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Permanental Representations of Negatively Subscripted Generalized Order-k Fibonacci Numbers

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Abstract

We introduce an extension to the negative indices of a family of generalized Fibonacci sequences of order-k, and for which we establish recurrence relations. We also give permanental representations of negatively subscripted generalized Fibonacci and Lucas sequences via Hessenberg matrices.

1 Introduction

A family of k sequences of the generalized order-k Fibonacci numbers was defined and studied by Er [3], for $n > 0$ and $1 \leq i \leq k$, as follows:

$$g_n^{(i)} = c_1 g_{n-1}^{(i)} + c_2 g_{n-2}^{(i)} + \cdots + c_k g_{n-k}^{(i)}, \tag{1}$$

where $c_1, c_2, \ldots, c_k$ are real constant coefficients, with $c_k \neq 0$, and for $1 - k \leq n \leq 0$,

$$g_n^{(i)} = \begin{cases} 1, & \text{for } i = 1 - n; \\ 0, & \text{otherwise.} \end{cases}$$

Several sequences were derived from (1). Lee and Lee [6] studied the sequence obtained from (1) by setting $(c_1, c_2, \ldots, c_k) = (1, 1, \ldots, 1)$ and $i = 1$, which is called in the present paper order-k Fibonacci sequence $(\widetilde{g}_n^{(k)})_n$. Also, Lee [8] studied order-k Lucas sequence denoted $(l_n^{(k)})_n$, for $k \geq 2$. The sequence $(l_n^{(k)})_n$ is obtained from (1) by setting $(c_1, c_2, \ldots, c_k) = (1, 1, \ldots, 1)$ with initial conditions $l_0^{(k)} = 2$, $l_j^{(k)} = 2^{j-1}$, for $j = 1, \ldots, k-1$, and $l_k^{(k)} = 2^{k-1} + 1$.

In the present paper, we consider the family of shifted generalized order-k Fibonacci sequences defined as (1), but with initial conditions, for $0 \leq n \leq k - 1$,

$$g_n^{(i)} = \begin{cases} 1, & \text{for } i = k - n; \\ 0, & \text{otherwise.} \end{cases}$$

The *permanent* of a square matrix $A = (a_{ij})_{n \times n}$ is defined as follows:

$$\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation runs over all permutations σ of the symmetric group $\mathfrak{S}_n$ of order n.

Next, we present the definition of matrix contraction; see for instance [2]. Let $A = (a_{ij})$ be an $m \times n$ real matrix with row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say that A is *contractible* on column (resp., row) k if column (resp., row) k contains exactly two nonzero entries. If A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$, then the $(m-1) \times (n-1)$ matrix $A_{ij}^{(k)}$, obtained from A by replacing row i by $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k, is called the contraction of A on column k relative to rows i and j.

Brualdi and Gibson [2] gave the following lemma.

Lemma 1. *Let A be a nonnegative integral matrix of order $n > 1$, and let B be a contraction of A. Then*

$$\operatorname{per} A = \operatorname{per} B.$$

Permanental representations of recurrent sequences have been investigated by several authors. Yilmaz and Bozkurt [11] gave permanental representations of sums of Fibonacci and Lucas numbers via Hessenberg matrices. Lee et al. [7] obtained $g_{n+k-1}^{(k)}$ as the permanent of an $n \times n$ upper triangular $(0,1)$-matrix; that is a matrix whose all entries are either 0 or 1. Lee [8] constructed a $(0,1)$-matrix of order n and established that the permanent of that matrix is the $(n-1)^{\text{th}}$ term $l_{n-1}^{(k)}$ of the order-k Lucas sequence.

The aim of this work is to propose an extension to negative indices of the family of generalized order-k Fibonacci sequences $(g_n^{(i)})_n$, $1 \leq i \leq k$. For instance, extensions to negative indices of order-k Fibonacci, order-k Lucas, order-k Pell, and order-k Jacobsthal are also given. As application, we provide some permanental representations.

The present paper is organized as follows: In Section 2, we give an extension of order-k Fibonacci, order-k Lucas, and the family of generalized order-k Fibonacci sequences $(g_n^{(i)})_n$, $1 \leq i \leq k$, to negatively subscripted indices. We also provide recurrence formulas for the n^{th} term of these negatively subscripted sequences. In Section 3, we present negatively subscripted generalized order-k Fibonacci numbers as the permanent of special Hessenberg matrix. In Section 4, we give more permanental representations of negatively subscripted order-k Fibonacci and order-k Lucas sequences. We conclude by Section 5, where we provide proofs of the results presented in Sections 3 and 4.

2 Extension to negative indices

Extension to negative indices of classical Fibonacci sequence $(F_n)_n$ with initial conditions $F_0 = 0$, $F_1 = 1$ is given for $n \geq 1$, by

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n. \tag{2}$$

The extension to negative indices of classical Lucas sequence $(L_n)_n$ with initial conditions $L_0 = 2$, $L_1 = 1$ is given for $n \geq 1$, by

$$L_{-n} = L_{-n+2} - L_{-n+1} = (-1)^n L_n. \tag{3}$$

Firstly, we propose an extension of order-k Fibonacci sequence $(\widetilde{g}_n^{(k)})_n$ to negatively subscripted indices, for $n \geq 1$, as follows:

$$\widetilde{g}_{-n}^{(k)} = -\widetilde{g}_{-n+1}^{(k)} - \widetilde{g}_{-n+2}^{(k)} - \cdots - \widetilde{g}_{-n+k-1}^{(k)} + \widetilde{g}_{-n+k}^{(k)}. \tag{4}$$

For example, for $k = 2$, $(\widetilde{g}_n^{(2)})_{n \in \mathbb{Z}}$ is the classical Fibonacci sequence.

For $k = 3$, $(\widetilde{g}_n^{(3)})_{n \in \mathbb{Z}}$ is the Tribonacci sequence

$$\ldots, -8, 4, 1, -3, 2, 0, -1, 1, 0, 0, 1, 1, 2, 4, 7, 13, \ldots$$

For $k = 4$, $(\widetilde{g}_n^{(4)})_{n\in\mathbb{Z}}$ is the Quadrabonacci sequence

$$\dots, 0, 1, -3, 2, 0, 0, -1, 1, 0, 0, 0, 1, 1, 2, 4, 8, 15, 29, \dots$$

An extension of order-k Lucas sequence $(l_n^{(k)})_n$ to negatively subscripted indices, for $n \geq 1$, is given by

$$l_{-n}^{(k)} = -l_{-n+1}^{(k)} - l_{-n+2}^{(k)} - \cdots - l_{-n+k-1}^{(k)} + l_{-n+k}^{(k)}. \tag{5}$$

For example, for $k = 2$, $(l_n^{(2)})_{n\in\mathbb{Z}}$ is the Lucas sequence.

