Journal of Integer Sequences, Vol. 25 (2022),
Article 22.4.2

On the Coefficients of the Distinct Monomials in the Expansion of $x_1(x_1 + x_2) \cdots (x_1 + x_2 + \cdots + x_n)$

Sela Fried[1]
Department of Computer Science
Ben-Gurion University of the Negev
David Ben Gurion Blvd 1
Be'er Sheva
Israel
friedsela@gmail.com

Abstract

We initiate the study of the coefficients of the distinct monomials in the expansion of the multivariate polynomials $x_1(x_1 + x_2) \cdots (x_1 + x_2 + \cdots + x_n), n \in \mathbb{N}$, the number of which was shown by Shallit to be counted by the Catalan numbers $C_n, n \in \mathbb{N}$. In particular, we obtain an exact formula for the coefficients and reduce the complexity of the search for their maximum from the order of C_n to the order of the number of partitions of n with distinct parts.

1 Introduction

Let $n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be indeterminates. It is well known that among the multitude of their combinatorial interpretations, the Catalan numbers also count the distinct monomials in the expansion of the multivariate polynomials

$$p_n = x_1(x_1 + x_2) \cdots (x_1 + x_2 + \cdots + x_n), \quad n \in \mathbb{N},$$

[1] The author is a postdoctoral fellow in the Department of Computer Science at the Ben-Gurion University of the Negev and a teaching fellow in the Department of Computer Science at the Israel Academic College in Ramat Gan.

a result that goes back at least to [8]. It seems that despite the naturalness of these polynomials, the coefficients of their distinct monomials have not been studied extensively. In particular, to the best of our knowledge, a closed formula for their maximum is not known. In this work, we initiate the study of these coefficients and accomplish the following: upon establishing several elementary properties including a closed formula for the coefficients (Lemma 4), we show that, for every $n \in \mathbb{N}$, a maximal coefficient in the expansion of p_n is attained at a monomial belonging to a certain set $\mathcal{M}_n$ that has the same cardinality as the set of all partitions of n with distinct parts (Theorem 17 and Lemma 18). Finally, we provide an algorithm (Algorithm 1) that successively (and greedily) generates a sequence of monomials $(r_n)_{n\in\mathbb{N}}$ such that $r_n \in \mathcal{M}_n$ for every $n \in \mathbb{N}$ and conjecture that the corresponding coefficients are actually maximal.

Our interest in the polynomials $p_n, n \in \mathbb{N}$, was triggered during our work [5] on the restrictiveness of stochastic orders in which we established a closed formula ([5, Lemma 3.6]) for the probability that a random probability distribution that is uniformly drawn from the probability n-simplex is greater than a fixed probability distribution, with respect to the usual stochastic order. The formula involves a sum over all distinct monomials of p_{n-1}. Thus, the distinct monomials of $p_n, n \in \mathbb{N}$, have a direct application in probability theory.

2 Main results

This section consists of two parts: in the first, we establish several elementary properties of the coefficients of $p_n, n \in \mathbb{N}$, including a closed formula for them and, in the second, we address the problem of finding the maximal coefficients of $p_n, n \in \mathbb{N}$. Before we begin, let us, for completeness, prove the claim stated in the introduction that the distinct monomials in the expansion of $p_n, n \in \mathbb{N}$, are counted by the Catalan numbers which have the explicit formula $C_n = \frac{1}{n+1}\binom{2n}{n}$ for every $n \in \mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (e.g., [9, Theorem 1.4.1]). To simplify formulations throughout this work, whenever we refer to a monomial of p_n, we mean a monomial in the expansion of p_n after combining like terms. In particular, in our terminology, the monomials of p_n are distinct. Likewise, whenever we refer to a coefficient (of a monomial) of p_n, we mean the coefficient (of the monomial) after expanding p_n and combining like terms. Finally, unless stated otherwise, $n \in \mathbb{N}$.

Lemma 1. *Let $\mathcal{P}_n$ denote the set of all monomials of p_n. Then $|\mathcal{P}_n| = C_n$.*

Proof. The Catalan numbers have several fundamental interpretations. Thus, it suffices to construct a bijection between $\mathcal{P}_n$ and a set consisting of the elements in one of these interpretations. Following [9], we shall prove that there is a bijection between $\mathcal{P}_n$ and the set $\mathcal{T}_n$ of plane trees with $n+1$ vertices (cf. [9, p. 6 and Theorem 1.5.1]). The bijection goes through two auxiliary sets $\mathcal{A}_n$ and $\mathcal{B}_n$.

Let us define

$$\mathcal{A}_n = \left\{ (a_1, \ldots, a_n) \in \mathbb{N}_0^n \mid \sum_{i=1}^{n} a_i = n \text{ and } \sum_{i=k+1}^{n} a_i \leq n-k \text{ for every } 1 \leq k \leq n-1 \right\}.$$

We shall show that there are two maps $\Theta\colon \mathcal{A}_n \to \mathcal{P}_n$ and $\Phi\colon \mathcal{P}_n \to \mathcal{A}_n$ such that $\Theta\circ\Phi = \mathrm{id}_{\mathcal{P}_n}$ and $\Phi \circ \Theta = \mathrm{id}_{\mathcal{A}_n}$. For $(a_1, \ldots, a_n) \in \mathcal{A}_n$, let

$$\Theta((a_1, \ldots, a_n)) = x_1^{a_1} \cdots x_n^{a_n}.$$

The proof that $x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{P}_n$ relies on the following observation: suppose $i_1, \ldots, i_n \in \{1, \ldots, n\}$ are such that $1 \le i_k \le k$ for every $1 \le k \le n$. Then $x_{i_1} \cdots x_{i_n} \in \mathcal{P}_n$. We define $i_1, \ldots, i_n$ as follows: first, if $a_n = 1$ then we set $i_n = n$. Now, suppose we have already defined $i_{n+1-\sum_{i=k}^n a_i}, \ldots, i_n$ for some $1 < k \le n$ such that $1 \le i_r \le r$ for every $n+1-\sum_{i=k}^n a_i \le r \le n$. Set $i_{n+1-\sum_{i=k-1}^n a_i} = \cdots = i_{n-\sum_{i=k}^n a_i} = k-1$. Since $(a_1, \ldots, a_n) \in \mathcal{A}_n$, we have

$$\sum_{i=k-1}^{n} a_i \le n+2-k \iff n+1-\sum_{i=k-1}^{n} a_i \ge k-1.$$

It follows that $1 \le i_r \le r$ for $n+1-\sum_{i=k-1}^n a_i \le r \le n$.

