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Convolutions of Sequences with Similar Linear Recurrence Formulas

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Abstract

We show that a collection of convolution formulas involving the Fibonacci, Tribonacci, Pell, Jacobsthal, Narayana, and other number sequences can all be derived from a single elegant formula so long as the sequences follow certain mild requirements. We give many examples, both old and new.

1 Introduction

Let us dive right into some interesting convolution formulas. From Szakács [12, Corollary 2.4] and two years later in Koshy and Griffiths [10, equation (2.2)] we have this lovely convolution formula that connects the Fibonacci numbers F_n (A000045 in the On-Line Encyclopedia of Integer Sequences [7]) with the seemingly-unrelated Jacobsthal numbers J_n (A001045),

$$\sum_{i=0}^{n} F_i J_{n-i} = J_{n+1} - F_{n+1}. \tag{1}$$

This formula can also be found at A094687.

From Frontczak [5, Theorem 2.1] we have another lovely formula, this one with the Fibonacci numbers and the Tribonacci numbers T_n (A000073) except that our Tribonacci numbers start with $T_0 = 0, T_1 = 1, T_2 = 1$,

$$\sum_{i=0}^{n} F_i T_{n-i} = T_{n+2} - F_{n+2}. \tag{2}$$

The terms $T_{n+2} - F_{n+2}$ appear at sequence A000100.

Dresden and Wang [4] noted the similarity between formulas 1 and 2, and asked if there were other convolution formulas of this type. A bit of investigation reveals quite a few more, such as this next little gem that involves just the Fibonacci numbers,

$$\sum_{i=0}^{n} F_{3i} F_{n-i} = \frac{1}{3}\left(F_{3n} - 2F_n\right). \tag{3}$$

The above formula can be found at A049674.

Here is an adorable equation that connects the Narayana numbers N_n (A000930) and the triangular numbers t_n (A000217),

$$\sum_{i=1}^{n} N_{3i-1} t_{n-i} = N_{3n-1} - t_n. \tag{4}$$

We believe this formula is new; we added it to the OEIS at A350498.

This next one gives us another surprising connection (first proved by Szakács [12, Corollary 2.7]), this time between the Jacobsthal numbers and Pell numbers (A000129),

$$\sum_{i=0}^{n} J_i P_{n-i} = \frac{1}{2}\left(P_{n+1} + P_n - J_{n+2}\right). \tag{5}$$

Here are some formulas with Fibonacci numbers and with powers that all seem rather similar:

$$\sum_{i=0}^{n} F_i \, 1^{n-i} = F_{n+2} - 1 \cdot 1^{n+1} \tag{6}$$

$$\sum_{i=0}^{n} F_{2i} \, 3^{n-i} = 1 \cdot 3^{n+1} - F_{2(n+2)} \tag{7}$$

$$\sum_{i=0}^{n} F_{3i} \, 4^{n-i} = F_{3(n+2)} - 2 \cdot 4^{n+1}. \tag{8}$$

The general rule is

$$\sum_{i=0}^{n} F_{ki} \left(L_k\right)^{n-i} = (-1)^k \left(F_k \cdot \left(L_k\right)^{n+1} - F_{k(n+2)}\right). \tag{9}$$

It should not come as a surprise that we can play a similar game with the Pell numbers:

$$\sum_{i=0}^{n} P_i \, 2^{n-i} = P_{n+2} - 1 \cdot 2^{n+1} \tag{10}$$

$$\sum_{i=0}^{n} P_{2i} \, 6^{n-i} = 2 \cdot 6^{n+1} - P_{2(n+2)} \tag{11}$$

$$\sum_{i=0}^{n} P_{3i} \, 14^{n-i} = P_{3(n+2)} - 5 \cdot 14^{n+1}. \tag{12}$$

The general rule for these is

$$\sum_{i=0}^{n} P_{ki} \, (Q_k)^{n-i} = (-1)^k \left(P_k \cdot (Q_k)^{n+1} - P_{k(n+2)} \right), \tag{13}$$

where the Q_n's are the Pell-Lucas numbers A002203. For k odd, these Fibonacci and Pell formulas can also be derived from a combinatorial formula by Benjamin and Quinn [2, Identity 99].