For $k = 3$, $(l_n^{(3)})_{n\in\mathbb{Z}}$ is the Tribonacci-Lucas sequence

$$\dots, 6, -11, 6, 1, -4, 3, 0, -1, 2, 1, 2, 5, 8, 15, 28, 51, \dots$$

For $k = 4$, $(l_n^{(4)})_{n\in\mathbb{Z}}$ is the Quadrabonacci-Lucas sequence

$$\dots, -11, 6, 0, 1, -4, 3, 0, 0, -1, 2, 1, 2, 4, 9, 16, 31, 60, \dots$$

Secondly, we establish the extension of the family of generalized order-k Fibonacci sequences, $(g_n^{(i)})_n$, $1 \leq i \leq k$, to negatively subscripted indices, in the following way:

$$g_{-n}^{(i)} = -\frac{c_{k-1}}{c_k} g_{-n+1}^{(i)} - \frac{c_{k-2}}{c_k} g_{-n+2}^{(i)} - \cdots - \frac{c_1}{c_k} g_{-n+k-1}^{(i)} + \frac{1}{c_k} g_{-n+k}^{(i)}. \tag{6}$$

where, $c_j \in \mathbb{R}$ for $1 \leq j \leq k$ and $c_k \neq 0$. And $g_{-n}^{(i)}$ is said to be the n^{th} negatively subscripted generalized order-k Fibonacci number of the sequence number i of the family.

For instance, we give the extension to negative indices of classical order-k sequences. We start by a family of order-k Fibonacci sequences defined for $1 \leq i \leq k$ and $n \geq k$ as follows:

$$f_n^{(i)} = f_{n-1}^{(i)} + f_{n-2}^{(i)} + \cdots + f_{n-k}^{(i)},$$

with initial conditions; for $0 \leq n \leq k-1$,

$$f_n^{(i)} = \begin{cases} 1, & \text{for } i = k-n; \\ 0, & \text{otherwise.} \end{cases}$$

For $n \geq 1$, we give the extension of the family of order-k Fibonacci sequences to negative indices as follows:

$$f_{-n}^{(i)} = -f_{-n+1}^{(i)} - f_{-n+2}^{(i)} - \cdots - f_{-n+k-1}^{(i)} + f_{-n+k}^{(i)}. \tag{7}$$

The following table gives the first terms of negatively subscripted order-k Fibonacci sequences for $n \geq 1$,

	$k=2$	$k=3$	$k=4$
$i=1$	$1,-1,2,-3,5,-8,13,-21,34,-55\dots$	$1,-1,0,2,-3,1,4,-8,5,7\dots$	$1,-1,0,0,2,-3,1,0,4,-8\dots$
$i=2$	$-1,2,-3,5,-8,13,-21,34,-55,89\dots$	$-1,2,-1,-2,5,-4,-3,12,-13,-2\dots$	$-1,2,-1,0,-2,5,-4,1,-4,12\dots$
$i=3$		$-1,0,2,-3,1,4,-8,5,7,-20\dots$	$-1,0,2,-1,-2,1,4,-4,-3,4\dots$
$i=4$			$-1,0,0,2,-3,1,0,4,-8,5\dots$

Table 1: First terms of negatively subscripted order-k Fibonacci sequences

Remark 2. For $k=2$, $(f_{-n}^{(1)})_n$ is the extension to negative indices of classical Fibonacci numbers.

Kiliç and Taşci [5] studied a family of k sequences of order-k Pell numbers defined for $n>0$ and $1\le i\le k$, as follows:

$$P_n^{(i)} = 2P_{n-1}^{(i)} + P_{n-2}^{(i)} + \cdots + P_{n-k}^{(i)}. \tag{8}$$

with initial conditions; for $1-k\le n\le 0$,

$$P_n^{(i)} = \begin{cases} 1, & \text{for } n = 1-i; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we consider k sequences of shifted order-k Pell numbers (8) for $n\ge k$ and $1\le i\le k$, with initial conditions, for $0\le n\le k-1$,

$$P_n^{(i)} = \begin{cases} 1, & \text{for } i = k-n; \\ 0, & \text{otherwise.} \end{cases}$$

Then for $n\ge 1$, we give extension of these sequences (8) to negative indices as follows:

$$P_{-n}^{(i)} = -P_{-n+1}^{(i)} - \cdots - P_{-n+k-2}^{(i)} - 2P_{-n+k-1}^{(i)} + P_{-n+k}^{(i)}. \tag{9}$$

The next table gives the first terms of negatively subscripted order-k Pell sequences for $n\ge 1$,

	$k=2$	$k=3$	$k=4$
$i=1$	$1,-2,5,-12,29,-70,169,-408\dots$	$1,-1,-1,4,-3,-6,16,-7,-31,61\dots$	$1,-1,0,-1,4,-4,2,-7,17,-18\dots$
$i=2$	$-2,5,-12,29,-70,169,-408,985\dots$	$-2,3,1,-9,10,9,-38,30,55,-153\dots$	$-2,3,-1,2,-9,12,-8,16,-41,53\dots$
$i=3$		$-2,3,-3,1,4,-12,21,-26,19,9\dots$	$-1,-1,3,0,-2,-5,10,-1,-1,-23\dots$
$i=4$			$-1,0,-1,4,-4,2,-7,17,-18,17\dots$

Table 2: First terms of negatively subscripted order-k Pell sequences

Remark 3. Some of the sequences in Table 2 are known in OEIS [9], as for example, A215936, A276229, and A078021.