In the other direction, for $x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{P}_n$ we let $\Phi(x_1^{a_1} \cdots x_n^{a_n}) = (a_1, \ldots, a_n)$. To see that $(a_1, \ldots, a_n) \in \mathcal{A}_n$, first notice that $a_n \le 1 = n-(n-1)$. Suppose now that we have already shown that $\sum_{i=k+1}^n a_i \le n-k$ for some $1 < k \le n-1$. By definition of p_n, the indeterminate x_k can appear in at most $n-k+1$ places, of which $\sum_{i=k+1}^n a_i$ places are already taken. Thus,

$$a_k \le n-k+1-\sum_{i=k+1}^{n} a_i \iff \sum_{i=k}^{n} a_i \le n-(k-1).$$

Now consider the set

$$\mathcal{B}_n = \left\{ (b_1, \ldots, b_n) \in \mathbb{N}_0 \cup \{-1\} \mid \sum_{i=1}^{n} b_i = 0 \text{ and } \sum_{i=1}^{k} b_i \ge 0 \text{ for every } 1 \le k \le n-1 \right\}.$$

One verifies immediately that the map $(a_1, \ldots, a_n) \mapsto (a_1 - 1, \ldots, a_n - 1)$ gives a bijection between $\mathcal{A}_n$ and $\mathcal{B}_n$.

It remains to show that there is a bijection between $\mathcal{B}_n$ and $\mathcal{T}_n$. By [9, 82 on p. 71], the following procedure provides such a bijection: perform a depth-first search through a plane tree with $n+1$ vertices and every time a vertex is encountered for the first time, record one less than its number of children, except that the last vertex is ignored. We prove by induction on n that the resulting sequence belongs to $\mathcal{B}_n$: for $n = 1$, the plane tree has 2 vertices and the corresponding sequence is necessarily (0) which obviously belongs to $\mathcal{B}_1$. Suppose that the claim holds for plane trees with n vertices and consider a plane tree T with $n+1$ vertices together with the corresponding sequence $(b_1, \ldots, b_n)$. Deleting the last vertex from T gives a plane tree T' with n vertices. Let $1 \le l \le n$ be the index of the parent of the last vertex of T. We distinguish between two cases:

1. Suppose that $l = n$. Then T must end with a sequence of three vertices that make a tree of depth two. Thus, $b_n = 0$ and $(b_1, \ldots, b_{n-1})$ is the sequence corresponding to T'. Using the induction hypothesis and putting back $b_n = 0$, we see that $(b_1, \ldots, b_n) \in \mathcal{B}_n$.

2. Suppose that $l < n$. Then the nth vertex of T is a leaf and therefore $b_n = -1$. Now, $b_l \geq 0$ and $(b_1, \ldots, b_{l-1}, b_l - 1, b_{l+1}, \ldots, b_{n-1})$ is the sequence corresponding to T', which, by the induction hypothesis, belongs to $\mathcal{B}_{n-1}$. Putting back $b_n = -1$ and replacing $b_l - 1$ with b_l, we obtain the original sequence and conclude that it belongs to $\mathcal{B}_n$.

One can show, again by induction, that any $(b_1, \ldots, b_n) \in \mathcal{B}_n$ induces a unique plane tree with $n+1$ vertices. We omit the details. □

Remark 2. It follows from the proof of Lemma 1 that we may identify $\mathcal{P}_n$ with $\mathcal{A}_n$ and we shall exploit this equivalent representation freely throughout this work. In particular, we shall refer to the elements of $\mathcal{A}_n$ as 'monomials'.

2.1 Elementary properties of the coefficients of p_n

Our first result is an explicit formula for the coefficients of p_n. We shall use the following notation:

Definition 3. For $(a_1, \ldots, a_n) \in \mathcal{A}_n$, we denote by $c_{(a_1,\ldots,a_n)}$ the corresponding coefficient.

Lemma 4. *Let* $(a_1, \ldots, a_n) \in \mathcal{A}_n$. *Then*

$$c_{(a_1,\ldots,a_n)} = \prod_{k=1}^{n-1} \frac{n-k+1-\sum_{i=k+1}^{n} a_i}{a_k!}.$$

Proof. Beginning with a_n, we notice that x_n can be taken solely from the last term of the product in the definition of p_n. Thus, there are $\binom{1}{a_n}$ possibilities to do that. Proceeding to a_{n-1}, we notice that x_{n-1} can be taken only from the last two terms of the product, but not from those that contributed x_n. This gives $\binom{2-a_n}{a_{n-1}}$ possibilities. Continuing so until we reach a_1, we conclude that the number of possibilities to obtain $(a_1, \ldots, a_n)$ is given by

$$\begin{aligned}
\prod_{k=0}^{n-1} \binom{k+1-\sum_{i=n-k+1}^{n} a_i}{a_{n-k}} &= \prod_{k=0}^{n-1} \frac{\left(k-\sum_{i=n-(k-1)}^{n} a_i\right)! \left(k+1-\sum_{i=n-(k-1)}^{n} a_i\right)}{a_{n-k}! \left(k+1-\sum_{i=n-k}^{n} a_i\right)!} \\
&= \prod_{k=0}^{n-1} \frac{k+1-\sum_{i=n-(k-1)}^{n} a_i}{a_{n-k}!} \\
&= \prod_{k=1}^{n-1} \frac{n-k+1-\sum_{i=k+1}^{n} a_i}{a_k!}.
\end{aligned}$$

□

It is desirable, when writing down the expansion of the different $p_n, n \in \mathbb{N}$, to maintain consistency regarding the order of their terms. To this end, we define an ordering on $\mathcal{A}_n$. This, of course, induces an ordering of the corresponding coefficients. A reasonable choice is lexicographic and in decreasing order:

Definition 5. Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathcal{A}_n$ such that $a \neq b$ and let $k = \min\{1 \leq i \leq n \mid a_i \neq b_i\}$. We write $a \prec b$ if $a_k > b_k$.

