{ times}}) = \frac{n^{t+1} - (n-1)^{t+1}}{n^t}$$
$$= \frac{1}{n^t} \sum_{s=0}^{t} (-1)^s \binom{t+1}{s+1} n^{t-s}$$
$$= t + 1 + \sum_{s=1}^{t} (-1)^s \binom{t+1}{s+1} \frac{1}{n^s}$$

and the assertion follows. $\square$

Remark 8. The coefficients in the sum $\sum_{s=0}^{t} (-1)^s \binom{t+1}{s+1} n^{t-s}$ correspond to the $t+1$th row of Pascal's triangle with alternating signs, omitting the first 1. For example, for $t = 1, 2,$ and 3, the sum has the form

$$2n - 1,$$
$$3n^2 - 3n + 1, \text{ and}$$
$$4n^3 - 6n^2 + 4n - 1,$$

respectively.

Only a small modification of the proof of [1, Theorem 3.4] is necessary to prove Theorem 4. We give the full proof for completeness.

Proof of Theorem 4. Let $z_1, \ldots, z_r \in Z$ be such that $f(g(X)) = \{z_1, \ldots, z_r\}$. For every $1 \le i \le r$, let $k_i = |f^{-1}(z_i)|$ and let $y_{i1}, \ldots, y_{ik_i} \in Y$ be such that $f^{-1}(z_i) = \{y_{i1}, \ldots, y_{ik_i}\}$. Furthermore, for $1 \le i \le r$ and $1 \le j \le k_i$, let $\ell_{ij} = |g^{-1}(y_{ij})|$. We have

$$\deg(f \circ g) = \frac{1}{n} \sum_{i=1}^{r} |g^{-1}(f^{-1}(z_i))|^2$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{r}\left(\sum_{j=1}^{k_i}\ell_{ij}\right)^2 \\
&\overset{\text{(a)}}{\leq} \frac{1}{n}\sum_{i=1}^{r}k_i\sum_{j=1}^{k_i}\ell_{ij}^2 \\
&\leq \max_{z\in Z}|f^{-1}(z)|\overbrace{\frac{1}{n}\sum_{i=1}^{r}\sum_{j=1}^{k_i}\ell_{ij}^2}^{\overset{\text{(b)}}{=}\deg(g)} \\
&= \max_{z\in Z}|f^{-1}(z)|\deg(g),
\end{aligned}
$$

where in (a) we used the Cauchy-Schwarz inequality and (b) is due to the fact that

$$
g(X) \subseteq f^{-1}(f(g(X))) = \{y_{ij} : 1 \leq i \leq r, 1 \leq j \leq k_i\}. \qquad \square
$$

Proof of Lemma 5. Assume that Y is of size m. Applying the inequality $||x||_\infty \leq ||x||_2$, which holds for every $x \in \mathbb{R}^m$, on the vector in $\mathbb{R}^m$, whose entries correspond to the sizes of the preimages under f of all the elements of Y, we obtain

$$
\max_{y\in Y}|f^{-1}(y)| \leq \sqrt{\sum_{y\in Y}|f^{-1}(y)|^2} = \sqrt{n}\sqrt{\frac{1}{n}\sum_{y\in Y}|f^{-1}(y)|^2} = \sqrt{n}\sqrt{\deg(f)}. \qquad \square
$$

Proof of Theorem 6. We proceed by induction on p and prove that, for every $m, n \in \mathbb{N}$ and $p \in \mathbb{N}_0$, we have

$$
\begin{aligned}
&\sum_{\mathbf{k}\in O(n,m)} \mathrm{mult}(\mathbf{k})||\mathbf{k}||_p^p \\
&= \begin{cases} nm^{n-(p-1)}\sum_{k=1}^{p}{p \brace k}\left(\sum_{j=1}^{k}(-1)^{k-j}{k \brack j}n^{j-1}\right)m^{p-k}, & \text{if } p \geq 1; \\ m^{n+1}, & \text{otherwise.}\end{cases}
\end{aligned}
$$