In 2009, Yilmaz and Bozkurt [10] studied a family of k sequences of order-k Jacobsthal numbers defined for $n > 0$ and $1 \leq i \leq k$, as follows:

$$J_n^{(i)} = J_{n-1}^{(i)} + 2J_{n-2}^{(i)} + \cdots + J_{n-k}^{(i)}, \tag{10}$$

with initial conditions for $1 - k \leq n \leq 0$,

$$J_n^{(i)} = \begin{cases} 1, & \text{for } n = 1 - i; \\ 0, & \text{otherwise.} \end{cases}$$

As before, we consider k sequences of shifted order-k Jacobsthal numbers (10) for $n \geq k$ and $1 \leq i \leq k$, with initial conditions for $0 \leq n \leq k - 1$,

$$J_n^{(i)} = \begin{cases} 1, & \text{for } i = k - n; \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $n \geq 1$, we give the extension of these sequences (10) to negative indices as follows:

$$J_{-n}^{(i)} = -\frac{1}{2}J_{-n+1}^{(i)} + \frac{1}{2}J_{-n+2}^{(i)}, \text{ for } k = 2,\ 1 \leq i \leq 2, \tag{11}$$

$$J_{-n}^{(i)} = -J_{-n+1}^{(i)} - \cdots - J_{-n+k-3}^{(i)} - 2J_{-n+k-2}^{(i)} - J_{-n+k-1}^{(i)} + J_{-n+k}^{(i)}. \text{ for } k \geq 3 \tag{12}$$

The next table gives the first terms of negatively subscripted order-k Jacobsthal sequences for $n \geq 1$,

	$k=2$	$k=3$	$k=4$
$i=1$	$\frac{1}{2}, -\frac{1}{4}, \frac{3}{8}, -\frac{5}{16}, \frac{11}{32}, -\frac{21}{64}, \frac{43}{128}, -\frac{85}{256} \ldots$	$1, -2, 3, -3, 1, 4, -12, 21, -26, 19 \ldots$	$1, -1, -1, 2, 2, -6, -1, 13, -3 \ldots$
$i=2$	$-\frac{1}{2}, \frac{3}{4}, -\frac{5}{8}, \frac{11}{16}, -\frac{21}{32}, \frac{43}{64}, -\frac{85}{128}, \frac{171}{256} \ldots$	$-1, 3, -5, 6, -4, -3, 16, -33, 47 \ldots$	$-1, 2, 0, -3, 0, 8, -5, -14, 16, 25 \ldots$
$i=3$		$-2, 3, -3, 1, 4, -12, 21, -26, 19, 9 \ldots$	$-2, 1, 4, -4, -7, 12, 10, -31, -8 \ldots$
$i=4$			$-1, -1, 2, 2, -6, -1, 13, -3, -28 \ldots$

Table 3: First terms of negatively subscripted order-k Jacobsthal sequences

Remark 4. The sequences of $k = 3$ are known in the OEIS as A077990, -A078064, and A077990 respectively.

We now give some results on the n^{th} negatively subscripted generalized order-k Fibonacci number $g_{-n}^{(i)}$ using matrix methods.

Theorem 5. *For $n \geq 0$ and $2 \leq i \leq k$, we have*

$$g_{-n-1}^{(i)} = -\frac{c_{i-1}}{c_k} g_{-n}^{(k)} + g_{-n}^{(i-1)}, \tag{13}$$

and for $i = 1$,

$$g_{-n-1}^{(1)} = \frac{1}{c_k} g_{-n}^{(k)}. \tag{14}$$

Proof. For the proof, we use the matrix approach. The matrix approach was used, for example, by Kalman [4] and Er [3] to study generalized Fibonacci numbers. Let

$$A := \begin{pmatrix} -\frac{c_{k-1}}{c_k} & -\frac{c_{k-2}}{c_k} & -\frac{c_{k-3}}{c_k} & \cdots & -\frac{c_1}{c_k} & \frac{1}{c_k} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

We have the following n order-k recurrence relations

$$\begin{pmatrix} g^{(i)}_{-n-1} \\ g^{(i)}_{-n} \\ \vdots \\ g^{(i)}_{-n+k-2} \end{pmatrix} = A \begin{pmatrix} g^{(i)}_{-n} \\ g^{(i)}_{-n+1} \\ \vdots \\ g^{(i)}_{-n+k-1} \end{pmatrix}. \tag{15}$$

In order to deal with k sequences of negatively subscripted generalized order-k Fibonacci sequences at the same time, we construct a $k \times k$ square matrix G_{-n} as follows:

$$G_{-n} = \begin{pmatrix} g^{(k)}_{-n} & g^{(k-1)}_{-n} & g^{(k-2)}_{-n} & \cdots & g^{(1)}_{-n} \\ g^{(k)}_{-n+1} & g^{(k-1)}_{-n+1} & g^{(k-2)}_{-n+1} & \cdots & g^{(1)}_{-n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g^{(k)}_{-n+k-1} & g^{(k-1)}_{-n+k-1} & g^{(k-2)}_{-n+k-1} & \cdots & g^{(1)}_{-n} \end{pmatrix}.$$

It is clear that, $A = G_{-1}$.

Next, (15) becomes

$$G_{-n-1} = AG_{-n}.$$

By induction, it is also clear that

$$G_{-n-1} = A^n A = G_{-n}A.$$

Finally, we have

$$G_{-n-1} = AG_{-n} = G_{-n}A. \tag{16}$$

Thus, Equations (13) and (14) are deduced from (16). □

From Theorem 5, we get the following results:

Corollary 6. *Let $f^{(i)}_{-n}$ be the n^{th} negatively subscripted order-k Fibonacci number of the sequence number i. Then*

$$f^{(1)}_{-n-1} = f^{(k)}_{-n};$$
$$f^{(i)}_{-n-1} = -f^{(k)}_{-n} + f^{(i-1)}_{-n}; 2 \le i \le k.$$

Corollary 7. *Let $P_{-n}^{(i)}$ be the n^{th} negatively subscripted order-k Pell number of the sequence number i. Then*

$$\begin{aligned}
&P_{-n-1}^{(1)} = P_{-n}^{(k)};\\
&P_{-n-1}^{(2)} = -2P_{-n}^{(k)} + P_{-n}^{(1)};\\
&P_{-n-1}^{(i)} = -P_{-n}^{(k)} + P_{-n}^{(i-1)};\ 3 \leq i \leq k.
\end{aligned}$$

Corollary 8. *Let $J_{-n}^{(i)}$ be the n^{th} negatively subscripted generalized order-k Jacobsthal number of the sequence number i. Then for $k \geq 3$,*

$$\begin{aligned}
&J_{-n-1}^{(1)} = J_{-n}^{(k)};\\
&J_{-n-1}^{(2)} = -J_{-n}^{(k)} + J_{-n}^{(1)};\\
&J_{-n-1}^{(3)} = -2J_{-n}^{(k)} + J_{-n}^{(2)};\\
&J_{-n-1}^{(i)} = -J_{-n}^{(k)} + J_{-n}^{(i-1)};\ 4 \leq i \leq k.
\end{aligned}$$

We conclude this section by defining a generalized order-k Fibonacci sequence $(V_n)_n$ with arbitrary initial conditions and we give the extension of the sequence $(V_n)_n$ to negative indices. Then we give the n^{th} negatively subscripted number V_{-n} as the sum of k terms of negatively subscripted generalized order-k Fibonacci numbers defined in (6).