Example 6. The elements of $\mathcal{A}_3$ are ordered as follows:

$$(3,0,0) \prec (2,1,0) \prec (2,0,1) \prec (1,2,0) \prec (1,1,1).$$

Example 7. In the following "triangle", we present the coefficients of $p_1, \ldots, p_5$ ordered according to Definition 5 (cf. A347917 in the *On-Line Encyclopedia of Integer Sequences* (OEIS):

1
1 1
1 2 1 1 1
1 3 2 1 3 4 2 1 1 1 2 1 1 1
1 4 3 2 1 6 9 6 3 3 4 2 1 1 4 9 6 3 6 8 4 2 2 1 2 1 1 1 1 3 2 1 3 4 2 1 1 1 2 1 1 1.

Example 8. Consider $c_{(n,0,\ldots,0)}$ and $c_{(1,\ldots,1)}$ which are, respectively, the first and the last coefficients of p_n. Then

$$c_{(n,0,\ldots,0)} = c_{(1,\ldots,1)} = 1.$$

Indeed,

$$c_{(n,0,\ldots,0)} = \frac{1}{n!}\prod_{k=1}^{n-1}(n-k+1) = 1 \quad \text{and}$$
$$c_{(1,\ldots,1)} = \prod_{k=1}^{n-1}(n-k+1-(n-k)) = 1.$$

Lemma 9. *The coefficients of p_{n+1} contain (at least) two copies of the coefficients of p_n.*

Proof. Suppose that $n \geq 2$ and let $(a_1, \ldots, a_{n-1}) \in \mathcal{A}_{n-1}$. Clearly,

$$(1, a_1, \ldots, a_{n-1}), (a_1, \ldots, a_{n-1}, 1) \in \mathcal{A}_n.$$

Now,

$$\begin{aligned} c_{(a_1,\ldots,a_{n-1},1)} &= \prod_{k=1}^{n-1} \frac{n-1-k+1-\sum_{i=k+1}^{n-1} a_i}{a_k!} \\ &= \prod_{k=1}^{n-2} \frac{n-1-k+1-\sum_{i=k+1}^{n-1} a_i}{a_k!} \\ &= c_{(a_1,\ldots,a_{n-1})}, \end{aligned}$$

where in the second equality we used that $a_{n-1}! = 1$. Similarly,

$$\begin{aligned} c_{(1,a_1,\ldots,a_{n-1})} &= \frac{n-1+1-\sum_{i=1}^{n-1} a_i}{1!} \prod_{k=2}^{n-1} \frac{n-k+1-\sum_{i=k+1}^{n} a_{i-1}}{a_{k-1}!} \\ &= (n-(n-1)) \prod_{k=1}^{n-2} \frac{n-1-k+1-\sum_{i=k+1}^{n-1} a_i}{a_k!} \\ &= c_{(a_1,\ldots,a_{n-1})}. \end{aligned}$$

□

The coefficients of p_n sum to $n!$:

Lemma 10. *We have* $\sum_{(a_1,\ldots,a_n)\in\mathcal{A}_n} c_{(a_1,\ldots,a_n)} = n!$.

Proof. The assertion follows immediately from specializing $(x_1,\ldots,x_n)\mapsto(1,\ldots,1)$ in the definition of p_n. □

In the following lemma, we calculate the sum of the coefficients of the monomials of p_n that contain x_i. It provides additional interpretation to several known sequences (cf. Table 1).

Lemma 11. *Let* $1\le i\le n$. *Then*

$$\sum_{\substack{(a_1,\ldots,a_n)\in\mathcal{A}_n\\ a_i>0}} c_{(a_1,\ldots,a_n)} = (n-i+1)(n-1)!.$$

Proof. Specializing, $f_n(x_i) := p_n(1,\ldots,1,x_i,1,\ldots,1)$ is a (univariate) polynomial in x_i whose free coefficient r is the sum of the coefficients corresponding to the monomials that do not contain x_i. Thus,

$$\begin{aligned} r &= f_n(0)\\ &= p_n(1,\ldots,1,x_i,1,\ldots,1)|_{x_i=0}\\ &= p_n(1,\ldots,1,0,1,\ldots,1)\\ &= 1\cdot 2\cdots(i-1)(i-1)i\cdots(n-1)\\ &= (i-1)(n-1)!. \end{aligned} \tag{1}$$

It follows that

$$\begin{aligned} \sum_{\substack{(a_1,\ldots,a_n)\in\mathcal{A}_n\\ a_i>0}} c_{(a_1,\ldots,a_n)} &= \sum_{(a_1,\ldots,a_n)\in\mathcal{A}_n} c_{(a_1,\ldots,a_n)} - \sum_{\substack{(a_1,\ldots,a_n)\in\mathcal{A}_n\\ a_i=0}} c_{(a_1,\ldots,a_n)}\\ &= n! - r = (n-i+1)(n-1)!, \end{aligned}$$

where the second and third equalities are due to Lemma 10 and (1). □

Sequence	OEIS Number
$n!$	A000142
$(n-1)(n-1)!$	A001563
$(n-2)(n-1)!$	A062119
$(n-3)(n-1)!$	A052571

Table 1: Several sequences for which Lemma 11 provides an additional interpretation.

Combining Lemma 1 together with Lemma 10 we immediately obtain the following result.

Corollary 12. *The average of the coefficients of* p_n *is* $\frac{n!}{C_n}$.

