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Permanental Representations of Negatively Subscripted Generalized Order-k Fibonacci Numbers

Hacène Belbachir
USTHB
Faculty of Mathematics
RECITS Laboratory
P. O. Box 32
El Alia, 16111, Algiers
Algeria
hbelbachir@usthb.dz

Ihab-Eddine Djellas
USTHB
Faculty of Mathematics
RECITS Laboratory
P. O. Box 32
El Alia, 16111, Algiers
Algeria
Centre de Recherche sur l'Information Scientifique et Technique–CERIST
Algiers, Algeria
idjellas@usthb.dz

Fariza Krim
Oran 1 University
Faculty of Sciences
GEAN Laboratory
P. O. Box 1524
El Menaouer, 31000, Oran
Algeria USTHB
Faculty of Mathematics
RECITS Laboratory
P. O. Box 32
El Alia, 16111, Algiers
Algeria
fkrim@usthb.dz

Abstract

We introduce an extension to the negative indices of a family of generalized Fibonacci sequences of order-k, and for which we establish recurrence relations. We also give permanental representations of negatively subscripted generalized Fibonacci and Lucas sequences via Hessenberg matrices.

1 Introduction

A family of k sequences of the generalized order-k Fibonacci numbers was defined and studied by Er [3], for $n > 0$ and $1 \leq i \leq k$, as follows:

$$g_n^{(i)} = c_1 g_{n-1}^{(i)} + c_2 g_{n-2}^{(i)} + \cdots + c_k g_{n-k}^{(i)}, \tag{1}$$

where $c_1, c_2, \ldots, c_k$ are real constant coefficients, with $c_k \neq 0$, and for $1 - k \leq n \leq 0$,

$$g_n^{(i)} = \begin{cases} 1, & \text{for } i = 1 - n; \\ 0, & \text{otherwise.} \end{cases}$$

Several sequences were derived from (1). Lee and Lee [6] studied the sequence obtained from (1) by setting $(c_1, c_2, \ldots, c_k) = (1, 1, \ldots, 1)$ and $i = 1$, which is called in the present paper order-k Fibonacci sequence $(\widetilde{g}_n^{(k)})_n$. Also, Lee [8] studied order-k Lucas sequence denoted $(l_n^{(k)})_n$, for $k \geq 2$. The sequence $(l_n^{(k)})_n$ is obtained from (1) by setting $(c_1, c_2, \ldots, c_k) = (1, 1, \ldots, 1)$ with initial conditions $l_0^{(k)} = 2$, $l_j^{(k)} = 2^{j-1}$, for $j = 1, \ldots, k-1$, and $l_k^{(k)} = 2^{k-1} + 1$.

In the present paper, we consider the family of shifted generalized order-k Fibonacci sequences defined as (1), but with initial conditions, for $0 \leq n \leq k - 1$,

$$g_n^{(i)} = \begin{cases} 1, & \text{for } i = k - n; \\ 0, & \text{otherwise.} \end{cases}$$

The *permanent* of a square matrix $A = (a_{ij})_{n \times n}$ is defined as follows:

$$\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation runs over all permutations σ of the symmetric group $\mathfrak{S}_n$ of order n.

Next, we present the definition of matrix contraction; see for instance [2]. Let $A = (a_{ij})$ be an $m \times n$ real matrix with row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say that A is *contractible* on column (resp., row) k if column (resp., row) k contains exactly two nonzero entries. If A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$, then the $(m-1) \times (n-1)$ matrix $A_{ij}^{(k)}$, obtained from A by replacing row i by $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k, is called the contraction of A on column k relative to rows i and j.

Brualdi and Gibson [2] gave the following lemma.

Lemma 1. *Let A be a nonnegative integral matrix of order $n > 1$, and let B be a contraction of A. Then*

$$\operatorname{per} A = \operatorname{per} B.$$

Permanental representations of recurrent sequences have been investigated by several authors. Yilmaz and Bozkurt [11] gave permanental representations of sums of Fibonacci and Lucas numbers via Hessenberg matrices. Lee et al. [7] obtained $g_{n+k-1}^{(k)}$ as the permanent of an $n \times n$ upper triangular $(0,1)$-matrix; that is a matrix whose all entries are either 0 or 1. Lee [8] constructed a $(0,1)$-matrix of order n and established that the permanent of that matrix is the $(n-1)^{\text{th}}$ term $l_{n-1}^{(k)}$ of the order-k Lucas sequence.

The aim of this work is to propose an extension to negative indices of the family of generalized order-k Fibonacci sequences $(g_n^{(i)})_n$, $1 \leq i \leq k$. For instance, extensions to negative indices of order-k Fibonacci, order-k Lucas, order-k Pell, and order-k Jacobsthal are also given. As application, we provide some permanental representations.

The present paper is organized as follows: In Section 2, we give an extension of order-k Fibonacci, order-k Lucas, and the family of generalized order-k Fibonacci sequences $(g_n^{(i)})_n$, $1 \leq i \leq k$, to negatively subscripted indices. We also provide recurrence formulas for the n^{th} term of these negatively subscripted sequences. In Section 3, we present negatively subscripted generalized order-k Fibonacci numbers as the permanent of special Hessenberg matrix. In Section 4, we give more permanental representations of negatively subscripted order-k Fibonacci and order-k Lucas sequences. We conclude by Section 5, where we provide proofs of the results presented in Sections 3 and 4.

2 Extension to negative indices

Extension to negative indices of classical Fibonacci sequence $(F_n)_n$ with initial conditions $F_0 = 0$, $F_1 = 1$ is given for $n \geq 1$, by

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n. \tag{2}$$

The extension to negative indices of classical Lucas sequence $(L_n)_n$ with initial conditions $L_0 = 2$, $L_1 = 1$ is given for $n \geq 1$, by

$$L_{-n} = L_{-n+2} - L_{-n+1} = (-1)^n L_n. \tag{3}$$

Firstly, we propose an extension of order-k Fibonacci sequence $(\widetilde{g}_n^{(k)})_n$ to negatively subscripted indices, for $n \geq 1$, as follows:

$$\widetilde{g}_{-n}^{(k)} = -\widetilde{g}_{-n+1}^{(k)} - \widetilde{g}_{-n+2}^{(k)} - \cdots - \widetilde{g}_{-n+k-1}^{(k)} + \widetilde{g}_{-n+k}^{(k)}. \tag{4}$$

For example, for $k = 2$, $(\widetilde{g}_n^{(2)})_{n\in\mathbb{Z}}$ is the classical Fibonacci sequence.

