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Tornheim-Like Series, Harmonic Numbers, and Zeta Values

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Abstract

We obtain explicit evaluations of the Tornheim-like multiple series involving harmonic numbers. We give a new relationship between harmonic numbers and $\zeta(2)$. We also present closed-form formulas of some multiple series in terms of zeta values.

1 Introduction

The Riemann zeta function is defined by

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}, \quad (\operatorname{Re} z > 1).$$

For even positive integers, one has the well-known relationship between zeta values and Bernoulli numbers:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}. \tag{1}$$

Here

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \dots .$$

This result was proved for the first time by Euler in 1740.

For odd positive integers, no simple expression like (1) is known. Roger Apéry [1] proved the irrationality of $\zeta(3)$, and after that $\zeta(3)$ was named Apéry's constant. Rivoal [17] showed that infinitely many of the numbers $\zeta(2n+1)$ must be irrational. Zudilin [23] showed that at least one of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational.

For a positive integer n and an integer m the nth partial sum of $\zeta(m)$ is called the nth generalized harmonic number of order m, and is denoted by $H_n^{(m)}$, i.e.,

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}.$$

The special case $m = 1$ is the classical harmonic number

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \;\; H_0 = 0.$$

It is well known that there are deep relationships between Tornheim-like series, harmonic numbers and zeta values. The Tornheim double series [19] (or the so-called Witten zeta function [22]) is defined by

$$S(a,b,c) := \sum_{m,n=1}^{\infty} \frac{1}{m^a n^b (m+n)^c}. \tag{2}$$

The following equation is a simple and nice example of a connection between Tornheim-like series, harmonic numbers, and zeta values (see [4, 5, 10, 11, 16]):

$$\sum_{n,m=1}^{\infty} \frac{1}{nm(n+m)} = \sum_{m=1}^{\infty} \frac{H_m}{m^2} = 2\zeta(3). \tag{3}$$

Tornheim-like series have attracted increasing attention in recent years and they have proven to be a powerful tool for finding many interesting relationships between various zeta values ([2, 3, 4, 5, 6, 10, 11, 13, 15, 18, 20]).

Boyadzhiev [7, 8] described a simple method for evaluating double series of the form (2) in terms of zeta values. Kuba [13] considered the following general sum:

$$V = \sum_{j,k=1}^{\infty} \frac{H_{j+k}^{(u)}}{j^r k^s (j+k)^t}.$$

This sum includes the Tornheim double series (2) as a special case. Kuba [13] proved that whenever $w = r+s+t+u$ is even, for $r, s, t, w \in \mathbb{N}$, the series V can be explicitly evaluated in terms of zeta functions.

On the other hand, Xu and Li [20] used the Tornheim type series for the evaluation of non-linear Euler sums. Among other results they obtained

$$\sum_{m=1}^{\infty} \frac{H_{m+k}}{m(m+k)} = \frac{H_k^2 + H_k^{(2)}}{k}, k \in \mathbb{N} = \{1, 2, 3, \dots\}. \tag{4}$$

From (3) and (4) it is easy to see that the value of the series

$$a(k) = \sum_{m=1}^{\infty} \frac{H_{m+k}}{m(m+k)}, k \in \mathbb{N} \cup \{0\}$$

is irrational for $k = 0$ and rational for every $k \in \mathbb{N}$. Hence the following questions naturally arise: for integers $s \in \mathbb{N} \cup \{0\}$, are the values of the double series

$$\sum_{n,m=1}^{\infty} \frac{H_{n+m+s}}{nm(n+m+s)},$$

and more generally, the multiple series

$$A_n(s) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_{n-1}=1}^{\infty} \frac{H_{k_1+\cdots+k_{n-1}+s}}{k_1 \cdots k_{n-1}(k_1 + \cdots + k_{n-1} + s)}$$

rational or irrational numbers? This question is studied in the second section. Namely, in the case when n is odd, we have solved this question exactly. If n is even, we give a partial solution depending on the odd zeta values.

In the third section, some new relationships between harmonic numbers and $\zeta(2)$ are given and explicit evaluation formulas for some double series via zeta values are established.