Remark 13. The sequence $\left(\frac{n!}{C_n}\right)_{n\in\mathbb{N}}$ is rational and the sequences of the corresponding numerators and denominators are A144187 and A144186 in the OEIS, respectively. Corollary 12 provides an additional interpretation for these sequences that correspond to the denominator and numerator in the series expansion of the EGF for the Catalan numbers, respectively.

In the following lemma, we calculate the sum of the coefficients of p_n whose corresponding monomials have x_i as a variable with maximal index. This result seems to provides the first interpretation for A299504 in the OEIS. These monomials were used in the proof of [5, Lemma 3.6].

Lemma 14. *Let* $1 \leq i \leq n$. *Then*

$$\sum_{\substack{(a_1,\dots,a_n)\in\mathcal{A}_n \\ a_{i+1}=\dots=a_n=0}} c_{(a_1,\dots,a_n)} = i!i^{n-i}.$$

Proof. Clearly,

$$\sum_{\substack{(a_1,\dots,a_n)\in\mathcal{A}_n \\ a_{i+1}=\dots=a_n=0}} = p_n(\overbrace{1,\dots,1}^{i\text{ times}},\overbrace{0,\dots,0}^{n-i\text{ times}}) = i!i^{n-i}.$$

□

2.2 The maximal coefficients of $p_n, n \in \mathbb{N}$

Let $m \in \mathbb{N}$. The multinomial theorem (e.g., [3, Theorem 3.9]) states that

$$(x_1 + \dots + x_n)^m = \sum \binom{m}{k_1,\dots,k_n} x_1^{k_1}\cdots x_n^{k_n}, \tag{2}$$

where the sum is over all nonnegative integers $k_1,\dots,k_n$ such that $k_1 + \dots + k_n = m$. It seems to be folklore that the maximal coefficient on the right-hand side of (2) is obtained whenever r of the $k_1,\dots,k_n$ are equal to $q+1$ and the rest are equal to q where $m = qn+r$, $q \in \mathbb{N}_0$ and $0 \leq r < n$. In this section, we address the analogous problem of finding the maximal coefficients of $p_n, n \in \mathbb{N}$, the sequence of which we denote by $(m_n)_{n\in\mathbb{N}}$ (A349404 in the OEIS). For example,

$$(m_n)_{n\in\mathbb{N}} = 1, 1, 2, 4, 9, 27, 96, 384, 1536, \dots$$

The quotients of consecutive elements of $(m_n)_{n\in\mathbb{N}}$ exhibit a nontrivial pattern and the induced sequence is denoted by $(q_n)_{n\in\mathbb{N}}$, i.e., $q_n = \frac{m_{n+1}}{m_n}, n \in \mathbb{N}$. Table 2 lists $(m_n)_{n=1}^{29}$ and $(q_n)_{n=1}^{28}$, which were established by brute force. The last column in the table lists a monomial of p_n whose coefficient is m_n. Notice that such a monomial is, in general, not unique and in Table 3 we list all other monomials of $p_1, \dots, p_{29}$ whose coefficients are maximal.

The problem of finding a maximal coefficient of p_n may be formulated as a combinatorial optimization problem as follows:

$$\begin{aligned} &\text{maximize } \prod_{k=1}^{n-1} \frac{n-k+1-\sum_{i=k+1}^{n} a_i}{a_k!} \\ &\text{subject to } (a_1, \dots, a_n) \in \mathcal{A}_n. \end{aligned}$$

This problem is intractable already for small values of n since C_n is asymptotically $\frac{4^n}{\sqrt{\pi} n^{3/2}}$ (e.g., [2, Problem 12-4]). The following two lemmas reduce the complexity of the problem. More precisely, in Lemma 15, we show that it suffices to perform the search over elements of $(a_1, \dots, a_n) \in \mathcal{A}_n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and, in Lemma 16, we show that the search may be performed over $(a_1, \dots, a_n) \in \mathcal{A}_n$ with consecutive differences bounded by 1, i.e., $a_i - a_{i+1} \leq 1$ for every $1 \leq i \leq n-1$. In Lemma 18, we show that the subset $\mathcal{M}_n$ of $\mathcal{A}_n$ consisting of elements that have both properties has the same cardinality as the set of all partitions of n with distinct parts, reducing the complexity of the problem to the order of $\frac{3^{3/4}}{12n^{3/4}} e^{\pi\sqrt{n/3}}$ (e.g., [4, (50) on p. 48]).

Lemma 15. *Let $(a_1, \dots, a_n) \in \mathcal{A}_n$ such that $a_i < a_{i+1}$ for some $1 \leq i \leq n-1$. For every $1 \leq k \leq n$ we define*

$$a'_k = \begin{cases} a_k, & k \neq i, i+1; \\ a_{i+1}, & k = i; \\ a_i, & k = i+1. \end{cases}$$

Then $(a'_1, \dots, a'_n) \in \mathcal{A}_n$ and $c_{(a'_1,\dots,a'_n)} > c_{(a_1,\dots,a_n)}$.

Proof. Clearly, $\prod_{k=1}^{n} a_k! = \prod_{k=1}^{n} a'_k!$. Furthermore, for every $k \in \{1, \dots, i-1, i+1, \dots, n\}$, we have

$$n - k + 1 - \sum_{j=k+1}^{n} a_j = n - k + 1 - \sum_{j=k+1}^{n} a'_j.$$

It follows from Lemma 4 that

$$\begin{aligned} c_{(a'_1,\dots,a'_n)} > c_{(a_1,\dots,a_n)} &\iff n - i + 1 - \sum_{j=i+1}^{n} a'_j > n - i + 1 - \sum_{j=i+1}^{n} a_j \\ &\iff -a'_{i+1} - \sum_{j=i+2}^{n} a'_j > -\sum_{j=i+1}^{n} a_j \\ &\iff a_i < a_{i+1}. \end{aligned}$$

□

Lemma 16. *Let $(a_1, \dots, a_n) \in \mathcal{A}_n$ such that $a_i > a_{i+1} + 1$ for some $1 \leq i \leq n-1$. For every $1 \leq k \leq n$ we define*

$$a'_k = \begin{cases} a_k, & k \neq i, i+1; \\ a_i - 1, & k = i; \\ a_{i+1} + 1, & k = i+1. \end{cases}$$

Then $(a'_1, \dots, a'_n) \in \mathcal{A}_n$ and $c_{(a'_1,\dots,a'_n)} \geq c_{(a_1,\dots,\mathcal{A}_n)}$.