At this point, it is probably not surprising that we can also find convolution formulas for Fibonacci numbers and Pell numbers together, such as the well-known formula

$$\sum_{i=0}^{n} F_i P_{n-i} = P_n - F_n \tag{14}$$

and the less familiar formula

$$\sum_{i=0}^{n} F_{3i} P_{n-i} = F_{3n}/2 - P_n. \tag{15}$$

With not much effort, we can prove a general formula for the Fibonacci and Pell numbers involving the Lucas numbers L_n (A000032) and the Pell-Lucas numbers Q_n,

$$\sum_{i=0}^{n} F_{ki} P_{m(n-i)} = \frac{1}{L_k - Q_m} \left(F_{kn} P_m - F_k P_{mn} \right), \qquad \text{for } k, m \text{ same parity.} \tag{16}$$

(It takes a bit more effort to show that $L_k = Q_m$ only for $k = 0$ and $m = 0, 1$; this follows from a procedure of Alekseyev [1, eq. (10)] which gives us the Diophantine system $x^2 - 5y^2 = \pm 4$ and $x^2 - 8z^2 = \pm 4$ which by a bit of algebra and the method of infinite descent gives only $(x, y, z) = (2, 0, 0)$ as the single non-negative solution and hence $L_k = Q_m$ only when both equal 2.)

Along the lines of equation (16), we have another general convolution formula, this one just involving the Fibonacci numbers which is a nice generalization of equation (3) from earlier:

$$\sum_{i=0}^{n} F_{ki}F_{m(n-i)} = \frac{1}{L_k - L_m}\left(F_{kn}F_m - F_kF_{mn}\right), \qquad \text{for } k \neq m \text{ but of same parity.} \tag{17}$$

It should not come as a surprise to learn that we have a version of equation (17) but for the Pell numbers in place of the Fibonacci numbers,

$$\sum_{i=0}^{n} P_{ki}P_{m(n-i)} = \frac{1}{Q_k - Q_m}\left(P_{kn}P_m - P_kP_{mn}\right), \qquad \text{for } k \neq m \text{ but of same parity.} \tag{18}$$

We note that for k, m both odd, then equations (16), (17), and (18) can all be derived from a theorem of Bramham and Griffiths [3, Theorem 3.1], but we believe the extension to when k, m are both even is original to this paper.

Finally, here is a rather complicated formula that does not seem related to the others (but as we will reveal later it is indeed in the same family),

$$\sum_{i=0}^{2n}(-1)^i F_iF_{i+1}F_{2n-i-1} \;=\; \frac{2}{3}\sum_{i=0}^{n} F_{2i-1}F_{4(n-i)} \;=\; \frac{1}{2}\left(F_{4n+1} - F_{2n-1}\right). \tag{19}$$

This formula is completely new, and it illustrates how our approach can produce novel results.

The right-hand sides of all these equations (1) through (19) exhibit a striking pattern. Is there a single theorem that will allow us to derive all these formulas? Of course there is, and we give two proofs of this theorem, one using generating functions and the other using a tiling argument. Our work follows closely the tiling techniques studied by Benjamin and Quinn [2] and elaborated by Bramham and Griffiths [3], and also the generating function techniques illustrated by Wilf [13].

We should note that in general, most convolution formulas are not as elegant as the ones seen above. For example, the convolution of F_n with itself is the rather unpleasant $((n-1)F_n + 2nF_{n-1})/5$, and for the convolution of $(F_n)^2$ with F_{3n} we have the quite unsavory equation

$$\sum_{i=0}^{n}(F_i)^2F_{3(n-i)} = \frac{1}{11}\Big(F_{3n} + L_{3n+1} - 5(F_n)^2 - (F_{n+1})^2 - 3F_nF_{n+1}\Big).$$

By comparison, our convolution formulas (1) through (19) are delightfully simple and graceful, with all but equation (5) involving just two terms. Furthermore, they all derive from a single theorem, as we now explain.