For $k = 3$, $(\widetilde{g}_n^{(3)})_{n\in\mathbb{Z}}$ is the Tribonacci sequence

$$\ldots, -8, 4, 1, -3, 2, 0, -1, 1, 0, 0, 1, 1, 2, 4, 7, 13, \ldots$$

For $k=4$, $(\widetilde{g}_n^{(4)})_{n\in\mathbb{Z}}$ is the Quadrabonacci sequence

$$\ldots,0,1,-3,2,0,0,-1,1,0,0,0,1,1,2,4,8,15,29,\ldots$$

An extension of order-k Lucas sequence $(l_n^{(k)})_n$ to negatively subscripted indices, for $n\geq 1$, is given by

$$l_{-n}^{(k)}=-l_{-n+1}^{(k)}-l_{-n+2}^{(k)}-\cdots-l_{-n+k-1}^{(k)}+l_{-n+k}^{(k)}. \tag{5}$$

For example, for $k=2$, $(l_n^{(2)})_{n\in\mathbb{Z}}$ is the Lucas sequence.

For $k=3$, $(l_n^{(3)})_{n\in\mathbb{Z}}$ is the Tribonacci-Lucas sequence

$$\ldots,6,-11,6,1,-4,3,0,-1,2,1,2,5,8,15,28,51,\ldots$$

For $k=4$, $(l_n^{(4)})_{n\in\mathbb{Z}}$ is the Quadrabonacci-Lucas sequence

$$\ldots,-11,6,0,1,-4,3,0,0,-1,2,1,2,4,9,16,31,60,\ldots$$

Secondly, we establish the extension of the family of generalized order-k Fibonacci sequences, $(g_n^{(i)})_n$, $1\leq i\leq k$, to negatively subscripted indices, in the following way:

$$g_{-n}^{(i)}=-\frac{c_{k-1}}{c_k}g_{-n+1}^{(i)}-\frac{c_{k-2}}{c_k}g_{-n+2}^{(i)}-\cdots-\frac{c_1}{c_k}g_{-n+k-1}^{(i)}+\frac{1}{c_k}g_{-n+k}^{(i)}. \tag{6}$$

where, $c_j\in\mathbb{R}$ for $1\leq j\leq k$ and $c_k\neq 0$. And $g_{-n}^{(i)}$ is said to be the n^{th} negatively subscripted generalized order-k Fibonacci number of the sequence number i of the family.

For instance, we give the extension to negative indices of classical order-k sequences. We start by a family of order-k Fibonacci sequences defined for $1\leq i\leq k$ and $n\geq k$ as follows:

$$f_n^{(i)}=f_{n-1}^{(i)}+f_{n-2}^{(i)}+\cdots+f_{n-k}^{(i)},$$

with initial conditions; for $0\leq n\leq k-1$,

$$f_n^{(i)}=\begin{cases}1, & \text{for } i=k-n;\\ 0, & \text{otherwise.}\end{cases}$$

For $n\geq 1$, we give the extension of the family of order-k Fibonacci sequences to negative indices as follows:

$$f_{-n}^{(i)}=-f_{-n+1}^{(i)}-f_{-n+2}^{(i)}-\cdots-f_{-n+k-1}^{(i)}+f_{-n+k}^{(i)}. \tag{7}$$

The following table gives the first terms of negatively subscripted order-k Fibonacci sequences for $n\geq 1$,

	$k=2$	$k=3$	$k=4$
$i=1$	$1,-1,2,-3,5,-8,13,-21,34,-55\ldots$	$1,-1,0,2,-3,1,4,-8,5,7\ldots$	$1,-1,0,0,2,-3,1,0,4,-8\ldots$
$i=2$	$-1,2,-3,5,-8,13,-21,34,-55,89\ldots$	$-1,2,-1,-2,5,-4,-3,12,-13,-2\ldots$	$-1,2,-1,0,-2,5,-4,1,-4,12\ldots$
$i=3$		$-1,0,2,-3,1,4,-8,5,7,-20\ldots$	$-1,0,2,-1,-2,1,4,-4,-3,4\ldots$
$i=4$			$-1,0,0,2,-3,1,0,4,-8,5\ldots$

Table 1: First terms of negatively subscripted order-k Fibonacci sequences

Remark 2. For $k=2$, $(f_{-n}^{(1)})_n$ is the extension to negative indices of classical Fibonacci numbers.

Kiliç and Taşci [5] studied a family of k sequences of order-k Pell numbers defined for $n>0$ and $1\leq i\leq k$, as follows:

$$P_n^{(i)}=2P_{n-1}^{(i)}+P_{n-2}^{(i)}+\cdots+P_{n-k}^{(i)}. \tag{8}$$

with initial conditions; for $1-k\leq n\leq 0$,

$$P_n^{(i)}=\begin{cases}1, & \text{for } n=1-i;\\ 0, & \text{otherwise.}\end{cases}$$

Next, we consider k sequences of shifted order-k Pell numbers (8) for $n\geq k$ and $1\leq i\leq k$, with initial conditions, for $0\leq n\leq k-1$,

$$P_n^{(i)}=\begin{cases}1, & \text{for } i=k-n;\\ 0, & \text{otherwise.}\end{cases}$$

Then for $n\geq 1$, we give extension of these sequences (8) to negative indices as follows:

$$P_{-n}^{(i)}=-P_{-n+1}^{(i)}-\cdots-P_{-n+k-2}^{(i)}-2P_{-n+k-1}^{(i)}+P_{-n+k}^{(i)}. \tag{9}$$

The next table gives the first terms of negatively subscripted order-k Pell sequences for $n\geq 1$,

	$k=2$	$k=3$	$k=4$
$i=1$	$1,-2,5,-12,29,-70,169,-408\ldots$	$1,-1,-1,4,-3,-6,16,-7,-31,61\ldots$	$1,-1,0,-1,4,-4,2,-7,17,-18\ldots$
$i=2$	$-2,5,-12,29,-70,169,-408,985\ldots$	$-2,3,1,-9,10,9,-38,30,55,-153\ldots$	$-2,3,-1,2,-9,12,-8,16,-41,53\ldots$
$i=3$		$-2,3,-3,1,4,-12,21,-26,19,9\ldots$	$-1,-1,3,0,-2,-5,10,-1,-1,-23\ldots$
$i=4$			$-1,0,-1,4,-4,2,-7,17,-18,17\ldots$

Table 2: First terms of negatively subscripted order-k Pell sequences

Remark 3. Some of the sequences in Table 2 are known in OEIS [9], as for example, A215936, A276229, and A078021.