2 Explicit evaluations of the Tornheim-like multiple series involving harmonic numbers

Theorem 1. *Consider the double series*

$$A(s) = \sum_{n,m=1}^{\infty} \frac{H_{n+m+s}}{nm(n+m+s)}, \quad s \in \mathbb{N} \cup \{0\}.$$

For any $s \in \mathbb{N}$ the value of $A(s)$ is rational but $A(0)$ is irrational. More precisely,

$$A(s) = \begin{cases} 6\zeta(4), & \text{if } s = 0; \\ 6\sum_{j=0}^{s-1}(-1)^j \binom{s-1}{j} \frac{1}{(j+1)^4}, & \text{if } s \geq 1. \end{cases}$$

Proof. Using telescoping series, we have

$$\begin{aligned}1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+m+s} &= \sum_{k=1}^{\infty}(\frac{1}{k}-\frac{1}{k+n+m+s})\\ &= (n+m+s)\sum_{k=1}^{\infty}\frac{1}{k(k+n+m+s)}.\end{aligned}$$

It then follows that

$$\begin{aligned}A(s) &= \sum_{n,m=1}^{\infty}\frac{1}{nm(n+m+s)}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+m+s})\\ &= \sum_{n,m,k=1}^{\infty}\frac{1}{nmk(n+m+k+s)}\\ &= \sum_{n,m,k=1}^{\infty}\left(\int_0^1 x^{n-1}dx\right)\left(\int_0^1 y^{m-1}dy\right)\left(\int_0^1 z^{k-1}dz\right)\left(\int_0^1 t^{n+m+k+s-1}dt\right)\\ &= \int_0^1 t^{s+2}\left[\int_0^1\left(\sum_{n=1}^{\infty}(xt)^{n-1}\right)dx\int_0^1\left(\sum_{m=1}^{\infty}(yt)^{m-1}\right)dy\int_0^1\left(\sum_{k=1}^{\infty}(zt)^{k-1}\right)dz\right]dt\\ &= \int_0^1 t^{s+2}\left[\int_0^1\frac{1}{1-xt}dx\int_0^1\frac{1}{1-yt}dy\int_0^1\frac{1}{1-zt}dz\right]dt.\end{aligned}$$

Since

$$\int_0^1\frac{1}{1-ut}du = -\frac{1}{t}\ln(1-t),$$

we have

$$A(s) = -\int_0^1 t^{s-1}\ln^3(1-t)dt = -\int_0^1(1-t)^{s-1}\ln^3 t dt. \tag{5}$$

Setting $s=0$, it follows that

$$\begin{aligned}A(0) &= -\int_0^1\frac{1}{1-t}\ln^3 t dt = -\sum_{j=0}^{\infty}\int_0^1 t^j\ln^3 t dt\\ &= -\sum_{j=0}^{\infty}(-\frac{6}{(j+1)^4}) = 6\zeta(4) = \frac{\pi^4}{16}.\end{aligned}$$

On the other hand, if $s\geq 1$, then utilizing the formulas

$$(1-t)^{s-1} = \sum_{j=0}^{s-1}(-1)^j\binom{s-1}{j}t^j$$

and

$$\int_0^1 t^j \ln^3 t dt = -\frac{3!}{(j+1)^4},$$

(5) can be computed explicitly as

$$\begin{aligned} A(s) &= -\int_0^1 (1-t)^{s-1} \ln^3 t dt \\ &= 3! \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \frac{1}{(j+1)^4}. \end{aligned}$$

This proves the stated result. □

In the same way as in Theorem 1, by making use of the formulas

$$(1-t)^{s-1} = \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} t^j \text{ and } \int_0^1 t^j \ln^k t dt = (-1)^k \frac{k!}{(j+1)^{k+1}},$$

one can prove the following more general result.

Theorem 2. *Let*

$$A_n(s) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_{n-1}=1}^{\infty} \frac{H_{k_1+\cdots+k_{n-1}+s}}{k_1 \cdots k_{n-1}(k_1+\cdots+k_{n-1}+s)}, \ s \in \mathbb{N} \cup \{0\}, n \geq 2.$$

Then

$$A_n(s) = \begin{cases} n!\zeta(n+1), & \text{if } s = 0; \\ n! \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \frac{1}{(j+1)^{n+1}}, & \text{if } s \geq 1. \end{cases} \tag{6}$$

Two special cases of the theorem are as follows:

$$A_2(s) = \sum_{k=1}^{\infty} \frac{H_{k+s}}{k(k+s)} = \begin{cases} 2!\zeta(3), & \text{if } s = 0; \\ 2! \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \frac{1}{(j+1)^3}, & \text{if } s \geq 1, \end{cases}$$

and

$$A_4(s) = \sum_{k,m,n=1}^{\infty} \frac{H_{k+m+n+s}}{kmn(k+m+n+s)} = \begin{cases} 4!\zeta(5), & \text{if } s = 0; \\ 4! \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \frac{1}{(j+1)^5}, & \text{if } s \geq 1. \end{cases}$$