2 Definitions and the main theorem

To begin, we suppose that $(a_n)_{n\geq 0}$ is a linear recurrence sequence with initial values $a_n = 0$ for $n \leq 0$ and $a_1 = 1$, and with recurrence formula

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_L a_{n-L} \tag{20}$$

for $n > 1$, with not all c_i equal to zero. We say that this sequence has *signature* $\{c_1, c_2, \ldots, c_L\}$ and has *order* L. It is not hard to show that this sequence (a_n) has generating function $A(x)$, where

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{x}{1 - c_1 x - c_2 x^2 - \cdots - c_L x^L}. \tag{21}$$

As an example, the Fibonacci sequence $(F_n)_{n\geq 0}$ has signature $\{1, 1\}$, order 2, and generating function $x/(1 - x - x^2)$.

Next, we suppose that (b_n) is another linear recurrence sequence with the same initial values as (a_n) and whose signature is $\{c_1, c_2, \ldots, (c_j + d_j), \ldots, c_L\}$, differing from the signature for (a_n) only in position j by a non-zero amount d_j. It is convenient but not necessary to assume that $d_j > 0$. The sequence (b_n) has generating function $B(x)$, where

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = \frac{x}{1 - c_1 x - c_2 x^2 - \cdots - (c_j + d_j)x^j - \cdots - c_L x^L}. \tag{22}$$

With all this in mind, we have the following theorem.

Theorem 1. *For any two sequences (a_n) and (b_n) as defined above with initial values of 0 for $n \leq 0$ and 1 for $n = 1$ and whose recurrence signatures differ in only the j^{th} term by an amount d_j, we have the convolution formula*

$$\sum_{i=0}^{n} a_i b_{n-i} \;=\; \sum_{i=0}^{n} b_i a_{n-i} \;=\; \frac{1}{d_j}\Big(b_{n+j-1} - a_{n+j-1}\Big). \tag{23}$$

We note that Szakács [12, Theorem 2.1] covers our Theorem 1 but only for second-order linear recurrences. In the next two sections, we present two separate proofs of our theorem.

3 Proving the convolution theorem using generating functions

Recall that we have

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{x}{1 - c_1 x - c_2 x^2 - \cdots - c_L x^L},$$
$$B(x) = \sum_{n=0}^{\infty} b_n x^n = \frac{x}{1 - c_1 x - c_2 x^2 - \cdots - (c_j + d_j)x^j - \cdots - c_L x^L}.$$

A bit of algebra (and a common denominator) reveals that

$$B(x) - A(x) \;=\; d_j x^{j-1} A(x) B(x),$$

which we simplify to

$$A(x)B(x) \;=\; \frac{1}{d_j}\left(\frac{B(x)}{x^{j-1}} - \frac{A(x)}{x^{j-1}}\right). \tag{24}$$

Here we make two crucial observations. First, we note that $A(x)B(x)$ is the generating function for the convolution sequence $\left(\sum_{i=0}^{n} a_i b_{n-i}\right)_{n\geq 0}$ which if we replace i with $n-i$ gives us the identical expression $\left(\sum_{i=0}^{n} b_i a_{n-i}\right)_{n\geq 0}$, and second, we note that dividing a generating function by a power of x shifts the index in the positive direction by that same power. In other words,

$$\frac{A(x)}{x^{j-1}} \;=\; \sum_{n=-(j-1)}^{\infty} a_{n+(j-1)} x^n,$$

with a similar statement for $B(x)/x^{j-1}$. Hence, equation (24) can be rewritten in terms of coefficients as

$$\sum_{i=0}^{n} a_i b_{n-i} = \frac{1}{d_j}\Big(b_{n+j-1} - a_{n+j-1}\Big) \qquad \text{for all } n \geq 0.$$

Therefore, we have proved Theorem 1.

4 Proving the convolution theorem using tilings

Note that tiling proofs for convolution formulas are quite common. Equation (6) is the very first identity in Benjamin and Quinn's book [2]. Versions of Theorem 1 for degree-two recurrence sequences can be found in that same book [2, Identities 99, 100] and also in an article by Bramham and Griffiths [3, Theorems 3.1, 3.2], and a limited version for degree three also appears in Benjamin and Quinn [2, p. 47, Ex. 4].