Proof. Arguing as in the proof of Lemma 15, we have

$$\begin{aligned} c_{(a'_1,\dots,a'_n)} \geq c_{(a_1,\dots,a_n)} &\iff \frac{n - i - \sum_{j=i+1}^n a_j}{a_{i+1}+1} \geq \frac{n - i - \sum_{j=i+1}^n a_j + 1}{a_i} \\ &\iff n - i - \sum_{j=i+1}^n a_j \geq \frac{a_{i+1}+1}{a_i - (a_{i+1}+1)} \\ &\iff n - (i-1) - \sum_{j=i+1}^n a_j \geq \frac{a_i}{a_i - (a_{i+1}+1)}. \end{aligned}$$

Since $a_i - (a_{i+1}+1) \geq 1$, it suffices to show that $n - (i-1) - \sum_{j=i+1}^n a_j \geq a_i$ or, equivalently, that $\sum_{j=i}^n a_j \leq n - (i-1)$, which holds true by the definition of $\mathcal{A}_n$. □

Combining Lemma 15 and Lemma 16, we obtain the following theorem.

Theorem 17. *A maximal coefficient of p_n is attained at a monomial belonging to $\mathcal{M}_n$, where*

$$\mathcal{M}_n = \{(a_1, \dots, a_n) \in \mathcal{A}_n \mid a_{i+1} \leq a_i \leq a_{i+1} + 1 \text{ for every } 1 \leq i \leq n-1\}.$$

Lemma 18.

1. *There is a bijection between the set*

$$\{(a_1, \dots, a_n) \in \mathcal{A}_n \mid a_1 \geq a_2 \geq \dots \geq a_n\}$$

   considered in Lemma 15 and the set of all partitions of n. In particular, Lemma 15 reduces the complexity of the problem to the order of $\frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$.

2. *There is a bijection between $\mathcal{M}_n$ and the set of all partitions of n with distinct parts. In particular, Theorem 17 reduces the complexity of the problem to the order of $\frac{3^{3/4}}{12n^{3/4}} e^{\pi\sqrt{n/3}}$.*

Proof. Recall that, by definition (e.g., [1, Definition 1.1]), a partition of n is a nonincreasing sequence $a_1 \geq a_2 \geq \dots \geq a_r$ of $r \in \mathbb{N}$ natural numbers such that $\sum_{i=1}^r a_i = n$. One also writes $(a_1, \dots, a_r)$ for such a partition.

1. Let $(a_1, \ldots, a_n) \in \mathcal{A}_n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and let $r = \max\{1 \leq i \leq n \mid a_i > 0\}$. Then $(a_1, \ldots, a_r)$ is a partition of n. Conversely, let $a_1, \ldots, a_r \in \mathbb{N}$ be a partition of n with $r \in \mathbb{N}$ parts. Extend $(a_1, \ldots, a_r)$ with $n - r$ zeros to obtain $(a_1, \ldots, a_r, 0, \ldots, 0) \in \mathbb{N}_0^n$. Let $1 \leq l \leq r-1$ and suppose $\sum_{i=l+1}^{n} a_i > n - l$. Since $a_r \geq 1$, by monotonicity, also $a_1, \ldots, a_{r-1} \geq 1$. Thus,

$$n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{l} a_i + \sum_{i=l+1}^{n} a_i > l + n - l = n,$$

a contradiction. This shows that $(a_1, \ldots, a_r, 0, \ldots, 0) \in \mathcal{A}_n$. The claim regarding the complexity is due to [6, 7, 10].

2. Let $(a_1, \ldots, a_n) \in \mathcal{M}_n$ and let $r = \max\{1 \leq i \leq n \mid a_i > 0\}$. Then $(a_1, \ldots, a_r)$ is a partition of n such that $a_i - a_{i+1} \leq 1$ for every $1 \leq i \leq r-1$ and $a_r = 1$. This means that each of the numbers $1, 2, \ldots, a_1 =: m$ appears as a part in $(a_1, \ldots, a_r)$. We claim that the conjugate $(a_1', \ldots, a_m')$ of $(a_1, \ldots, a_r)$ (cf. [1, Definition 1.8]) has distinct parts. To see that, let $1 \leq i \leq m$ and recall that a_i' is defined to be the number of parts of $(a_1, \ldots, a_r)$ that are $\geq i$. Suppose $a_k' = a_{k+1}'$ for some $1 \leq k \leq m-1$. This means that for every $1 \leq i \leq r$, if $a_i \geq k$, then also $a_i \geq k+1$. Thus, k cannot be a part of $(a_1, \ldots, a_r)$, a contradiction.

   Conversely, let $(a_1', \ldots, a_m')$ be a partition of n with $1 \leq m \leq n$ distinct parts and let $(a_1, \ldots, a_r)$ be its conjugate, where $1 \leq r \leq n$. Suppose that $a_i \geq a_{i+1} + 2$ for some $1 \leq i \leq m-1$. Thus, there are distinct $1 \leq k, l \leq m$ such that $i \leq a_k'$ and $a_l' < i+1$. It follows that $a_k' = i = a_l'$, a contradiction. It remains to show that $a_r = 1$. Since the parts of $(a_1', \ldots, a_m')$ are distinct, $a_1' > a_i'$ for every $2 \leq i \leq m$. Thus, $r = a_1'$ and $a_r = 1$ since a_r is the number of parts of $(a_1', \ldots, a_m')$ that are $\geq r$.

   The claim regarding the complexity may be found, for example, in [4, (50) on p. 48].