In 2009, Yilmaz and Bozkurt [10] studied a family of k sequences of order-k Jacobsthal numbers defined for $n > 0$ and $1 \leq i \leq k$, as follows:

$$J_n^{(i)} = J_{n-1}^{(i)} + 2J_{n-2}^{(i)} + \cdots + J_{n-k}^{(i)}, \tag{10}$$

with initial conditions for $1 - k \leq n \leq 0$,

$$J_n^{(i)} = \begin{cases} 1, & \text{for } n = 1 - i; \\ 0, & \text{otherwise.} \end{cases}$$

As before, we consider k sequences of shifted order-k Jacobsthal numbers (10) for $n \geq k$ and $1 \leq i \leq k$, with initial conditions for $0 \leq n \leq k - 1$,

$$J_n^{(i)} = \begin{cases} 1, & \text{for } i = k - n; \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $n \geq 1$, we give the extension of these sequences (10) to negative indices as follows:

$$J_{-n}^{(i)} = -\frac{1}{2}J_{-n+1}^{(i)} + \frac{1}{2}J_{-n+2}^{(i)}, \text{ for } k = 2,\ 1 \leq i \leq 2, \tag{11}$$

$$J_{-n}^{(i)} = -J_{-n+1}^{(i)} - \cdots - J_{-n+k-3}^{(i)} - 2J_{-n+k-2}^{(i)} - J_{-n+k-1}^{(i)} + J_{-n+k}^{(i)}. \text{ for } k \geq 3 \tag{12}$$

The next table gives the first terms of negatively subscripted order-k Jacobsthal sequences for $n \geq 1$,

	$k = 2$	$k = 3$	$k = 4$
$i = 1$	$\frac{1}{2}, -\frac{1}{4}, \frac{3}{8}, -\frac{5}{16}, \frac{11}{32}, -\frac{21}{64}, \frac{43}{128}, -\frac{85}{256} \ldots$	$1, -2, 3, -3, 1, 4, -12, 21, -26, 19 \ldots$	$1, -1, -1, 2, 2, -6, -1, 13, -3 \ldots$
$i = 2$	$-\frac{1}{2}, \frac{3}{4}, -\frac{5}{8}, \frac{11}{16}, -\frac{21}{32}, \frac{43}{64}, -\frac{85}{128}, \frac{171}{256} \ldots$	$-1, 3, -5, 6, -4, -3, 16, -33, 47 \ldots$	$-1, 2, 0, -3, 0, 8, -5, -14, 16, 25 \ldots$
$i = 3$		$-2, 3, -3, 1, 4, -12, 21, -26, 19, 9 \ldots$	$-2, 1, 4, -4, -7, 12, 10, -31, -8 \ldots$
$i = 4$			$-1, -1, 2, 2, -6, -1, 13, -3, -28 \ldots$

Table 3: First terms of negatively subscripted order-k Jacobsthal sequences

Remark 4. The sequences of $k = 3$ are known in the OEIS as A077990, -A078064, and A077990 respectively.

We now give some results on the n^{th} negatively subscripted generalized order-k Fibonacci number $g_{-n}^{(i)}$ using matrix methods.

Theorem 5. *For $n \geq 0$ and $2 \leq i \leq k$, we have*

$$g_{-n-1}^{(i)} = -\frac{c_{i-1}}{c_k}g_{-n}^{(k)} + g_{-n}^{(i-1)}, \tag{13}$$

and for $i = 1$,

$$g_{-n-1}^{(1)} = \frac{1}{c_k}g_{-n}^{(k)}. \tag{14}$$

Proof. For the proof, we use the matrix approach. The matrix approach was used, for example, by Kalman [4] and Er [3] to study generalized Fibonacci numbers. Let

$$A := \begin{pmatrix} -\frac{c_{k-1}}{c_k} & -\frac{c_{k-2}}{c_k} & -\frac{c_{k-3}}{c_k} & \cdots & -\frac{c_1}{c_k} & \frac{1}{c_k} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

We have the following n order-k recurrence relations

$$\begin{pmatrix} g_{-n-1}^{(i)} \\ g_{-n}^{(i)} \\ \vdots \\ g_{-n+k-2}^{(i)} \end{pmatrix} = A \begin{pmatrix} g_{-n}^{(i)} \\ g_{-n+1}^{(i)} \\ \vdots \\ g_{-n+k-1}^{(i)} \end{pmatrix}. \tag{15}$$

In order to deal with k sequences of negatively subscripted generalized order-k Fibonacci sequences at the same time, we construct a $k \times k$ square matrix G_{-n} as follows:

$$G_{-n} = \begin{pmatrix} g_{-n}^{(k)} & g_{-n}^{(k-1)} & g_{-n}^{(k-2)} & \cdots & g_{-n}^{(1)} \\ g_{-n+1}^{(k)} & g_{-n+1}^{(k-1)} & g_{-n+1}^{(k-2)} & \cdots & g_{-n+1}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{-n+k-1}^{(k)} & g_{-n+k-1}^{(k-1)} & g_{-n+k-1}^{(k-2)} & \cdots & g_{-n}^{(1)} \end{pmatrix}.$$

It is clear that, $A = G_{-1}$.

Next, (15) becomes

$$G_{-n-1} = AG_{-n}.$$

By induction, it is also clear that

$$G_{-n-1} = A^n A = G_{-n}A.$$

Finally, we have

$$G_{-n-1} = AG_{-n} = G_{-n}A. \tag{16}$$

Thus, Equations (13) and (14) are deduced from (16). □

From Theorem 5, we get the following results:

Corollary 6. *Let $f_{-n}^{(i)}$ be the n^{th} negatively subscripted order-k Fibonacci number of the sequence number i. Then*

$$f_{-n-1}^{(1)} = f_{-n}^{(k)};$$
$$f_{-n-1}^{(i)} = -f_{-n}^{(k)} + f_{-n}^{(i-1)}; 2 \leq i \leq k.$$

Corollary 7. *Let $P_{-n}^{(i)}$ be the n^{th} negatively subscripted order-k Pell number of the sequence number i. Then*

$$\begin{aligned}
P_{-n-1}^{(1)} &= P_{-n}^{(k)};\\
P_{-n-1}^{(2)} &= -2P_{-n}^{(k)} + P_{-n}^{(1)};\\
P_{-n-1}^{(i)} &= -P_{-n}^{(k)} + P_{-n}^{(i-1)};\ 3 \le i \le k.
\end{aligned}$$