To begin our tiling proof, we turn once more to Benjamin and Quinn [2, Section 3.1], where we find the following theorem:

Theorem 2 (Benjamin, Quinn)**.** *Let $c_1, c_2, \ldots c_L$ be nonnegative integers, and let $a_0, a_1, \ldots$ be the sequence of numbers defined by the recurrence*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_L a_{n-L}$$

with "ideal" initial conditions $a_1 = 1$ and $a_n = 0$ for $n \leq 0$. Then for all $n \geq 1$, a_n counts the **colored tilings** *of a board of length $n-1$, using tiles of length at most L, where for $1 \leq i \leq L$ each tile of length i is assigned one of c_i colors.*

With this in mind, we can now prove Theorem 1 for two sequences (a_n) and (b_n) in the case that all their associated constants are non-negative integers. (Benjamin and Quinn [2, Section 3.5] also discussed how to extend tiling proofs to deal with complex coefficients, but we will leave that as an exercise for the reader.)

From Theorem 2 and our definitions of (a_n) and (b_n) we know that a_n measures the number of ways to tile a board of length $n-1$ and that b_n measures the same except that the tiles of length j (henceforth called j*-minos*) have an additional d_j colors.

We now consider a board of length $n+j-2$ and we ask, how many ways are there to tile this board with our colored tiles of length 1 through L, and using at least one of the "additional" d_j colors for the j-mino?

On the one hand, by Theorem 2 there are b_{n+j-1} ways using all c_j+d_j colors for the j-mino and a_{n+j-1} ways that do not use the additional d_j colors, so their difference $b_{n+j-1}-a_{n+j-1}$ gives us the desired count.

On the other hand, we can condition on the location of the last j-mino that uses one of the additional d_j colors (we are given that there must be at least one such j-mino). We can think of this j-mino as stretching from position i to position $i+j-1$ on our board of length $n+j-2$ as i ranges from 1 to $n-1$; see Figure 1.

Figure 1: Conditioning on the location of the last j-mino.

At each position of the j-mino, there would be b_i possible tilings for the board of length $i-1$ to the left of the j-mino (these tilings could use one of these "additional" d_j colors) and a_{n-i} possible tilings for the board of length $n-i-1$ to the right (these tilings would not use one of the additional colors), and of course the j-mino itself has d_j possible colors. This gives us $b_i \cdot d_j \cdot a_{n-i}$ tilings for this configuration, and summing over i from 1 to $n-1$ gives

$$d_j \cdot \sum_{i=1}^{n-1} b_i a_{n-i},$$

and since $a_0 = b_0 = 0$ by our initial conditions, we can extend the sum from 0 to n without changing its value, giving us

$$d_j \cdot \sum_{i=0}^{n} b_i a_{n-i}$$

total number of tilings. Comparing this with our expression $b_{n+j-1}-a_{n+j-1}$ from earlier will give us our desired statement.

5 Examples

We now show how all our equations (1) through (19) follow from Theorem 1.

For our first equation (1), we recall that the generating functions for the Fibonacci numbers and Jacobsthal numbers are

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$$
$$\sum_{n=0}^{\infty} J_n x^n = \frac{x}{1-x-2x^2},$$

and so they have signatures $\{1,1\}$ and $\{1,2\}$ respectively. We can now apply Theorem 1 with $a_n = F_n$, $b_n = J_n$, $j = 2$, and $d_j = 1$ to obtain

$$\sum_{i=0}^{n} F_i J_{n-i} = J_{n+1} - F_{n+1},$$

which is our equation (1).

Moving on to equation (2) with the Fibonacci numbers and Tribonacci numbers, we recall that our definition of the Tribonacci numbers has them starting with $T_0 = 0$ and $T_1 = T_2 = 1$ so as to match the requirements for initial conditions in Theorem 1. These two sequences have signature $\{1,1\}$ and $\{1,1,1\}$ respectively, and so we use $a_n = F_n$ and $b_n = T_n$, which means $j = 3$ and $d_j = 1$ and so equation (23) of Theorem 1 gives us

$$\sum_{i=0}^{n} T_i F_{n-i} = T_{n+2} - F_{n+2},$$

which is a perfect match for our equation (2).