The cases $p = 0, 1$ or $n = 1$ are immediate. Suppose that the assertion holds for every $m, n \in \mathbb{N}$ and every $0 \leq r \leq p$. Assume that $n \geq 2$. We have

$$
\begin{aligned}
&\sum_{\mathbf{k}\in O(n,m)} \mathrm{mult}(\mathbf{k})||\mathbf{k}||_{p+1}^{p+1} \\
&= n\sum_{i=1}^{m}\sum_{\substack{\mathbf{k}\in O(n,m)\\ k_i\geq 1}}\binom{n-1}{k_1,\ldots,k_i-1,\ldots,k_m}(k_i-1+1)^p \\
&= n\sum_{r=0}^{p}\binom{p}{r}\sum_{i=1}^{m}\sum_{\substack{\mathbf{k}\in O(n,m)\\ k_i\geq 1}}\binom{n-1}{k_1,\ldots,k_i-1,\ldots,k_m}(k_i-1)^r
\end{aligned}
$$

$$
\begin{aligned}
&= n\sum_{r=0}^{p}\binom{p}{r}\sum_{\mathbf{k}\in O(n-1,m)} \mathrm{mult}(\mathbf{k})||\mathbf{k}||_r^r \\
&\overset{\text{(c)}}{=} n\sum_{r=1}^{p}\binom{p}{r}(n-1)m^{n-r}\sum_{k=1}^{r}\left\{ {r \atop k} \right\}\left(\sum_{j=1}^{k}(-1)^{k-j}\left[{k \atop j} \right](n-1)^{j-1}\right)m^{r-k} + nm^n \\
&= nm^{n-t}\sum_{k=1}^{p}\overbrace{\sum_{r=k}^{p}\binom{p}{r}\left\{ {r \atop k} \right\}}^{\overset{\text{(d)}}{=}\left\{ {p+1 \atop k+1} \right\}}\left(\sum_{j=1}^{k}(-1)^{k-j}\left[{k \atop j} \right]\overbrace{(n-1)^j}^{=\sum_{l=0}^{j}\binom{j}{l}(-1)^l n^{j-l}}\right)m^{p-k} + nm^n \\
&= nm^{n-p}\sum_{k=1}^{p}\left\{ {p+1 \atop k+1} \right\}\left(\overbrace{\sum_{j=1}^{k}\sum_{l=0}^{j}(-1)^{k+l-j}\binom{j}{l}\left[{k \atop j} \right]n^{j-l}}^{\overset{\text{(e)}}{=}\sum_{j=0}^{k}(-1)^{k-j}\left[{k+1 \atop j+1} \right]n^j}\right)m^{p-k} + nm^{n-p}m^p \\
&= nm^{n-p}\left(\sum_{k=1}^{p}\left\{ {p+1 \atop k+1} \right\}\left(\sum_{j=0}^{k}(-1)^{k-j}\left[{k+1 \atop j+1} \right]n^j\right)m^{p-k} + m^p\right) \\
&= nm^{n-p}\sum_{k=1}^{p+1}\left\{ {p+1 \atop k} \right\}\left(\sum_{j=1}^{k}(-1)^{k-j}\left[{k \atop j} \right]n^{j-1}\right)m^{p+1-k},
\end{aligned}
$$

where, in (c), we used the induction hypothesis, (d) is due to [3, Eq. (6.15)], and (e) follows after several algebraic manipulations, together with [3, Eq. (6.16)]. □

4 The limiting behavior of the degrees of noninvertibility of several specific dynamical systems

Let X be a set of size n. By Corollary 3, the expected degree of noninvertibility of a dynamical system $f: X \to X$ tends to 2, as $n \to \infty$. Defant and Propp [1] established the degrees of noninvertibility of several specific dynamical systems. Table 1 below summarizes their limiting behavior. Interestingly, in the cases where the limit is finite, it is not far from the expected value.

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compositions of functions.

The dynamical system f	The set X	$\lim_{n\to\infty} \deg(f)$	Remarks
Bubble sort (for permutations)	$\mathrm{Sym}(n)$	∞	
Stack-sorting	$\mathrm{Sym}(n)$	∞	
Nibble sort (for permutations)	$\mathrm{Sym}(n)$	$4e-9$	≈ 1.873
Nibble sort (for binary words)	$\{0,1\}^n$	$3/2$	
Binary chip-firing on an $n+1$-cycle	$\{0,1\}^n$	$3/2$	
Bulgarian solitaire	partitions of n	3	conjectured
Carolina solitaire	compositions of n	∞	

Table 1: The limiting behavior of the degrees of noninvertibility of several specific dynamical systems, established by Defant and Propp [1].

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