Corollary 8. *Let $J_{-n}^{(i)}$ be the n^{th} negatively subscripted generalized order-k Jacobsthal number of the sequence number i. Then for $k \ge 3$,*

$$\begin{aligned}
J_{-n-1}^{(1)} &= J_{-n}^{(k)};\\
J_{-n-1}^{(2)} &= -J_{-n}^{(k)} + J_{-n}^{(1)};\\
J_{-n-1}^{(3)} &= -2J_{-n}^{(k)} + J_{-n}^{(2)};\\
J_{-n-1}^{(i)} &= -J_{-n}^{(k)} + J_{-n}^{(i-1)};\ 4 \le i \le k.
\end{aligned}$$

We conclude this section by defining a generalized order-k Fibonacci sequence $(V_n)_n$ with arbitrary initial conditions and we give the extension of the sequence $(V_n)_n$ to negative indices. Then we give the n^{th} negatively subscripted number V_{-n} as the sum of k terms of negatively subscripted generalized order-k Fibonacci numbers defined in (6).

For $n \ge k$, let V_n be as follows:

$$V_n = c_1V_{n-1} + c_2V_{n-2} + \cdots + c_kV_{n-k},$$

with integer initial conditions $V_0, V_1, \ldots, V_{k-1}$, where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Belbachir and Bencherif [1] gave the explicit formula of the n^{th} term of the sequence $(V_n)_n$.

Then for $n \ge 1$, the value V_{-n} is as follows:

$$V_{-n} = -\frac{c_{k-1}}{c_k}V_{-n+1} - \frac{c_{k-2}}{c_k}V_{-n+2} - \cdots - \frac{c_1}{c_k}V_{-n+k-1} + \frac{1}{c_k}V_{-n+k}. \tag{17}$$

Note that for $(V_0, \ldots, V_{k-i-1}, V_{k-i}, V_{k-i+1}, \ldots, V_{k-1}) = (0, \ldots, 0, 1, 0, \ldots, 0)$, $1 \le i \le k$, we have $V_n = g_n^{(i)}$.

The n^{th} term V_{-n} is given by

Theorem 9. *For $n \ge 1$,*

$$V_{-n} = \sum_{i=0}^{k-1} a_i g_{-n}^{(k-i)}, \tag{18}$$

where $g_{-n}^{(j)}$ is the n^{th} negatively subscripted generalized order-k Fibonacci number of the sequence number i (6).

Proof. By induction on n.

For $n=1$, $V_{-1}=-\frac{c_{k-1}}{c_k}V_0-\cdots-\frac{c_1}{c_k}V_{k-2}+\frac{1}{c_k}V_{k-1}=-\frac{c_{k-1}}{c_k}a_0-\cdots-\frac{c_1}{c_k}a_{k-2}+\frac{1}{c_k}a_{k-1}=\sum_{i=0}^{k-1}a_i g_{-1}^{(k-i)}$.

Now suppose that (18) is true for $2\le j\le n$ and we prove the formula for $n+1$:

$$V_{-n-1}=-\frac{c_{k-1}}{c_k}V_{-n}-\frac{c_{k-2}}{c_k}V_{-n+1}-\cdots-\frac{c_1}{c_k}V_{-n+k-2}+\frac{1}{c_k}V_{-n+k-1}.$$

and

$$\begin{aligned}
V_{-n-1}&=-\frac{c_{k-1}}{c_k}\sum_{i=0}^{k-1}a_i g_{-n}^{(k-i)}-\frac{c_{k-2}}{c_k}\sum_{i=0}^{k-1}a_i g_{-n+1}^{(k-i)}-\cdots-\frac{c_1}{c_k}\sum_{i=0}^{k-1}a_i g_{-n+k-2}^{(k-i)}+\frac{1}{c_k}\sum_{i=0}^{k-1}a_i g_{-n+k-1}^{(k-i)}\\
&=a_0\left(-\frac{c_{k-1}}{c_k}g_{-n}^{(k)}-\frac{c_{k-2}}{c_k}g_{-n+1}^{(k)}-\cdots-\frac{c_1}{c_k}g_{-n+k-2}^{(k)}+\frac{1}{c_k}g_{-n+k-1}^{(k)}\right)\\
&+a_1\left(-\frac{c_{k-1}}{c_k}g_{-n}^{(k-1)}-\frac{c_{k-2}}{c_k}g_{-n+1}^{(k-1)}-\cdots-\frac{c_1}{c_{k-1}}g_{-n+k-2}^{(k-1)}+\frac{1}{c_{k-1}}g_{-n+k-1}^{(k-1)}\right)\\
&+\cdots+\\
&+a_{k-1}\left(-\frac{c_{k-1}}{c_k}g_{-n}^{(1)}-\frac{c_{k-2}}{c_k}g_{-n+1}^{(1)}-\cdots-\frac{c_1}{c_k}g_{-n+k-2}^{(1)}+\frac{1}{c_k}g_{-n+k-1}^{(1)}\right)\\
&=a_0g_{-n-1}^{(k)}+a_1g_{-n-1}^{(k-1)}+\cdots+a_{k-1}g_{-n-1}^{(1)}\\
&=\sum_{i=0}^{k-1}a_i g_{-n-1}^{(k-i)}.
\end{aligned}$$

Thus (18) is true for all $n\ge 1$. □

3 Representation of negatively subscripted generalized order-k Fibonacci numbers

In this section, we give a representation of the family of negatively subscripted generalized order-k Fibonacci numbers (6) using the permanent of Hessenberg matrices.

We introduce the $n\times n$ matrix $W_n=(w_{st})$ with $w_{1t}=0$, for $1\le t\le n$ and for all $1<i\le n$,

$$w_{st}=\begin{cases}1, & \text{if } s=t+1;\\ -\frac{c_{k-(t-s)-1}}{c_k}, & \text{if } 0\le t-s\le k-2;\\ \frac{1}{c_k}, & \text{if } s=t-k+1;\\ 0, & \text{elsewhere.}\end{cases}$$

And we construct, for $n\ge 1$, the $n\times n$ matrix A_n^i, for $1\le i\le k$, as follows:

$$A_n^i=W_n+\sum_{t=1}^{n}\left(\frac{\delta_{i,k}^t(k)}{c_k}-\sum_{j=1}^{k-1}\delta_{i,j}^t(k)\frac{c_{k-j}}{c_k}\right)E_{1t}$$

with $\delta^c_{a,b}(d)$ defined for $1 \leq a, b, c \leq d$ as

$$\delta^c_{a,b}(d) = \begin{cases} 1, & \text{if } a+b = c+d; \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

And E_{st} denotes the $n \times n$ matrix with 1 in position (s,t) and 0 elsewhere.