For equation (3) on two Fibonacci sequences, the sequence $(F_{3n})_{n\geq 0}$ presents a bit of a problem since its generating function is

$$\sum_{n=0}^{\infty} F_{3n} x^n = \frac{2x}{1-4x-x^2}$$

and the numerator does not match the numerators in equations (21) and (22). We simply divide by 2 to obtain

$$\sum_{n=0}^{\infty} (F_{3n}/2) x^n = \frac{x}{1-4x-x^2},$$

and so we now use $a_n = F_n$, $b_n = F_{3n}/2$, $j = 1$, and $d_j = 3$ and so Theorem 1 gives us

$$\sum_{i=0}^{n} (F_{3i}/2) F_{n-i} = \frac{1}{3}\left((F_{3n}/2) - F_n\right),$$

and we simply multiply through by 2 to obtain equation (3). We will use this technique again in what follows, and in particular when we prove the general Fibonacci convolution formula (17).

To derive equation (4) involving the triangular numbers and Narayana's numbers, we begin with two generating functions,

$$\frac{x}{(1-x)^3} = \frac{x}{1-3x+3x^2-x^3} \qquad \text{and} \qquad \frac{x}{1-4x+3x^2-x^3}. \tag{25}$$

The first generating function gives us the triangular numbers $0, 1, 3, 6, 10, 15, \ldots$ from A000217. The second gives us the sequence $0, 1, 4, 13, 41, 129$ which (aside from the first term) is a trisection of Narayana's numbers as seen at A052529. To be precise, if we let a_n and b_n represent these two sequences then $a_n = t_n$ for all n, and $b_n = N_{3n-1}$ for $n > 0$ (with $b_0 = 0$). We can now apply Theorem 1 with $j = 1$ and $d_j = 1$, dropping the first term $b_0 a_n$ from the sum, to obtain

$$\sum_{i=1}^{n} b_i a_{n-i} = b_n - a_n, \tag{26}$$

and we replace a_n and b_n with t_n and N_{3n-1} to obtain our equation (4).

To obtain equation (5) with the Jacobsthal and the Pell sequences, we begin with the Lichtenberg numbers $(\ell_n)_{n\geq 0}$ (A000975) as discussed by Hinz [6]. These have generating function

$$\sum_{n=0}^{\infty} \ell_n x^n = \frac{x}{1-2x-x^2+2x^3} \tag{27}$$

and so when we look at

$$\sum_{i=0}^{n} \ell_i P_{n-i}$$

we see that we can apply Theorem 1 if we let $a_n = \ell_n$, $b_n = P_n$, $j = 3$, and $d_j = 2$, giving us

$$\sum_{i=0}^{n} \ell_i P_{n-i} = \frac{1}{2}\left(P_{n+2} - \ell_{n+2}\right). \tag{28}$$

From A000975 we learn that $\ell_n = J_{n+1} - \pi_{n+1}$ where π_n is $n \bmod 2$, the parity of n. We substitute this into equation (28) to obtain

$$\sum_{i=0}^{n} (J_{i+1} - \pi_{i+1}) P_{n-i} = \frac{1}{2}\left(P_{n+2} - J_{n+3} + \pi_{n+3}\right). \tag{29}$$

If we separate the sum on the left, the above equation becomes

$$\sum_{i=0}^{n} J_{i+1} P_{n-i} \;-\; (P_n + P_{n-2} + P_{n-4} + \cdots) = \frac{1}{2}\left(P_{n+2} - J_{n+3} + \pi_{n+3}\right). \tag{30}$$

From an article by Koshy and Pell [8, Section 10.2] we learn that $(P_n + P_{n-2} + P_{n-4} + \cdots)$ equals $(P_{n+1} - \pi_{n+1})/2$, and so moving this over to the right of equation (30) gives us

$$\sum_{i=0}^{n} J_{i+1} P_{n-i} = \frac{1}{2}\left(P_{n+2} - J_{n+3} + \pi_{n+3} + P_{n+1} - \pi_{n+1}\right), \tag{31}$$

and since $\pi_{n+3} = \pi_{n+1}$ then all this simplifies to

$$\sum_{i=0}^{n} J_{i+1} P_{n-i} = \frac{1}{2}\left(P_{n+2} + P_{n+1} - J_{n+3}\right). \tag{32}$$

The expression on the left is $J_1 P_n + J_2 P_{n-1} + \cdots + J_{n+1} P_0$. Since $J_0 = 0$ we can add $J_0 P_{n+1}$ in front without changing the value, giving us $\sum_{i=0}^{n+1} J_i P_{n+1-i}$ on the left of equation (32). We now replace all n's with $n-1$'s in (32) to obtain

$$\sum_{i=0}^{n} J_i P_{n-i} = \frac{1}{2}\left(P_{n+1} + P_n - J_{n+2}\right), \tag{33}$$

which is equation (5).