For $n \geq k$, let V_n be as follows:

$$V_n = c_1 V_{n-1} + c_2 V_{n-2} + \cdots + c_k V_{n-k},$$

with integer initial conditions $V_0, V_1, \ldots, V_{k-1}$, where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Belbachir and Bencherif [1] gave the explicit formula of the n^{th} term of the sequence $(V_n)_n$.

Then for $n \geq 1$, the value V_{-n} is as follows:

$$V_{-n} = -\frac{c_{k-1}}{c_k} V_{-n+1} - \frac{c_{k-2}}{c_k} V_{-n+2} - \cdots - \frac{c_1}{c_k} V_{-n+k-1} + \frac{1}{c_k} V_{-n+k}. \tag{17}$$

Note that for $(V_0, \ldots, V_{k-i-1}, V_{k-i}, V_{k-i+1}, \ldots, V_{k-1}) = (0, \ldots, 0, 1, 0, \ldots, 0)$, $1 \leq i \leq k$, we have $V_n = g_n^{(i)}$.

The n^{th} term V_{-n} is given by

Theorem 9. *For $n \geq 1$,*

$$V_{-n} = \sum_{i=0}^{k-1} a_i g_{-n}^{(k-i)}, \tag{18}$$

where $g_{-n}^{(j)}$ is the n^{th} negatively subscripted generalized order-k Fibonacci number of the sequence number i (6).

Proof. By induction on n.

For $n=1$, $V_{-1} = -\frac{c_{k-1}}{c_k}V_0 - \cdots - \frac{c_1}{c_k}V_{k-2} + \frac{1}{c_k}V_{k-1} = -\frac{c_{k-1}}{c_k}a_0 - \cdots - \frac{c_1}{c_k}a_{k-2} + \frac{1}{c_k}a_{k-1} = \sum_{i=0}^{k-1} a_i g_{-1}^{(k-i)}$.

Now suppose that (18) is true for $2 \le j \le n$ and we prove the formula for $n+1$:

$$V_{-n-1} = -\frac{c_{k-1}}{c_k}V_{-n} - \frac{c_{k-2}}{c_k}V_{-n+1} - \cdots - \frac{c_1}{c_k}V_{-n+k-2} + \frac{1}{c_k}V_{-n+k-1}.$$

and

$$\begin{aligned}
V_{-n-1} &= -\frac{c_{k-1}}{c_k}\sum_{i=0}^{k-1} a_i g_{-n}^{(k-i)} - \frac{c_{k-2}}{c_k}\sum_{i=0}^{k-1} a_i g_{-n+1}^{(k-i)} - \cdots - \frac{c_1}{c_k}\sum_{i=0}^{k-1} a_i g_{-n+k-2}^{(k-i)} + \frac{1}{c_k}\sum_{i=0}^{k-1} a_i g_{-n+k-1}^{(k-i)} \\
&= a_0\left(-\frac{c_{k-1}}{c_k}g_{-n}^{(k)} - \frac{c_{k-2}}{c_k}g_{-n+1}^{(k)} - \cdots - \frac{c_1}{c_k}g_{-n+k-2}^{(k)} + \frac{1}{c_k}g_{-n+k-1}^{(k)}\right) \\
&+ a_1\left(-\frac{c_{k-1}}{c_k}g_{-n}^{(k-1)} - \frac{c_{k-2}}{c_k}g_{-n+1}^{(k-1)} - \cdots - \frac{c_1}{c_{k-1}}g_{-n+k-2}^{(k-1)} + \frac{1}{c_{k-1}}g_{-n+k-1}^{(k-1)}\right) \\
&+ \cdots + \\
&+ a_{k-1}\left(-\frac{c_{k-1}}{c_k}g_{-n}^{(1)} - \frac{c_{k-2}}{c_k}g_{-n+1}^{(1)} - \cdots - \frac{c_1}{c_k}g_{-n+k-2}^{(1)} + \frac{1}{c_k}g_{-n+k-1}^{(1)}\right) \\
&= a_0 g_{-n-1}^{(k)} + a_1 g_{-n-1}^{(k-1)} + \cdots + a_{k-1} g_{-n-1}^{(1)} \\
&= \sum_{i=0}^{k-1} a_i g_{-n-1}^{(k-i)}.
\end{aligned}$$

Thus (18) is true for all $n \ge 1$. □

3 Representation of negatively subscripted generalized order-k Fibonacci numbers

In this section, we give a representation of the family of negatively subscripted generalized order-k Fibonacci numbers (6) using the permanent of Hessenberg matrices.

We introduce the $n \times n$ matrix $W_n = (w_{st})$ with $w_{1t} = 0$, for $1 \le t \le n$ and for all $1 < i \le n$,

$$w_{st} = \begin{cases} 1, & \text{if } s = t+1; \\ -\frac{c_{k-(t-s)-1}}{c_k}, & \text{if } 0 \le t-s \le k-2; \\ \frac{1}{c_k}, & \text{if } s = t-k+1; \\ 0, & \text{elsewhere.} \end{cases}$$

And we construct, for $n \ge 1$, the $n \times n$ matrix A_n^i, for $1 \le i \le k$, as follows:

$$A_n^i = W_n + \sum_{t=1}^{n}\left(\frac{\delta_{i,k}^t(k)}{c_k} - \sum_{j=1}^{k-1}\delta_{i,j}^t(k)\frac{c_{k-j}}{c_k}\right)E_{1t}$$

with $\delta^c_{a,b}(d)$ defined for $1 \leq a, b, c \leq d$ as

$$\delta^c_{a,b}(d) = \begin{cases} 1, & \text{if } a + b = c + d; \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

And E_{st} denotes the $n \times n$ matrix with 1 in position (s, t) and 0 elsewhere.