□

Remark 19. The idea of the proof of the second statement in Lemma 18 is due to Grahl and Adams-Watters (see the comments to A000009 in the OEIS).

Despite the complexity reduction described above, the problem remains intractable. We provide a simple algorithm that for a prescribed $l \in \mathbb{N}$ successively generates a sequence of monomials $(r_n)_{n=1}^{l}$ such that $r_n \in \mathcal{A}_n$ for every $1 \leq n \leq l$. In Lemma 20, we show that actually $r_n \in \mathcal{M}_n$. The algorithm also returns a sequence $(s_n)_{n=1}^{l}$ of the corresponding coefficients. It uses a method Coefficient$(\cdot)$ that upon receiving a monomial of p_n as input returns its corresponding coefficient, e.g., by using the formula given by Lemma 4. We applied the algorithm with $l = 100$ and Table 4 lists the elements of $(s_n)_{n=29}^{100}$ returned by the algorithm (the first 29 elements of $(s_n)_{n=1}^{100}$ coincide with $(m_n)_{n=1}^{29}$ that were already listed in Table 2). We consider the correctness of the first 29 numbers returned by the algorithm and the preservation of the pattern that the elements of $(q_n)_{n=1}^{28}$ exhibit (this is illustrated in the third column of Table 4) to be a strong indication that the algorithm actually returns the true sequence $(m_n)_{n\in\mathbb{N}}$. Before we present the pseudocode of the algorithm, let us illustrate by an example its idea: assume that the algorithm arrived at, say, $r_{15} = (3,3,2,2,1,1,1,1,1,0,0,0,0,0,0)$. First, a zero is attached at the end of r_{15} and the result is denoted by r'_{16}. Thus, $r'_{16} = (3,3,2,2,1,1,1,1,1,0,0,0,0,0,0,0)$. Now, beginning at the end of r'_{16}, we iterate backwards and every time the value is about to increase, we add 1 at this point (actually, every time is wasteful and we also increase the first entry by 1 if it does not result in a gap of 2 between the first and the second entries). These are the candidates to pick r_{16} from. In our example, these are

(3,3,2,2,1,1,1,1,1,**1**,0,0,0,0,0,0), (3,3,2,2,**2**,1,1,1,1,0,0,0,0,0,0,0), (3,3,**3**,2,1,1,1,1,1,0,0,0,0,0,0,0), (**4**,3,2,2,1,1,1,1,1,0,0,0,0,0,0,0).

For each of the candidates we calculate the corresponding coefficient. Here, we obtain

$$370594350,\ 361267200,\ 321126400,\ 48168960.$$

We take r_{16} to be the monomial with the largest coefficient. In this case,

$$r_{16} = (3,3,2,2,1,1,1,1,1,1,0,0,0,0,0,0).$$

Now, we repeat the procedure described above with r_{16}.

Lemma 20. *Let $l \in \mathbb{N}$ and let $(r_n)_{n=1}^{l}$ be the sequence of monomials returned by Algorithm 1. Then $r_n \in \mathcal{M}_n$ for every $1 \le n \le l$.*

Proof. First notice that for each $2 \le n \le l$ the condition in the inner **for** loop is satisfied at least once. Indeed, if the first two conditions are not satisfied, then the third must be. Now, for $n = 1$ we have $r_1 = (1) \in \mathcal{M}_1$. Assume that $r_n = (a_1, \dots, a_n) \in \mathcal{M}_n$. The monomial r_{n+1} is chosen from the set R and therefore it suffices to show that $x \in \mathcal{M}_{n+1}$ for every $x \in R$. By definition of $\mathcal{M}_n$, we have $a_1 \ge \cdots \ge a_n$ and $a_i - a_{i+1} \le 1$ for every $1 \le i \le n-1$. We give the details only in the case that $x \in R$ due to $i > 1, a_i \ne a_{i-1}$ and $a_i = a_{i+1}$, the other two cases being similar. Let $a_{n+1} = 0$ and notice that, necessarily, $a_n \in \{0,1\}$. Thus, $a_n \ge a_{n+1}$ and $a_n - a_{n+1} \le 1$. Conclude that $x = (a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{n+1}) \in \mathcal{A}_{n+1}$. By the assumptions, $a_{i-1} = a_i + 1$ and $a_i + 1 - a_{i+1} = 1$. Therefore, $x \in \mathcal{M}_{n+1}$. □

3 Open questions

The following questions remain open and are left for future research:

1. Is there a closed formula for the maximal coefficients $(m_n)_{n\in\mathbb{N}}$?
2. Prove or disprove: the sequence $(s_n)_{n\in\mathbb{N}}$ returned by Algorithm 1 is equal to $(m_n)_{n\in\mathbb{N}}$.

Algorithm 1: Generate a sequence of monomials.

```
Input: The length l of the desired sequence
Output: A sequence (r_n)_{n=1}^l of monomials and the sequence (s_n)_{n=1}^l of their
        corresponding coefficients
r_1 ← (1)
s_1 ← 1
for n ← 2 to l do
    r'_n ← r_{n-1} ∪ 0
    R ← {}
    S ← {}
    for i ← n to 1 do
        if (i = n and r'_n[n] ≠ r'_n[n − 1]) or
        (i > 1 and r'_n[i] ≠ r'_n[i − 1] and r'_n[i] = r'_n[i + 1]) or
        (i = 1 and r'_n[1] = r'_n[2]) then
            temp ← r'_n
            temp[i] ← temp[i] + 1
            R ← R ∪ temp
            S ← S ∪ Coefficient(temp)
    k ← arg max S
    r_n ← R[k]
    s_n ← S[k]
return (r_n)_{n=1}^l, (s_n)_{n=1}^l
```

3. The sequence $(q_n)_{n\in\mathbb{N}}$ exhibits a nontrivial pattern. In particular, some of its elements are natural. Can we predict for which $n \in \mathbb{N}$ this happens? Furthermore, certain natural numbers seem to be missing from $(q_n)_{n\in\mathbb{N}}$. According to the output of Algorithm 1, for $n \leq 200$, these numbers are $15, 51, 54$ and 73. Is there a characterization for these numbers?