Then A^i_n is as follows:

$$A^i_n = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_k & 0 & 0 & \cdots & 0 \\ 1 & -\frac{c_{k-1}}{c_k} & -\frac{c_{k-2}}{c_k} & \cdots & -\frac{c_1}{c_k} & \frac{1}{c_k} & 0 & \cdots & 0 \\ 0 & 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & \frac{1}{c_k} \\ & & & & & & & \ddots & -\frac{c_1}{c_k} \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}, \tag{20}$$

where $A_t = \frac{\delta^t_{i,k}(k)}{c_k} - \sum_{j=1}^{k-1} \delta^t_{i,j}(k) \frac{c_{k-j}}{c_k}$, for $1 \leq t \leq k$.

We now give the representation of negatively subscripted generalized order-k Fibonacci numbers.

Theorem 10. *For $n \geq 1$, let A^i_n be as in (20) and $g^{(i)}_{-n}$ be the n^{th} negatively subscripted generalized order-k Fibonacci number of the sequence number i, then*

$$\operatorname{per} A^i_n = g^{(i)}_{-n}. \tag{21}$$

For instance, we give the following representations of k sequences of negatively subscripted order-k Fibonacci, order-k Pell, and order-k Jacobsthal numbers.

Let F^i_n be the $n \times n$ matrix defined, for $1 \leq i \leq k$, as follows:

$$F^i_n = \begin{pmatrix} \delta^1_{i,k}(k) - \sum_{j=1}^{k-1} \delta^1_{i,j}(k) & \delta^2_{i,k}(k) - \sum_{j=1}^{k-1} \delta^2_{i,j}(k) & \delta^3_{i,k}(k) - \sum_{j=1}^{k-1} \delta^3_{i,j}(k) & \cdots & \delta^k_{i,k} - \sum_{j=1}^{k-1} \delta^k_{i,j}(k) & 0 & 0 & \cdots & 0 \\ 1 & -1 & -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & -1 & -1 & 1 & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & 1 \\ & & & & & & & \ddots & -1 \\ & & & & & & & \ddots & \vdots \\ 0 & & \cdots & & & & \cdots & 0 & 1 & -1 \end{pmatrix}. \tag{22}$$

where $\delta^t_{i,k}(k)$ and $\delta^t_{i,j}(k)$ is defined as in (19) for all $1 \leq t \leq k$ and $1 \leq j \leq k-1$.

Corollary 11. *For $n \geq 1$, let F_n^i be as in (22) and $f_{-n}^{(i)}$ be the n^{th} negatively subscripted order-k Fibonacci number of the sequence number i, then*

$$\operatorname{per} F_n^i = f_{-n}^{(i)}. \tag{23}$$

Let $\mathcal{P}_n^i$ be the $n \times n$ matrix defined for $1 \leq i \leq k$ in the following way:

$$\mathcal{P}_n^i = \begin{pmatrix} S_1 & S_2 & \cdots & S_{k-1} & S_k & 0 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & -1 & -2 & 1 & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & 1 \\ & & & & & & & \ddots & -2 \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 1 & -1 \end{pmatrix}, \tag{24}$$

where $S_t = \delta_{i,k}^t - 2\delta_{i,k-1}^t - \sum_{j=1}^{k-2} \delta_{i,j}^t$ for $1 \leq t \leq k$ and $\delta_{i,j}^t(k)$ is defined as in (19).

Corollary 12. *For $n \geq 1$, let $\mathcal{P}_n^i$ be as in (24) and $P_{-n}^{(i)}$ be the n^{th} negatively subscripted order-k Pell number of the sequence number i, then*

$$\operatorname{per} \mathcal{P}_n^i = P_{-n}^{(i)}. \tag{25}$$

Let $\mathcal{J}_n^i$ be the $n \times n$ matrix defined for $1 \leq i \leq k$ and $k \geq 3$ as follows:

$$\mathcal{J}_n^i = \begin{pmatrix} T_1 & T_2 & T_3 & \cdots & T_{k-2} & T_{k-1} & T_k & 0 & 0 & \cdots & 0 \\ 1 & -1 & -1 & \cdots & -1 & -2 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & -1 & -1 & -2 & -1 & 1 & \ddots & \vdots \\ & & & & & & & & & \ddots & 0 \\ \vdots & & & & & & & & & \ddots & 1 \\ & & & & & & & & & \ddots & -1 \\ & & & & & & & & & \ddots & -2 \\ & & & & & & & & & \ddots & -1 \\ & & & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & & \cdots & 0 & 1 & 1 & -1 \end{pmatrix}, \tag{26}$$

where $T_t = \delta_{i,k}^t - 2\delta_{i,k-2}^t - \delta_{i,k-1}^t - \sum_{j=1}^{k-3} \delta_{i,j}^t$ for $1 \leq t \leq k$ and $\delta_{i,j}^t(k)$ is defined as in (19).

Corollary 13. *For $n \geq 1$, let $\mathcal{J}_n^i$ be as in (26) and $J_{-n}^{(i)}$ be the n^{th} negatively subscripted order-k Jacobsthal number of the sequence number i, then*

$$\operatorname{per} \mathcal{J}_n^i = J_{-n}^{(i)}. \tag{27}$$

4 More permanental representations of negatively subscripted order-k Fibonacci and order-k Lucas numbers

In this section, we give numerous types of matrices whose permanent are negatively subscripted order-k Fibonacci and order-k Lucas terms.