Then A^i_n is as follows:

$$A^i_n = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_k & 0 & 0 & \cdots & 0 \\ 1 & -\frac{c_{k-1}}{c_k} & -\frac{c_{k-2}}{c_k} & \cdots & -\frac{c_1}{c_k} & \frac{1}{c_k} & 0 & \cdots & 0 \\ 0 & 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & \frac{1}{c_k} \\ & & & & & & & \ddots & -\frac{c_1}{c_k} \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}, \tag{20}$$

where $A_t = \frac{\delta^t_{i,k}(k)}{c_k} - \sum_{j=1}^{k-1} \delta^t_{i,j}(k) \frac{c_{k-j}}{c_k}$, for $1 \leq t \leq k$.

We now give the representation of negatively subscripted generalized order-k Fibonacci numbers.

Theorem 10. *For $n \geq 1$, let A^i_n be as in (20) and $g^{(i)}_{-n}$ be the n^{th} negatively subscripted generalized order-k Fibonacci number of the sequence number i, then*

$$\operatorname{per} A^i_n = g^{(i)}_{-n}. \tag{21}$$

For instance, we give the following representations of k sequences of negatively subscripted order-k Fibonacci, order-k Pell, and order-k Jacobsthal numbers.

Let F^i_n be the $n \times n$ matrix defined, for $1 \leq i \leq k$, as follows:

$$F^i_n = \begin{pmatrix} \delta^1_{i,k}(k) - \sum_{j=1}^{k-1} \delta^1_{i,j}(k) & \delta^2_{i,k}(k) - \sum_{j=1}^{k-1} \delta^2_{i,j}(k) & \delta^3_{i,k}(k) - \sum_{j=1}^{k-1} \delta^3_{i,j}(k) & \cdots & \delta^k_{i,k} - \sum_{j=1}^{k-1} \delta^k_{i,j}(k) & 0 & 0 & \cdots & 0 \\ 1 & -1 & -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & -1 & -1 & 1 & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & 1 \\ & & & & & & & \ddots & -1 \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 1 & -1 \end{pmatrix}. \tag{22}$$

where $\delta^t_{i,k}(k)$ and $\delta^t_{i,j}(k)$ is defined as in (19) for all $1 \leq t \leq k$ and $1 \leq j \leq k - 1$.

Corollary 11. *For $n \geq 1$, let F_n^i be as in (22) and $f_{-n}^{(i)}$ be the n^{th} negatively subscripted order-k Fibonacci number of the sequence number i, then*

$$\operatorname{per} F_n^i = f_{-n}^{(i)}. \tag{23}$$

Let $\mathcal{P}_n^i$ be the $n \times n$ matrix defined for $1 \leq i \leq k$ in the following way:

$$\mathcal{P}_n^i = \begin{pmatrix} S_1 & S_2 & \cdots & S_{k-1} & S_k & 0 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & -1 & -2 & 1 & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & 1 \\ & & & & & & & \ddots & -2 \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 1 & -1 \end{pmatrix}, \tag{24}$$

where $S_t = \delta_{i,k}^t - 2\delta_{i,k-1}^t - \sum_{j=1}^{k-2} \delta_{i,j}^t$ for $1 \leq t \leq k$ and $\delta_{i,j}^t(k)$ is defined as in (19).

Corollary 12. *For $n \geq 1$, let $\mathcal{P}_n^i$ be as in (24) and $P_{-n}^{(i)}$ be the n^{th} negatively subscripted order-k Pell number of the sequence number i, then*

$$\operatorname{per} \mathcal{P}_n^i = P_{-n}^{(i)}. \tag{25}$$

Let $\mathcal{J}_n^i$ be the $n \times n$ matrix defined for $1 \leq i \leq k$ and $k \geq 3$ as follows:

$$\mathcal{J}_n^i = \begin{pmatrix} T_1 & T_2 & T_3 & \cdots & T_{k-2} & T_{k-1} & T_k & 0 & 0 & \cdots & 0 \\ 1 & -1 & -1 & \cdots & -1 & -2 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & -1 & -1 & -2 & -1 & 1 & \ddots & \vdots \\ & & & & & & & & & \ddots & 0 \\ \vdots & & & & & & & & & \ddots & 1 \\ & & & & & & & & & \ddots & -1 \\ & & & & & & & & & \ddots & -2 \\ & & & & & & & & & \ddots & -1 \\ & & & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & & & \cdots & 0 & 1 & -1 \end{pmatrix}, \tag{26}$$

where $T_t = \delta_{i,k}^t - 2\delta_{i,k-2}^t - \delta_{i,k-1}^t - \sum_{j=1}^{k-3} \delta_{i,j}^t$ for $1 \leq t \leq k$ and $\delta_{i,j}^t(k)$ is defined as in (19).

Corollary 13. *For $n \geq 1$, let $\mathcal{J}_n^i$ be as in (26) and $J_{-n}^{(i)}$ be the n^{th} negatively subscripted order-k Jacobsthal number of the sequence number i, then*

$$\operatorname{per} \mathcal{J}_n^i = J_{-n}^{(i)}. \tag{27}$$

4 More permanental representations of negatively subscripted order-k Fibonacci and order-k Lucas numbers

In this section, we give numerous types of matrices whose permanent are negatively subscripted order-k Fibonacci and order-k Lucas terms.

We first establish two kinds of permanental representations of the negatively subscripted order-k Fibonacci sequence $(\widetilde{g}_{-n})_n$. We introduce an $n \times n$ $(0, 1, -1)$-matrix $B_{(n,k)}$ defined by

$$b_{ij} = \begin{cases} 1, & \text{for } j - i = -1 \text{ or } k - 1; \\ -1, & \text{for } 0 \leq j - i \leq k - 2; \\ 0, & \text{elsewhere.} \end{cases} \qquad (\star)$$

And we construct for $n > k$, the $n \times n$ $(0, 1, -1)$-matrices $F_{(n,k)}$ and $\widetilde{F}_{(n,k)}$ as follows:

$$F_{(n,k)} = B_{(n,k)} + 2E_{11} + \sum_{j=2}^{k-1} E_{1j} - E_{1k},$$

$$\widetilde{F}_{(n,k)} = B_{(n,k)} + 2\sum\nolimits_{j=1}^{k-1} E_{1j},$$

where E_{ij} denotes the $n \times n$ matrix with 1 in position (i, j) and 0 elsewhere. $F_{(n,k)}$ and $\widetilde{F}_{(n,k)}$ correspond to

$$F_{(n,k)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & & \ddots & 0 \\ & & & & & & \ddots & 1 \\ & & & & & & & -1 \\ \vdots & & & & & \ddots & \ddots & \vdots \\ 0 & & & & & & 1 & -1 \end{pmatrix}$$

and

$$\widetilde{F}_{(n,k)} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & & \ddots & 0 \\ & & & & & & \ddots & 1 \\ & & & & & & & -1 \\ \vdots & & & & & \ddots & \ddots & \vdots \\ 0 & & & & & & 1 & -1 \end{pmatrix}.$$

Then we give the two permanental representations of the negatively subscripted order-k Fibonacci sequence.