4. For $n < 30$, the maximal coefficient of p_n is not uniquely attained only for $n = 2, 5, 6, 12, 13, 14$ and 15. Are these the only cases when this happens? If not, can we predict when?

n	m_n	q_n	$(a_1, \ldots, a_n)$
1	1	1	(1)
2	1	2	(1, 1)
3	2	2	(2, 1, 0)
4	4	$\frac{9}{4}$	(2, 1, 1, 0)
5	9	3	(2, 2, 1, 0, 0)
6	27	$\frac{32}{9}$	(2, 2, 1, 1, 0, 0)
7	96	4	(3, 2, 1, 1, 0, 0, 0)
8	384	4	(3, 2, 1, 1, 1, 0, 0, 0)
9	1536	$\frac{625}{128}$	(3, 2, 1, 1, 1, 1, 0, 0, 0)
10	7500	5	(3, 2, 2, 1, 1, 1, 0, 0, 0, 0)
11	37500	$\frac{648}{125}$	(3, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0)
12	194400	6	(3, 2, 2, 2, 1, 1, 1, 0, 0, 0, 0, 0)
13	1166400	$\frac{16807}{2592}$	(3, 2, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0)
14	7563150	7	(3, 3, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0)
15	52942050	$\frac{262144}{36015}$	(3, 3, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0)
16	385351680	8	(4, 3, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0)
17	3082813440	$\frac{531441}{65536}$	(4, 3, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0)
18	24998984640	9	(4, 3, 2, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)
19	224990861760	9	(4, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)
20	2024917755840	$\frac{5000000}{531441}$	(4, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)
21	19051200000000	10	(4, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)
22	190512000000000	$\frac{214358881}{21000000}$	(4, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)
23	1944663768432000	11	(4, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
24	21391301452752000	$\frac{214990848}{19487171}$	(4, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
25	235998033739776000	12	(4, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
26	2831976404877312000	12	(4, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
27	33983716858527744000	$\frac{10604499373}{859963392}$	(4, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
28	419064703766444736000	13	(4, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
29	5447841148963781568000		(4, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)

Table 2: The maximal coefficients of p_n and their consecutive quotients. The last column lists a monomial of p_n whose coefficient is equal to m_n.

n	$(a_1, \ldots, a_n)$
2	(2, 0)
5	(3, 1, 1, 0, 0)
6	(3, 1, 1, 1, 0, 0)
12	(3, 3, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0), (4, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0)
13	(4, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0), (3, 3, 2, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)
14	(4, 2, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0)
15	(4, 2, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0)

Table 3: Additional monomials of p_n whose coefficients are equal to m_n for $n < 30$.