We first establish two kinds of permanental representations of the negatively subscripted order-k Fibonacci sequence $(\widetilde{g}_{-n})_n$. We introduce an $n \times n$ $(0, 1, -1)$-matrix $B_{(n,k)}$ defined by

$$b_{ij} = \begin{cases} 1, & \text{for } j - i = -1 \text{ or } k - 1; \\ -1, & \text{for } 0 \leq j - i \leq k - 2; \\ 0, & \text{elsewhere.} \end{cases} \qquad (\star)$$

And we construct for $n > k$, the $n \times n$ $(0, 1, -1)$-matrices $F_{(n,k)}$ and $\widetilde{F}_{(n,k)}$ as follows:

$$F_{(n,k)} = B_{(n,k)} + 2E_{11} + \sum_{j=2}^{k-1} E_{1j} - E_{1k},$$

$$\widetilde{F}_{(n,k)} = B_{(n,k)} + 2\sum\nolimits_{j=1}^{k-1} E_{1j},$$

where E_{ij} denotes the $n \times n$ matrix with 1 in position (i, j) and 0 elsewhere. $F_{(n,k)}$ and $\widetilde{F}_{(n,k)}$ correspond to

$$F_{(n,k)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & & \ddots & 0 \\ & & & & & & \ddots & 1 \\ & & & & & & & -1 \\ \vdots & & & & & \ddots & \ddots & \vdots \\ 0 & & & & & & 1 & -1 \end{pmatrix}$$

and

$$\widetilde{F}_{(n,k)} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & & \ddots & 0 \\ & & & & & & \ddots & 1 \\ & & & & & & & -1 \\ \vdots & & & & & \ddots & \ddots & \vdots \\ 0 & & & & & & 1 & -1 \end{pmatrix}.$$

Then we give the two permanental representations of the negatively subscripted order-k Fibonacci sequence.

Theorem 14. *For $n > k$, we have*

$$\operatorname{per} F_{(n,k)} = \widetilde{g}_{-n}^{(k)} \tag{28}$$

and

$$\operatorname{per} \widetilde{F}_{(n,k)} = \widetilde{g}_{-n+k}^{(k)}. \tag{29}$$

Next, using the same approach given in Theorem 14, see next section, we give a representation of $l_{-n+1}^{(k)}$.

We use the $n \times n$ square matrix $B_{(n,k)} = (b_{ij})$ given by ($\star$), and we construct for $n > k$, the $n \times n$ matrix $A_{(n,k)}$ as follows:

$$A_{(n,k)} = B_{(n,k)} + 2E_{11} + \sum_{j=2}^{k-1} E_{1j} - E_{1k} + E_{1,k+1},$$

where E_{ij} is the $n \times n$ matrix with 1 in position (i, j) and 0 elsewhere.

The matrix $A_{(n,k)}$ corresponds to

$$A_{(n,k)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & -1 & 1 & 0 & \ddots & \vdots \\ & \ddots & \ddots & & \ddots & -1 & & & 0 \\ & & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & & & & 1 \\ & & & & & & & & -1 \\ & & & & & & & \ddots & \vdots \\ & & & & & & \ddots & \ddots & \vdots \\ 0 & & \cdots & & & 0 & & 1 & -1 \end{pmatrix}.$$

Theorem 15. *For $n > k$, we have*

$$\operatorname{per} A_{(n,k)} = l_{-n+1}^{(k)}.$$

Proof. The proof is the same as Theorem 14, see next section. □

Finally, let B_n be an $n \times n$ matrix defined by

$$B_n = T_n + 2(E_{11} + E_{22} + E_{33} - E_{43}) + E_{13} - E_{23} + E_{24} - E_{34},$$

where T_n is the $n \times n$ $(0, 1, -1)$-matrix such that $t_{ii} = -1$, $t_{ij} = 1$ if and only if $|j - i| = 1$.

The matrix B_n corresponds to

$$B_n = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & \cdots & & 0 \\ 1 & 1 & 0 & 1 & 0 & & & \vdots \\ 0 & 1 & 1 & 0 & 0 & & & \\ 0 & 0 & -1 & -1 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & -1 & 1 & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & & & & 0 & 1 & -1 \end{pmatrix}.$$

Then we obtain the following result.

Theorem 16. *For $n \geq 4$, we have*

$$\operatorname{per} B_n = l^{(2)}_{-n+1}.$$

5 Proofs

We start this section by giving the proof of Theorem 10,

Proof. We have

$$\frac{\delta^1_{i,k}}{c_k} - \sum_{t=1}^{k-1} \delta^1_{i,t} \frac{c_{k-t}}{c_k} = \begin{cases} \frac{1}{c_k}, & \text{for } i = 1; \\ -\frac{c_{i-1}}{c_k}, & \text{for } 2 \leq i \leq k. \end{cases}$$

On the other hand by Theorem 5, we have

$$g^{(i)}_{-1} = \begin{cases} \frac{1}{c_k}, & \text{for } i = 1; \\ -\frac{c_{i-1}}{c_k}, & \text{for } 2 \leq i \leq k. \end{cases}$$

Thus,

$$\frac{\delta^1_{i,k}}{c_k} - \sum_{t=1}^{k-1} \delta^1_{i,t} \frac{c_{k-t}}{c_k} = g^{(i)}_{-1}.$$

Next for $2 \leq j \leq k-1$, by the definition of shifted generalized order-k Fibonacci sequences (1), we have

$$\frac{\delta^j_{i,k}}{c_k} - \sum_{t=1}^{k-1} \delta^j_{i,t} \frac{c_{k-t}}{c_k} = \begin{cases} 0, & \text{for } 1 \leq i \leq j-1; \\ \frac{1}{c_k} g^{(i)}_{k-j}, & \text{for } i = j; \\ -\frac{c_{i-1}}{c_k} g^{(i)}_{k-i}, & \text{for } j+1 \leq i \leq k. \end{cases}$$

Then $\frac{\delta_{i,k}^j}{c_k} - \sum_{t=1}^{k-1} \delta_{i,t}^j \frac{c_{k-t}}{c_k}$ can be written as

$$\frac{\delta_{i,k}^j}{c_k} - \sum_{t=1}^{k-1} \delta_{i,t}^j \frac{c_{k-t}}{c_k} = \frac{1}{c_k} g_{k-j}^{(i)} - \sum_{t=j}^{k-1} \frac{c_{k-t}}{c_k} g_{t-j}^{(i)}.$$

Finally,

$$\frac{\delta_{i,k}^k}{c_k} - \sum_{t=1}^{k-1} \delta_{i,t}^k \frac{c_{k-t}}{c_k} = \begin{cases} \frac{1}{c_k} g_0^{(k)}, & \text{for } i = k; \\ 0, & \text{otherwise.} \end{cases}$$

Then for $1 \le i \le k$, we have

$$\frac{\delta_{i,k}^k}{c_k} - \sum_{t=1}^{k-1} \delta_{i,t}^k \frac{c_{k-t}}{c_k} = \frac{1}{c_k} g_0^{(i)}.$$