Theorem 14. *For $n > k$, we have*

$$\operatorname{per} F_{(n,k)} = \widetilde{g}_{-n}^{(k)} \tag{28}$$

and

$$\operatorname{per} \widetilde{F}_{(n,k)} = \widetilde{g}_{-n+k}^{(k)}. \tag{29}$$

Next, using the same approach given in Theorem 14, see next section, we give a representation of $l_{-n+1}^{(k)}$.

We use the $n \times n$ square matrix $B_{(n,k)} = (b_{ij})$ given by ($\star$), and we construct for $n > k$, the $n \times n$ matrix $A_{(n,k)}$ as follows:

$$A_{(n,k)} = B_{(n,k)} + 2E_{11} + \sum_{j=2}^{k-1} E_{1j} - E_{1k} + E_{1,k+1},$$

where E_{ij} is the $n \times n$ matrix with 1 in position (i, j) and 0 elsewhere.

The matrix $A_{(n,k)}$ corresponds to

$$A_{(n,k)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & -1 & 1 & 0 & \ddots & \vdots \\ & \ddots & \ddots & & \ddots & -1 & & & 0 \\ & & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & & & & 1 \\ & & & & & & & & -1 \\ & & & & & & & \ddots & \vdots \\ & & & & & & \ddots & \ddots & \vdots \\ 0 & & \cdots & & & 0 & & 1 & -1 \end{pmatrix}.$$

Theorem 15. *For $n > k$, we have*

$$\operatorname{per} A_{(n,k)} = l_{-n+1}^{(k)}.$$

Proof. The proof is the same as Theorem 14, see next section. □

Finally, let B_n be an $n \times n$ matrix defined by

$$B_n = T_n + 2(E_{11} + E_{22} + E_{33} - E_{43}) + E_{13} - E_{23} + E_{24} - E_{34},$$

where T_n is the $n \times n$ $(0, 1, -1)$-matrix such that $t_{ii} = -1$, $t_{ij} = 1$ if and only if $|j - i| = 1$.

The matrix B_n corresponds to

$$B_n = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & \cdots & & 0 \\ 1 & 1 & 0 & 1 & 0 & & & \vdots \\ 0 & 1 & 1 & 0 & 0 & & & \\ 0 & 0 & -1 & -1 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & -1 & 1 & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & & & & 0 & 1 & -1 \end{pmatrix}.$$

Then we obtain the following result.

Theorem 16. *For $n \geq 4$, we have*

$$\operatorname{per} B_n = l_{-n+1}^{(2)}.$$

5 Proofs

We start this section by giving the proof of Theorem 10,

Proof. We have

$$\frac{\delta_{i,k}^1}{c_k} - \sum_{t=1}^{k-1} \delta_{i,t}^1 \frac{c_{k-t}}{c_k} = \begin{cases} \frac{1}{c_k}, & \text{for } i = 1; \\ -\frac{c_{i-1}}{c_k}, & \text{for } 2 \leq i \leq k. \end{cases}$$

On the other hand by Theorem 5, we have

$$g_{-1}^{(i)} = \begin{cases} \frac{1}{c_k}, & \text{for } i = 1; \\ -\frac{c_{i-1}}{c_k}, & \text{for } 2 \leq i \leq k. \end{cases}$$

Thus,

$$\frac{\delta_{i,k}^1}{c_k} - \sum_{t=1}^{k-1} \delta_{i,t}^1 \frac{c_{k-t}}{c_k} = g_{-1}^{(i)}.$$

Next for $2 \leq j \leq k-1$, by the definition of shifted generalized order-k Fibonacci sequences (1), we have

$$\frac{\delta_{i,k}^j}{c_k} - \sum_{t=1}^{k-1} \delta_{i,t}^j \frac{c_{k-t}}{c_k} = \begin{cases} 0, & \text{for } 1 \leq i \leq j-1; \\ \frac{1}{c_k} g_{k-j}^{(i)}, & \text{for } i = j; \\ -\frac{c_{i-1}}{c_k} g_{k-i}^{(i)}, & \text{for } j+1 \leq i \leq k. \end{cases}$$

Then $\frac{\delta^j_{i,k}}{c_k} - \sum_{t=1}^{k-1} \delta^j_{i,t} \frac{c_{k-t}}{c_k}$ can be written as

$$\frac{\delta^j_{i,k}}{c_k} - \sum_{t=1}^{k-1} \delta^j_{i,t} \frac{c_{k-t}}{c_k} = \frac{1}{c_k} g^{(i)}_{k-j} - \sum_{t=j}^{k-1} \frac{c_{k-t}}{c_k} g^{(i)}_{t-j}.$$

Finally,

$$\frac{\delta^k_{i,k}}{c_k} - \sum_{t=1}^{k-1} \delta^k_{i,t} \frac{c_{k-t}}{c_k} = \begin{cases} \frac{1}{c_k} g^{(k)}_0, & \text{for } i = k; \\ 0, & \text{otherwise.} \end{cases}$$

Then for $1 \le i \le k$, we have

$$\frac{\delta^k_{i,k}}{c_k} - \sum_{t=1}^{k-1} \delta^k_{i,t} \frac{c_{k-t}}{c_k} = \frac{1}{c_k} g^{(i)}_0.$$

Hence, the matrix A^i_n can be written as follows:

$$A^i_n = \begin{pmatrix} g^{(i)}_{-1} & \frac{1}{c_k} g^{(i)}_{k-2} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g^{(i)}_{t-2} & \frac{1}{c_k} g^{(i)}_{k-3} - \sum_{t=3}^{k-1} \frac{c_{k-t}}{c_k} g^{(i)}_{t-3} & \cdots & \frac{1}{c_k} g^{(i)}_0 & 0 & 0 & \cdots & 0 \\ 1 & -\frac{c_{k-1}}{c_k} & -\frac{c_{k-2}}{c_k} & \cdots & -\frac{c_1}{c_k} & \frac{1}{c_k} & 0 & \cdots & 0 \\ 0 & 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & \frac{1}{c_k} \\ & & & & & & & \ddots & -\frac{c_1}{c_k} \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