Moving on to equations (6) through (9) on the convolution of powers and Fibonacci numbers, we need the identity

$$F_{k(n+2)} = L_k F_{k(n+1)} - (-1)^k F_{kn}$$

which Koshy [9, p. 112] credits to Ruggles [11]. This identity, along with the initial values of $F_{k0} = 0$ and $F_{k1} = F_k$, tells us that the generating function for $(F_{kn})_{n \geq 0}$ is

$$\sum_{n=0}^{\infty} F_{kn} x^n = \frac{F_k x}{1 - L_k x + (-1)^k x^2}. \tag{34}$$

As for the sequence $0, 1, L_k, (L_k)^2, (L_k)^3, \ldots$, this has generating function $x/(1 - L_k x)$, which means to apply Theorem 1 we would have to use $b_0 = 0$ and $b_n = (L_k)^{n-1}$ to have $\sum_{n=0}^{\infty} b_n x^n$ match the coefficients of $x/(1 - L_k x)$. For a_n we let $a_n = F_{kn}/F_k$, and then we assign $j = 2$ and $d_j = (-1)^k$ to obtain from Theorem 1 the intermediate step

$$\sum_{i=0}^{n} (F_{ki}/F_k) b_{n-i} = \frac{1}{(-1)^k}\left(b_{n+1} - (F_{k(n+1)}/F_k)\right). \tag{35}$$

Since $b_0 = 0$ and $b_n = (L_k)^{n-1}$ for $n > 0$, the sum on the left of equation (35) doesn't have an nth term and so we really have

$$\sum_{i=0}^{n-1} (F_{ki}/F_k)(L_k)^{n-1-i} = \frac{1}{(-1)^k}\left((L_k)^n - (F_{k(n+1)}/F_k)\right). \tag{36}$$

We now replace n with $n+1$, multiply both sides by F_k, and rewrite $1/(-1)^k$ as just $(-1)^k$, to obtain

$$\sum_{i=0}^{n} F_{ki}\,(L_k)^{n-i} = (-1)^k \left(F_k \cdot (L_k)^{n+1} - F_{k(n+2)}\right)$$

which is our desired equation (9), and this covers (6) through (8) by replacing k with $1, 2$, and 3 respectively.

As for equations (10) through (13) on powers and Pell numbers, we need the identity

$$P_{k(n+2)} = Q_k P_{k(n+1)} - (-1)^k P_{kn}$$

which can be obtained from Koshy's book on Pell numbers [8, (7.20)] although Koshy's definition of Q_n is slightly different from ours. This identity, along with the initial values of $P_{k0} = 0$ and $P_{k1} = P_k$, tells us that the generating function for $(P_{kn})_{n\geq 0}$ is

$$\sum_{n=0}^{\infty} P_{kn} x^n = \frac{P_k x}{1 - Q_k x + (-1)^k x^2}. \tag{37}$$

From here, the procedure is identical to the one before, except that here we use P_{kn}/P_k instead of F_{kn}/F_k and we use Q_k in place of L_k. This gives us equation (13), which covers (10) through (12) by replacing k with $1, 2$, and 3 respectively.

At this point, we have already done most of the work to establish equation (16) on the convolution of Fibonacci numbers and Pell numbers. Thanks to equations (34) and (37), we know that the signature for $(F_{kn}/F_k)_{n\geq 0}$ is $\{L_k, -(-1)^k\}$ and for $(P_{mn}/P_m)_{n\geq 0}$ is $\{Q_m, -(-1)^m\}$. So long as k and m have the same parity, we can apply Theorem 1 with $a_n = P_{mn}/P_m$, $b_n = F_{kn}/F_k$, $j = 1$, and $d_j = L_k - Q_m$. After multiplying both sides by $F_k \cdot Q_m$, we have our formula (16), which will then give us (14) and (15) with appropriate choices for k and m.