Hence, the matrix A_n^i can be written as follows:

$$A_n^i = \begin{pmatrix} g_{-1}^{(i)} & \frac{1}{c_k} g_{k-2}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{t-2}^{(i)} & \frac{1}{c_k} g_{k-3}^{(i)} - \sum_{t=3}^{k-1} \frac{c_{k-t}}{c_k} g_{t-3}^{(i)} & \cdots & \frac{1}{c_k} g_0^{(i)} & 0 & 0 & \cdots & 0 \\ 1 & -\frac{c_{k-1}}{c_k} & -\frac{c_{k-2}}{c_k} & \cdots & -\frac{c_1}{c_k} & \frac{1}{c_k} & 0 & \cdots & 0 \\ 0 & 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & \frac{1}{c_k} \\ & & & & & & & \ddots & -\frac{c_1}{c_k} \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

By Lemma 1, the matrix A_n^i is contractible on column 1 relative to rows 1 and 2. Let $(A_n^i)^1$ be the $(n-1) \times (n-1)$ contraction matrix, then we have $\operatorname{per} A_n^i = \operatorname{per}(A_n^i)^1$ and $(A_n^i)^1$ is as follows:

$$(A_n^i)^1 = \begin{pmatrix} g_{-2}^{(i)} & \frac{1}{c_k} g_{k-3}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{t-3}^{(i)} & \frac{1}{c_k} g_{k-4}^{(i)} - \sum_{t=3}^{k-1} \frac{c_{k-t}}{c_k} g_{t-4}^{(i)} & \cdots & \frac{1}{c_k} g_{-1}^{(i)} & 0 & 0 & \cdots & 0 \\ 1 & -\frac{c_{k-1}}{c_k} & -\frac{c_{k-2}}{c_k} & \cdots & -\frac{c_1}{c_k} & \frac{1}{c_k} & 0 & \cdots & 0 \\ 0 & 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & \frac{1}{c_k} \\ & & & & & & & \ddots & -\frac{c_1}{c_k} \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

Furthermore, the matrix $(A_n^i)^1$ is also contractible on column 1 relative to rows 1 and 2. Continuing the same contraction process we get by Lemma 1,

$$\operatorname{per} A_n^i = \operatorname{per}(A_n^i)^p$$

with $(A_n^i)^p$ defined for $p \leq n-k$ by

$$(A_n^i)^p = \begin{pmatrix} g_{-p-1}^{(i)} & \frac{1}{c_k} g_{-p+k-2}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{-p+t-2}^{(i)} & \cdots & \frac{1}{c_k} g_{-p+1}^{(i)} - \frac{c_1}{c_k} g_{-p}^{(i)} & \frac{1}{c_k} g_{-p}^{(i)} & 0 & 0 & \cdots & 0 \\ 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & -\frac{c_3}{c_k} & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & \frac{1}{c_k} \\ & & & & & & & \ddots & -\frac{c_1}{c_k} \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

And for $p \geq n-k+1$, $(A_n^i)^p$ is defined as follows:

$$(A_n^i)^p = \begin{pmatrix} g_{-p-1}^{(i)} & \frac{1}{c_k} g_{-p+k-2}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{-p+t-2}^{(i)} & \cdots & \frac{1}{c_k} g_{-n+k+1}^{(i)} - \sum_{t=n-p+1}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t+1}^{(i)} & \frac{1}{c_k} g_{-n+k}^{(i)} - \sum_{t=n-p}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t}^{(i)} \\ 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_{k-n+p+2}}{c_k} & -\frac{c_{k-n+p+1}}{c_k} \\ 0 & 1 & \cdots & -\frac{c_{k-n+p+3}}{c_k} & -\frac{c_{k-n+p+2}}{c_k} \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

Then

$$(A_n^i)^{n-2} = \begin{pmatrix} g_{-n+1}^{(i)} & \frac{1}{c_k} g_{-n+k}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t}^{(i)} \\ 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

By Lemma 1, we have

$$\begin{aligned} \operatorname{per} A_n^i = \operatorname{per}(A_n^i)^{n-2} &= -\frac{c_{k-1}}{c_k} g_{-n+1}^{(i)} + \frac{1}{c_k} g_{-n+k}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t}^{(i)} \\ &= \frac{1}{c_k} g_{-n+k}^{(i)} - \sum_{t=1}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t}^{(i)} \\ &= g_{-n}^{(i)}. \end{aligned}$$

So $\operatorname{per} A_n^i = g_{-n}^{(i)}$ and the proof is complete. □

Secondly, we provide the proof of Theorem 14,

Proof. Let $F^p_{(n,k)} = (f^p_{ij})$ be the p^{th} contraction of $F_{(n,k)}$, $1 \le p \le n-2$.

The matrix $F_{(n,k)} = (f_{ij})$, can be contracted on column 1 relative to rows 1 and 2.

We can easily verify that if $p = 1$,

$f^1_{11} = f^1_{12} = \cdots = f^1_{1k-1} = -1$, $f^1_{1k} = 1$ and $f^1_{1q} = 0$, for $q \ge k+1$; and for all $i = 2, \ldots, n-1$,

$$\begin{cases} f_{i,i-1} = 1, & \\ f_{ij} = -1, & \text{for } 0 \le j-i \le k-2; \\ f_{ij} = 1 & \text{for } j-i = k-1. \end{cases}$$

Hence

$$F^1_{(n,k)} = \begin{pmatrix} -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & \ddots & \vdots \\ & & & & \ddots & \ddots & 0 \\ & & & & & \ddots & 1 \\ \vdots & & & & & & -1 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & \cdots & & & 1 & -1 \end{pmatrix}$$

and

$$F^1_{(n,k)} = \begin{pmatrix} \widetilde{g}^{(k)}_{-2} & \widetilde{g}^{(k)}_{k-3} - \sum_{j=-1}^{k-4} \widetilde{g}^{(k)}_j & \widetilde{g}^{(k)}_{k-4} - \sum_{j=-1}^{k-5} \widetilde{g}^{(k)}_j & \cdots & -\widetilde{g}^{(k)}_{-1} + \widetilde{g}^{(k)}_0 & \widetilde{g}^{(k)}_{-1} & 0 & \cdots & 0 \\ 1 & -1 & -1 & \cdots & & -1 & 1 & \ddots & \vdots \\ 0 & 1 & -1 & \cdots & & & -1 & \ddots & 0 \\ & & & & & & & \ddots & 1 \\ & & & & & & & & -1 \\ & & & & & & \ddots & \ddots & \vdots \\ 0 & & & \cdots & & & 0 & 1 & -1 \end{pmatrix}.$$