n	s_n	$\frac{s_{n+1}}{s_n}$
30	5447841148963781568000	$\frac{289254654976}{22024729467}$
31	71547458245452693504000	14
32	1001664415436337709056000	14
33	14023301816108727926784000	$\frac{8649755859375}{578509309952}$
34	209673612792393750000000000	$\frac{962072674304}{64072265625}$
35	3148339635292233982279680000	16
36	50373434164675743716474880000	16
37	805974946634811899463598080000	$\frac{582622237229761}{35184372088832}$
38	13346235805315454304418995840000	17
39	226886008690362723175122929280000	$\frac{46273255257047040}{2638936015687741}$
40	3978404225024620607607354163200000	18
41	71611276050443170936932374937600000	$\frac{42052983462257059}{2313662762852352}$
42	1301601882440198387257545835699200000	19
43	24730435766363769357893370878284800000	19
44	469878279560911617799974046687411200000	$\frac{327680000000000000000}{16400663550280253 01}$
45	9388017391765133721600000000000000000000	20
46	187760347835302674432000000000000000000000	$\frac{6812231858295168230 1}{3276800000000000000 0}$
47	3903402780908908402684872776385162240000000	21
48	81971458399087076456382328304088407040000000	$\frac{68440034007706025984}{3243919932521508681}$
49	1729429060270920090607276147799029186560000000	22
50	38047439325960241993360075251578642104320000000	22
51	837043665171125323853921655534730126295040000000	$\frac{1410500395606629689261 03}{615960306069354233856 0}$
52	19167638064180062036564117435227926126077952000000	23
53	440855675476141426840974701010242300899792896000000	$\frac{1453988974913412787077 12}{613261041568099864896 1}$
54	10452307389872489948744789613053673200242655232000000	24
55	250855377356939758769874950713288156805823725568000000	$\frac{5820766091346740722656 25}{242331495818902131179 52}$
56	6025508444195271521816253662109375000000000000000000000	25
57	150637711104881788045406341552734375000000000000000000000	$\frac{411676753769725624766643 2}{162981450557708740234375}$
58	3804975577941702545004131009801025732678899466240000000000	26
59	98929365026484266170107406254826669049651386122240000000000	26
60	2572163490688590920422792562625493395290936039178240000000000	$\frac{1570042899082081611640534563}{589590204000278377288171 52}$
61	68495151317539451014974372699807632703067401288427560000000000	27
62	1849369085573565177404308062894806082982819834787544120000000000	$\frac{4386626230041171804059126988 8}{1570042899082081611640534563}$
63	51670504955929472544522433459271401555552182671108997120000000000	28
64	1446774138766025231246628136859599243555461114791051919360000000000	$\frac{1769945761511097531977866404 01}{626660890005881686294160998 4}$
65	40862798295083655283632061806834793855899184040599487771040000000000	29
66	1185021150557426003225329792398209021821076337177385145360160000000000	29
67	34365613366165354093534563979548061632811213778144169215444640000000000	$\frac{523017660150000000000000000000}{176994576151109753197786640401}$
68	1015501326834227574804448936477849769406960000000000000000000000000000000	30
69	3046503980502682724413346809433549308220880000000000000000000000000000000	$\frac{645590698195138073036733040138561}{2092070640600000000000000000000}$
70	940118652620125224647533652879072642271519651664470795263552602800000000000	31
71	2914367823122388196407354323925125191041710920159859465317013068680000000000000	$\frac{649037107316853453566312041152512}{2082550639339155074312042064963 1}$
72	908277006976838893495396839720732975992938383907565812097329240473600000000000000	32
73	29064864223258844591852698871063455231774028285042105987114535695155200000000000	32
74	930075655144283026939286363874030567416768905121347391587665142224966400000000000	$\frac{2781855434090103443811378243892171521}{851658379257333641101545081499811 84}$
75	30379974863093055499068023870097652840716436890381792175415887077743001600000000000000	33
76	1002539170482070831469244787713222543743642417382599141788724273565519052800000000000	$\frac{2847501839779123940187735784914157568}{84298649517881922539738734663399137}$
77	33864506118722234127534503334495450730573121152148488603345499618079355699200000000000	34
78	1151393208036555960336173113372845324839486119173048612513746987014698093772800000000000	$\frac{399669593472470313551127910614013671875}{115575074673387971689972805387692277 76}$
79	39816271517312974256100509002896515744268277555434859054687500000000000000000000000000000	35
80	1393569503105954098963517815101378051049389714440220066914062500000000000000000000000000000	$\frac{40414063873238203032156980022826 8146688}{114191312420705803871750831604003906 25}$
81	49320570642735368629992208025957484343031663764994032732691353227472207179939840000000000000000	36
82	1775540543138473270679719488934469436349139895539785178376888716188999458477834240000000000000000	36
83	63919459552985037744469901601640899708569036239432266421567993782803980505202032640000000000000000	$\frac{593259669852236877995997343980715813276 09}{161656255492952812128627920091307258675 2}$
84	2345769877936484815469854363747603640274357258394513512999736861193926236361921658880000000000000000	37
85	86793485483649938172384611458661334690151218560596999980990263864175270745391101378560000000000000000	$\frac{3680781460659747283443907150184716949387 6736}{970059730434063003209671332725224505491985}$
86	3293280220294377971347467731349401295685328549014404380417498855020965676295207000211456000000000000000000	38
87	125144648371186362911203773791277249236042484862547366455864956490796695699217866008035328000000000000000000	$\frac{909377887614652551975371341130628025063 9479}{237357799848111043775597979581466351239168}$
88	4794608647994409038997330943549548574735166856026307562390437024839722999393985472296755584000000000000000000	39
89	186989737271781952520895906798432394414671507385025994933227043968749196976365433419573467776000000000000000000	$\frac{3602879701896396800000000000000000000000}{9093778876146525519753713411306280250639479}$
90	7408378167700986380828325883329584618472057265089496740817451089920000000000000000000000000000000000000000	40
91	296335126708039455233133035333183384738882290603579869632698043596800000000000000000000000000000000000000000	40
92	11853405068321578209325321413327335389555291624143194785307921743872000000000000000000000000000000000000000000	$\frac{5898367729974440156007411567235998189066906676 1}{144115188075855872000000000000000000000000000}$
93	485137915571417159462028231314994899343248266747019135549714969297846930802852529386687060879513600000000000000000	41
94	19890654538428103537943157483914790873073178936627784557538313741211724162916953704854169496060057600000000000000000	$\frac{111527955370179152517685615438885015802763938554183 68}{26718886503646412145718939349387945455230518107772 5}$
95	830260659007146007772745689466656164663688800918177680838519253171995494099296780078316846245644075008000000000000000	42
96	34870947678300132326455318957599558915874929638563462595217808633223810752170464763289307542317051150336000000000000000	$\frac{1009377610923155579774054111680520981491981129024 9}{237279062189656720523796155594134696283911225344}$
97	1483399063252167028338360029439804956347076714866041438488642548695961414227705961250855012767516543942656000000000000000	43
98	6378615971984318221854948126591161312292429873923978185501162959392634081179135633378676554900321138953420800000000000000	$\frac{885182809206724429753029845421970028406525755654144}{20422291197747566381475048306094261718558687959341}$
99	276474424454132753982878286719696808687914519530412428664492321021500524235186979770138014266402785677488947200000000000000	44
100	12164874675981841175246644615666659582268238859338146861237662124946023066348227109886072627721722569809513676800000000000000	

Table 4: $s_{30}, \ldots, s_{100}$ returned by Algorithm 1 and their consecutive quotients.

4 Acknowledgments

We thank the anonymous referee for the careful reading of the manuscript and for the helpful suggestions.

References

[1] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1984.

[2] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, MIT Press, 2009.

[3] D. DeTemple and W. Webb, *Combinatorial Reasoning: An Introduction to the Art of Counting*, John Wiley & Sons, 2014.

[4] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[5] S. Fried, On the restrictiveness of the usual stochastic order and the likelihood ratio order, *Statist. Probab. Lett.* **170** (2021), Article 109012.

[6] G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, *Proc. Lond. Math. Soc.* **17** (1918), 75–115.

[7] H. Rademacher, On the partition function $p(n)$, *Proc. Lond. Math. Soc.* **43** (1938), 241–254.

[8] J. O. Shallit, Elementary Problem E2972: Another appearance of the Catalan numbers, *Amer. Math. Monthly* **93** (1982), 217–218.

[9] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

[10] Y. V. Uspensky, Asymptotic expressions of numerical functions occurring in problems concerning the partition of numbers into summands, *Bull. Acad. Sci. de Russie* **14** (1920), 199–218.

2010 *Mathematics Subject Classification*: Primary 05A99; Secondary 11P81, 90C27.

Keywords: Catalan number, combinatorial optimization problem, integer partition, multivariate polynomial.

(Concerned with sequences A000009, A000108, A000142, A001563, A052571, A062119, A144186, A144187, A299504, A347917 and A349404.)

Received November 20 2021; revised versions received November 21 2021; March 28 2022. Published in *Journal of Integer Sequences*, March 30 2022.

Return to Journal of Integer Sequences home page.