Moving on, let us discuss equation (17) on the convolution of two Fibonacci sequences. From our discussion earlier, we know that the signature for $(F_{kn}/F_k)_{n\geq 0}$ is $\{L_k, -(-1)^k\}$ and likewise for $(F_{mn}/F_m)_{n\geq 0}$ it is $\{L_m, -(-1)^m\}$. So long as $k \neq m$ have the same parity, we can use Theorem 1 with $j = 1$, $a_n = F_{mn}/F_m$, $b_n = F_{kn}/F_k$, and $d_j = L_k - L_m$ to obtain equation (17). The identical technique applies to equation (18) with the convolution of two Pell sequences and so we will not elaborate further on that.

Finally, we will derive equation (19), which we repeat here:

$$\sum_{i=0}^{2n} (-1)^i F_i F_{i+1} F_{2n-i-1} \;=\; \frac{2}{3} \sum_{i=0}^{n} F_{2i-1} F_{4(n-i)} \;=\; \frac{1}{2}\left(F_{4n+1} - F_{2n-1}\right). \tag{38}$$

We begin with the sum on the far left. We note that if we let $a_n = (-1)^{n-1} F_{n-1} + 1$ and $b_n = F_n F_{n+1}$ then the two sequences have generating functions $x/(1 - 2x^2 + x^3)$ and $x/(1 - 2x - 2x^2 + x^3)$ respectively, so we can apply Theorem 1 with $j = 1$ and $d_j = 2$ to obtain

$$\sum_{i=0}^{n} F_i F_{i+1} \left((-1)^{n-i-1} F_{n-i-1} + 1\right) = \frac{1}{2}\left(F_n F_{n+1} - ((-1)^{n-1} F_{n-1} + 1)\right). \tag{39}$$

We split up the sum on the left, and replace all n's with $2n$'s, to get

$$\sum_{i=0}^{2n} (-1)^{i+1} F_i F_{i+1} F_{2n-i-1} \;+\; \sum_{i=0}^{2n} F_i F_{i+1} \;=\; \frac{1}{2}\left(F_{2n} F_{2n+1} + F_{2n-1} - 1\right). \tag{40}$$

Next, from Koshy [9, p. 109] we learn that $\sum_{i=0}^{2n} F_i F_{i+1}$ equals $F_{2n+1}^2 - 1$, so moving that over to the right we have

$$\sum_{i=0}^{2n} (-1)^{i+1} F_i F_{i+1} F_{2n-i-1} \;=\; \frac{1}{2}\left(F_{2n} F_{2n+1} + F_{2n-1} - 1 - 2F_{2n+1}^2 + 2\right), \tag{41}$$

and after some tedious calculations the right-hand side simplifies to $\left(F_{2n-1} - F_{4n+1}\right)/2$, and so after multiplying everything by -1 we obtain our desired formula.

For the sum in the middle of equation (38), we note that $F_{2i-1} = F_{2(i+1)} - 2F_{2i}$ so we cam split up this middle sum as

$$\frac{2}{3}\sum_{i=0}^{n} F_{2i-1} F_{4(n-i)} \;=\; \frac{2}{3}\sum_{i=0}^{n} F_{2(i+1)} F_{4(n-i)} - \frac{4}{3}\sum_{i=0}^{n} F_{2i} F_{4(n-i)}. \tag{42}$$

The second sum on the right matches perfectly with equation (17), allowing us to rewrite the above equation as

$$\frac{2}{3}\sum_{i=0}^{n} F_{2i-1} F_{4(n-i)} \;=\; \frac{2}{3}\sum_{i=0}^{n} F_{2(i+1)} F_{4(n-i)} - \frac{1}{3}\left(F_{4n} - 3F_{2n}\right), \tag{43}$$

and for the first sum on the right, we first re-index the sum (replacing i with $i-1$) to get $(2/3)\sum_{i=1}^{n+1} F_{2i} F_{4(n+1-i)}$ and then since $F_{2\cdot 0} = 0$ we can change the starting value from $i = 1$ to $i = 0$ giving us $(2/3)\sum_{i=0}^{n+1} F_{2i} F_{4(n+1-i)}$ and this matches again with equation (17) but using $n+1$ instead of n, giving us $(1/6)\left(F_{4(n+1)} - 3F_{2(n+1)}\right)$. Thus, equation (43) becomes

$$\frac{2}{3}\sum_{i=0}^{n} F_{2i-1} F_{4(n-i)} \;=\; \frac{1}{6}\left(F_{4(n+1)} - 3F_{2(n+1)}\right) - \frac{1}{3}\left(F_{4n} - 3F_{2n}\right), \tag{44}$$

and after combining terms we have

$$\frac{2}{3}\sum_{i=0}^{n} F_{2i-1} F_{4(n-i)} \;=\; \frac{1}{6}\Big(\left(F_{4n+4} - 2F_{4n}\right) - 3\left(F_{2n+2} - 2F_{2n}\right)\Big). \tag{45}$$