Furthermore, the matrix $F^1_{(n,k)}$ can be contracted. From Lemma 1 we obtain

$$\operatorname{per} F_{(n,k)} = \operatorname{per} F^p_{(n,k)}$$

with $F^p_{(n,k)}$ defined for $p \le n-k$ by

$$F^p_{(n,k)} = \begin{pmatrix} \widetilde{g}^{(k)}_{-p-1} & \widetilde{g}^{(k)}_{-p+k-2} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-p+j} & \cdots & -\widetilde{g}^{(k)}_{-p} + \widetilde{g}^{(k)}_{-p+1} & \widetilde{g}^{(k)}_{-p} & 0 & \cdots & 0 \\ 1 & -1 & \cdots & \cdots & -1 & 1 & \ddots & \vdots \\ & & & & & \ddots & \ddots & 0 \\ & & & & & & \ddots & 1 \\ & & & & & & & -1 \\ & & & & & \ddots & \ddots & \vdots \\ 0 & & \cdots & & & 0 & 1 & -1 \end{pmatrix}.$$

And for $p \geq n-k+1$, $F^p_{(n,k)}$ is defined as

$$F^p_{(n,k)} = \begin{pmatrix} \widetilde{g}^{(k)}_{-p-1} & \widetilde{g}^{(k)}_{-p+k-2} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-p+j} & \cdots & \widetilde{g}^{(k)}_{k-n+1} - \sum_{j=0}^{k-(n-p)} \widetilde{g}^{(k)}_{-p+j} & \widetilde{g}^{(k)}_{k-n} - \sum_{j=0}^{k-(n-p+1)} \widetilde{g}^{(k)}_{-p+j} \\ 1 & -1 & \cdots & \cdots & -1 \\ 0 & 1 & \cdots & & -1 \\ \vdots & & & & \vdots \\ 0 & & \cdots & 1 & -1 \end{pmatrix}.$$

So we have

$$F^{n-2}_{(n,k)} = \begin{pmatrix} \widetilde{g}^{(k)}_{-n+1} & \widetilde{g}^{(k)}_{-n+k} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-n+2+j} \\ 1 & -1 \end{pmatrix}.$$

Lemma 1 gives

$$\begin{aligned} \operatorname{per} F_{(n,k)} &= \operatorname{per} \begin{pmatrix} \widetilde{g}^{(k)}_{-n+1} & \widetilde{g}^{(k)}_{-n+k} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-n+2+j} \\ 1 & -1 \end{pmatrix} \\ &= -\widetilde{g}^{(k)}_{-n+1} + \widetilde{g}^{(k)}_{-n+k} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-n+2+j} \\ &= \widetilde{g}^{(k)}_{-n+k} - \left(\widetilde{g}^{(k)}_{-n+1} + \widetilde{g}^{(k)}_{-n+2} + \cdots + \widetilde{g}^{(k)}_{-n+k-1} \right). \end{aligned}$$

So $\operatorname{per} F_{(n,k)} = \widetilde{g}^{(k)}_{-n}$ and the proof of Identity (28) is complete.

By Lemma 1, we can write

$$\operatorname{per} \widetilde{F}_{(n,k)} = \operatorname{per} \widetilde{F}^1_{(n,k)}, \tag{30}$$

where $\widetilde{F}^1_{(n,k)}$ is the first contraction of $\widetilde{F}_{(n,k)}$.

Then $\widetilde{f}^1_{11} = \widetilde{f}^1_{12} = \cdots = \widetilde{f}^1_{1k-1} = 0$, $\widetilde{f}^1_{1k} = 1$ and $\widetilde{f}^1_{1q} = 0$ $(q \geq k+1)$; and for $i = 2, \ldots, n-1$, $\widetilde{f}^1_{i,i-1} = 1$, $\widetilde{f}^1_{ij} = -1$ for $(0 \leq j-i \leq k-2)$ and $\widetilde{f}^1_{ij} = 1$ for $(j-i = k-1)$.

That is,

$$\operatorname{per} \widetilde{F}^1_{(n,k)} = \operatorname{per} \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & \ddots & \vdots \\ 0 & 1 & & & \ddots & \ddots & 0 \\ & & & & & \ddots & 1 \\ & & & & & & -1 \\ & & & & & & \vdots \\ 0 & & \cdots & & 0 & 1 & -1 \end{pmatrix}_{n-1}.$$

Computing per $\widetilde{F}^1_{(n,k)}$ by the Laplace expansion with respect to the first column, we obtain

$$\operatorname{per}\widetilde{F}^1_{(n,k)} = \operatorname{per}\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & \ddots & \vdots \\ & & & & \ddots & \ddots & 0 \\ & & & & & \ddots & 1 \\ & & & & & & -1 \\ & & & & \ddots & \ddots & \vdots \\ & & \cdots & & & 1 & -1 \end{pmatrix}_{n-k}.$$

From identity (28), $\operatorname{per}\widetilde{F}^1_{(n,k)} = \operatorname{per} F_{(n-k,k)} = \widetilde{g}^{(k)}_{-(n-k)}$, and from (30), we have, $\operatorname{per}\widetilde{F}_{(n,k)} = \widetilde{g}^{(k)}_{-n+k}$. The proof is complete. □

We conclude this section with the proof of Theorem 16.

Proof. If $n = 4$,

$$\operatorname{per} B_4 = \operatorname{per}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} = -4 = l^{(2)}_{-3}.$$

By induction on n, we assume that $\operatorname{per} B_n = l^{(2)}_{-n+1}$ and we establish that $\operatorname{per} B_{n+1} = l^{(2)}_{-n}$.

Let

$$F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$B_{n+1} = \begin{pmatrix} B_n & F \\ F^t & -1 \end{pmatrix}.$$

Let $B^{(p)}_n$ be the p^{th} contraction of B_n. Since

$$B^{(n-2)}_n = \begin{pmatrix} l^{(2)}_{-n+2} & l^{(2)}_{-n+3} \\ 1 & -1 \end{pmatrix},$$

we get

$$\operatorname{per} B_{n+1} = \operatorname{per} B^{(n-2)}_{n+1} = \operatorname{per}\begin{pmatrix} l^{(2)}_{-n+2} & l^{(2)}_{-n+3} & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = l^{(2)}_{-n+2} - l^{(2)}_{-n+3} + l^{(2)}_{-n+2} = l^{(2)}_{-n+2} - l^{(2)}_{-n+1} = l^{(2)}_{-n}.$$

The proof is completed. □

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