By Lemma 1, the matrix A^i_n is contractible on column 1 relative to rows 1 and 2. Let $(A^i_n)^1$ be the $(n-1) \times (n-1)$ contraction matrix, then we have $\operatorname{per} A^i_n = \operatorname{per}(A^i_n)^1$ and $(A^i_n)^1$ is as follows:

$$(A^i_n)^1 = \begin{pmatrix} g^{(i)}_{-2} & \frac{1}{c_k} g^{(i)}_{k-3} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g^{(i)}_{t-3} & \frac{1}{c_k} g^{(i)}_{k-4} - \sum_{t=3}^{k-1} \frac{c_{k-t}}{c_k} g^{(i)}_{t-4} & \cdots & \frac{1}{c_k} g^{(i)}_{-1} & 0 & 0 & \cdots & 0 \\ 1 & -\frac{c_{k-1}}{c_k} & -\frac{c_{k-2}}{c_k} & \cdots & -\frac{c_1}{c_k} & \frac{1}{c_k} & 0 & \cdots & 0 \\ 0 & 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & \frac{1}{c_k} \\ & & & & & & & \ddots & -\frac{c_1}{c_k} \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

Furthermore, the matrix $(A_n^i)^1$ is also contractible on column 1 relative to rows 1 and 2. Continuing the same contraction process we get by Lemma 1,

$$\operatorname{per} A_n^i = \operatorname{per}(A_n^i)^p$$

with $(A_n^i)^p$ defined for $p \leq n-k$ by

$$(A_n^i)^p = \begin{pmatrix} g_{-p-1}^{(i)} & \frac{1}{c_k} g_{-p+k-2}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{-p+t-2}^{(i)} & \cdots & \frac{1}{c_k} g_{-p+1}^{(i)} - \frac{c_1}{c_k} g_{-p}^{(i)} & \frac{1}{c_k} g_{-p}^{(i)} & 0 & 0 & \cdots & 0 \\ 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & -\frac{c_3}{c_k} & -\frac{c_2}{c_k} & -\frac{c_1}{c_k} & \frac{1}{c_k} & \ddots & \vdots \\ & & & & & & & \ddots & 0 \\ \vdots & & & & & & & \ddots & \frac{1}{c_k} \\ & & & & & & & \ddots & -\frac{c_1}{c_k} \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

And for $p \geq n-k+1$, $(A_n^i)^p$ is defined as follows:

$$(A_n^i)^p = \begin{pmatrix} g_{-p-1}^{(i)} & \frac{1}{c_k} g_{-p+k-2}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{-p+t-2}^{(i)} & \cdots & \frac{1}{c_k} g_{-n+k+1}^{(i)} - \sum_{t=n-p+1}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t+1}^{(i)} & \frac{1}{c_k} g_{-n+k}^{(i)} - \sum_{t=n-p}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t}^{(i)} \\ 1 & -\frac{c_{k-1}}{c_k} & \cdots & -\frac{c_{k-n+p+2}}{c_k} & -\frac{c_{k-n+p+1}}{c_k} \\ 0 & 1 & \cdots & -\frac{c_{k-n+p+3}}{c_k} & -\frac{c_{k-n+p+2}}{c_k} \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

Then

$$(A_n^i)^{n-2} = \begin{pmatrix} g_{-n+1}^{(i)} & \frac{1}{c_k} g_{-n+k}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t}^{(i)} \\ 1 & -\frac{c_{k-1}}{c_k} \end{pmatrix}.$$

By Lemma 1, we have

$$\begin{aligned} \operatorname{per} A_n^i = \operatorname{per}(A_n^i)^{n-2} &= -\frac{c_{k-1}}{c_k} g_{-n+1}^{(i)} + \frac{1}{c_k} g_{-n+k}^{(i)} - \sum_{t=2}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t}^{(i)} \\ &= \frac{1}{c_k} g_{-n+k}^{(i)} - \sum_{t=1}^{k-1} \frac{c_{k-t}}{c_k} g_{-n+t}^{(i)} \\ &= g_{-n}^{(i)}. \end{aligned}$$

So $\operatorname{per} A_n^i = g_{-n}^{(i)}$ and the proof is complete. □

Secondly, we provide the proof of Theorem 14, *Proof.* Let $F^p_{(n,k)} = (f^p_{ij})$ be the p^{th} contraction of $F_{(n,k)}$, $1 \le p \le n-2$.

The matrix $F_{(n,k)} = (f_{ij})$, can be contracted on column 1 relative to rows 1 and 2.

We can easily verify that if $p = 1$,

$f^1_{11} = f^1_{12} = \cdots = f^1_{1k-1} = -1$, $f^1_{1k} = 1$ and $f^1_{1q} = 0$, for $q \ge k+1$; and for all $i = 2, \ldots, n-1$,

$$\begin{cases} f_{i,i-1} = 1, & \\ f_{ij} = -1, & \text{for } 0 \le j-i \le k-2; \\ f_{ij} = 1 & \text{for } j-i = k-1. \end{cases}$$

Hence

$$F^1_{(n,k)} = \begin{pmatrix} -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & \ddots & \vdots \\ & & & & \ddots & \ddots & 0 \\ & & & & & \ddots & 1 \\ \vdots & & & & & & -1 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & \cdots & & & 1 & -1 \end{pmatrix}$$

and

$$F^1_{(n,k)} = \begin{pmatrix} \widetilde{g}^{(k)}_{-2} & \widetilde{g}^{(k)}_{k-3} - \sum_{j=-1}^{k-4} \widetilde{g}^{(k)}_j & \widetilde{g}^{(k)}_{k-4} - \sum_{j=-1}^{k-5} \widetilde{g}^{(k)}_j & \cdots & -\widetilde{g}^{(k)}_{-1} + \widetilde{g}^{(k)}_0 & \widetilde{g}^{(k)}_{-1} & 0 & \cdots & 0 \\ 1 & -1 & -1 & \cdots & & -1 & 1 & \ddots & \vdots \\ 0 & 1 & -1 & \cdots & & & -1 & \ddots & 0 \\ & & & & & & & \ddots & 1 \\ & & & & & & & & -1 \\ & & & & & & \ddots & \ddots & \vdots \\ 0 & & & \cdots & & & 0 & 1 & -1 \end{pmatrix}.$$