Since $F_{4n+4} - 2F_{4n} = 3F_{4n+1}$ and $F_{2n+2} - 2F_{2n} = F_{2n-1}$, we get

$$\frac{2}{3}\sum_{i=0}^{n} F_{2i-1} F_{4(n-i)} \;=\; \frac{1}{6}\Big(3F_{4n+1} - 3F_{2n-1}\Big) \;=\; \frac{1}{2}\left(F_{4n+1} - F_{2n-1}\right), \tag{46}$$

which gives us the second part of equation (38), as desired.

6 Conclusion

What other "universal convolution" formulas can we find? As we noted earlier, Szakács has a version of our Theorem 1 that applies to order-two recurrences with completely different signatures; the formulas are rather complicated when done in such generality. Dresden and Wang [4] have a surprisingly simple universal convolution formula that applies to Fibonacci-type and Lucas-type sequences with the same signatures but different initial values, as follows:

Theorem 3 (Dresden, Wang)**.** *For a given Fibonacci-type sequence* $(\mathcal{F}_n^{(k)})_{n\geq 0}$ *and companion Lucas-type sequence* $(\mathcal{L}_n^{(k)})_{n\geq 0}$ *which count the number of ways to tile a strip and a bracelet respectively with the same collection of single-color tiles of lengths* 1 *through* k*, we have*

$$\sum_{i=0}^{n-1} \mathcal{F}_i^{(k)} \mathcal{L}_{n-i}^{(k)} \;=\; (n-1)\mathcal{F}_n^{(k)}.$$

Going back to our Theorem 1, there are plenty of additional convolutions that we could write down. For example, the Tripell nuumbers (A077939), the Tribonacci numbers, the Padovan numbers (A000931), and the sum of every other Tribonacci number (A113300) have signatures $\{2, 1, 1\}$, $\{1, 1, 1\}$, $\{0, 1, 1\}$ and $\{3, 1, 1\}$ respectively, and so it would be easy to write down convolution formulas for these numbers. Likewise, the Tribonacci numbers and the Tetranacci numbers (A000078) have signatures $\{1, 1, 1\}$ and $\{1, 1, 1, 1\}$ and so they too would have an elegant convolution formula. We also mention the Lucas-Lehmer sequence A107920 with signature $\{1, -2\}$ that can be matched to the Fibonacci numbers (signature $\{1, 1\}$), the Jacobsthal numbers (signature $\{1, 2\}$), and the Mersenne numbers $2^n - 1$ (signature $\{3, -2\}$).

We finish with one last example along these lines. If we define a_n as $F_{n/2}F_{n/2+1}$ for n even, and $(F_{(n+1)/2})^2$ for n odd, then $(a_n)_{n\geq 0}$ is A006498 with indexing offset by 1. We then define U_n to be the Tetranacci numbers with indexing offset by 2, so that $U_n = 0$ for $n \leq 0$ and $U_1 = 1$. These two sequences have signatures $\{1, 0, 1, 1\}$ and $\{1, 1, 1, 1\}$ respectively. Then, Theorem 1 with $2n + 1$ in place of n tells us that

$$\sum_{i=0}^{2n+1} a_i U_{2n+1-i} \;=\; U_{2n+2} - a_{2n+2}. \tag{47}$$

Using our definition of a_n, and splitting up the sum on the left into two sums over even indices and odd indices respectively that we then re-index, we obtain this fascinating relationship between the Fibonacci numbers and the Tetranacci numbers,

$$\sum_{i=0}^{n} F_i F_{i+1} U_{2n-2i+1} + \sum_{i=0}^{n} (F_{i+1})^2 U_{2n-2i} \;=\; U_{2n+2} - F_{n+1}F_{n+2}. \tag{48}$$

We can only imagine that there are many other interesting convolution formulas just waiting to be found.

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