Furthermore, the matrix $F^1_{(n,k)}$ can be contracted. From Lemma 1 we obtain

$$\operatorname{per} F_{(n,k)} = \operatorname{per} F^p_{(n,k)}$$

with $F^p_{(n,k)}$ defined for $p \le n-k$ by

$$F^p_{(n,k)} = \begin{pmatrix} \widetilde{g}^{(k)}_{-p-1} & \widetilde{g}^{(k)}_{-p+k-2} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-p+j} & \cdots & -\widetilde{g}^{(k)}_{-p} + \widetilde{g}^{(k)}_{-p+1} & \widetilde{g}^{(k)}_{-p} & 0 & \cdots & 0 \\ 1 & -1 & \cdots & \cdots & -1 & 1 & \ddots & \vdots \\ & & & & & \ddots & \ddots & 0 \\ & & & & & & \ddots & 1 \\ & & & & & & & -1 \\ & & & & & \ddots & \ddots & \vdots \\ 0 & & \cdots & & & 0 & 1 & -1 \end{pmatrix}.$$

And for $p \geq n-k+1$, $F^p_{(n,k)}$ is defined as

$$F^p_{(n,k)} = \begin{pmatrix} \widetilde{g}^{(k)}_{-p-1} & \widetilde{g}^{(k)}_{-p+k-2} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-p+j} & \cdots & \widetilde{g}^{(k)}_{k-n+1} - \sum_{j=0}^{k-(n-p)} \widetilde{g}^{(k)}_{-p+j} & \widetilde{g}^{(k)}_{k-n} - \sum_{j=0}^{k-(n-p+1)} \widetilde{g}^{(k)}_{-p+j} \\ 1 & -1 & \cdots & \cdots & -1 \\ 0 & 1 & \cdots & & -1 \\ \vdots & & & & \vdots \\ 0 & & \cdots & 1 & -1 \end{pmatrix}.$$

So we have

$$F^{n-2}_{(n,k)} = \begin{pmatrix} \widetilde{g}^{(k)}_{-n+1} & \widetilde{g}^{(k)}_{-n+k} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-n+2+j} \\ 1 & -1 \end{pmatrix}.$$

Lemma 1 gives

$$\begin{aligned} \operatorname{per} F_{(n,k)} &= \operatorname{per} \begin{pmatrix} \widetilde{g}^{(k)}_{-n+1} & \widetilde{g}^{(k)}_{-n+k} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-n+2+j} \\ 1 & -1 \end{pmatrix} \\ &= -\widetilde{g}^{(k)}_{-n+1} + \widetilde{g}^{(k)}_{-n+k} - \sum_{j=0}^{k-3} \widetilde{g}^{(k)}_{-n+2+j} \\ &= \widetilde{g}^{(k)}_{-n+k} - \left(\widetilde{g}^{(k)}_{-n+1} + \widetilde{g}^{(k)}_{-n+2} + \cdots + \widetilde{g}^{(k)}_{-n+k-1} \right). \end{aligned}$$

So $\operatorname{per} F_{(n,k)} = \widetilde{g}^{(k)}_{-n}$ and the proof of Identity (28) is complete.

By Lemma 1, we can write

$$\operatorname{per} \widetilde{F}_{(n,k)} = \operatorname{per} \widetilde{F}^1_{(n,k)}, \tag{30}$$

where $\widetilde{F}^1_{(n,k)}$ is the first contraction of $\widetilde{F}_{(n,k)}$.

Then $\widetilde{f}^1_{11} = \widetilde{f}^1_{12} = \cdots = \widetilde{f}^1_{1k-1} = 0$, $\widetilde{f}^1_{1k} = 1$ and $\widetilde{f}^1_{1q} = 0$ ($q \geq k+1$); and for $i = 2, \ldots, n-1$, $\widetilde{f}^1_{i,i-1} = 1$, $\widetilde{f}^1_{ij} = -1$ for $(0 \leq j-i \leq k-2)$ and $\widetilde{f}^1_{ij} = 1$ for $(j-i = k-1)$.

That is,

$$\operatorname{per} \widetilde{F}^1_{(n,k)} = \operatorname{per} \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & \ddots & \vdots \\ 0 & 1 & & & \ddots & \ddots & 0 \\ & & & & & \ddots & 1 \\ & & & & & & -1 \\ & & & & & & \vdots \\ 0 & & \cdots & & 0 & 1 & -1 \end{pmatrix}_{n-1}.$$

Computing $\operatorname{per} \widetilde{F}^1_{(n,k)}$ by the Laplace expansion with respect to the first column, we obtain

$$\operatorname{per} \widetilde{F}^1_{(n,k)} = \operatorname{per} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 & 1 & \ddots & \vdots \\ & & & & \ddots & \ddots & 0 \\ & & & & & \ddots & 1 \\ & & & & & & -1 \\ & & & & \ddots & \ddots & \vdots \\ & & \cdots & & & 1 & -1 \end{pmatrix}_{n-k}.$$

From identity (28), $\operatorname{per} \widetilde{F}^1_{(n,k)} = \operatorname{per} F_{(n-k,k)} = \widetilde{g}^{(k)}_{-(n-k)}$, and from (30), we have, $\operatorname{per} \widetilde{F}_{(n,k)} = \widetilde{g}^{(k)}_{-n+k}$. The proof is complete. □

We conclude this section with the proof of Theorem 16.

Proof. If $n = 4$,

$$\operatorname{per} B_4 = \operatorname{per} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} = -4 = l^{(2)}_{-3}.$$

By induction on n, we assume that $\operatorname{per} B_n = l^{(2)}_{-n+1}$ and we establish that $\operatorname{per} B_{n+1} = l^{(2)}_{-n}$.

Let

$$F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$B_{n+1} = \begin{pmatrix} B_n & F \\ F^t & -1 \end{pmatrix}.$$

Let $B_n^{(p)}$ be the p^{th} contraction of B_n. Since

$$B_n^{(n-2)} = \begin{pmatrix} l^{(2)}_{-n+2} & l^{(2)}_{-n+3} \\ 1 & -1 \end{pmatrix},$$

we get

$$\operatorname{per} B_{n+1} = \operatorname{per} B^{(n-2)}_{n+1} = \operatorname{per} \begin{pmatrix} l^{(2)}_{-n+2} & l^{(2)}_{-n+3} & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = l^{(2)}_{-n+2} - l^{(2)}_{-n+3} + l^{(2)}_{-n+2} = l^{(2)}_{-n+2} - l^{(2)}_{-n+1} = l^{(2)}_{-n}.$$

The proof is completed. □

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