Remark 3. It can be easily seen from (6) that the expression $A_n(s)$ is a rational number for all $s \geq 1$ and $n \geq 2$. However $A_2(0) = 2!\zeta(3)$ is irrational (Apéry). If $n \geq 4$ and even, it is not known whether the numbers $A_n(0) = n!\zeta(n+1)$ are irrational or not. On the other hand, for any odd $n \in \mathbb{N}$ we have $A_n(0) = n!\zeta(n+1) = r_n \pi^{n+1}$ (see (1)), which is irrational because r_n is rational and π^{n+1} is irrational. Notice that, as is well known, the irrationality of π^n is a consequence of the transcendence of π.

Remark 4. There is an interesting connection between the multiple series $A_n(s)$ and the Bell polynomials $b_n(x)$. For a given sequence $x = (x_1, x_2, \ldots)$, the Bell polynomials $b_n(x) = b_n(x_1, x_2, \ldots)$ are defined by the generating function (see [9] or [14])

$$\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} = \exp\left\{ \sum_{k=1}^{\infty} x_k \frac{t^k}{k!} \right\}.$$

As a result of this definition one has the following explicit expression [9]:

$$b_n(x) = \sum_{\sigma(n)} \frac{n!}{a_1! a_2! \cdots a_n!} \left(\frac{x_1}{1}\right)^{a_1} \left(\frac{x_2}{2}\right)^{a_2} \cdots \left(\frac{x_n}{n}\right)^{a_n},$$

where the summation ranges over the set $\sigma(n)$ of all partitions of n. Using this explicit expression we can give a list of the first few Bell polynomials as

$$\begin{aligned} b_0(x) &= 1, & b_1(x) &= x_1, \\ b_2(x) &= x_1^2 + x_2, & b_3(x) &= x_1^3 + 3x_1x_2 + 2x_3. \end{aligned} \tag{7}$$

Considering the well-known harmonic number identity (see [21, Eq. (3.56)] or [12, Corollary 2.2])

$$\begin{aligned} &i!m\binom{m+n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(m+k)^{i+1}} \\ &= b_i(H_{m+n} - H_{m-1}, H_{m+n}^{(2)} - H_{m-1}^{(2)}, \ldots, H_{m+n}^{(i)} - H_{m-1}^{(i)}) \end{aligned}$$

we have

$$n!s \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \frac{1}{(j+1)^{n+1}} = b_n(H_s, H_s^{(2)}, \ldots, H_s^{(n)}), \tag{8}$$

which corresponds to (6) for $s \geq 1$. Hence we can restate $A_n(s)$ in terms of the Bell polynomials as

$$A_n(s) = \begin{cases} n!\zeta(n+1), & \text{if } s = 0; \\ \frac{1}{s} b_n(H_s, H_s^{(2)}, \ldots, H_s^{(n)}), & \text{if } s \geq 1. \end{cases}$$

Thanks to this formula, considering (7) we can write $A_n(s)$ as a finite combination of the harmonic and generalized harmonic numbers. For instance, for $n = 2$ we have

$$A_2(s) = \begin{cases} 2\zeta(3), & \text{if } s = 0; \\ \frac{1}{s}((H_s)^2 + H_s^{(2)}), & \text{if } s \geq 1, \end{cases}$$

which coincides with (4). For $n = 3$ we have

$$A_3(s) = \begin{cases} 6\zeta(4), & \text{if } s = 0; \\ \frac{1}{s}((H_s)^3 + 3H_s H_s^{(2)} + 2H_s^{(3)}), & \text{if } s \geq 1. \end{cases}$$

3 Explicit evaluations of some double series in zeta values

The next theorem gives a new relationship between harmonic numbers and $\zeta(2)$.

Theorem 5. *Let* $O_m = \sum_{k=1}^{m} \frac{1}{2k-1} = H_{2m} - \frac{1}{2}H_m$. *Then*

$$\sum_{m=1}^{\infty} \frac{O_m}{2m(2m+1)} = \frac{1}{4}\zeta(2). \tag{9}$$

Proof. Let

$$A = \sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+1)}$$

and

$$B = \sum_{m,n=1}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+1)}.$$

From the equation

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8},$$

we have

$$A = \frac{\pi^2}{4} - 1 + B. \tag{10}$$

Further, using telescoping series, we have

$$\begin{aligned}
B &= \sum_{m=1}^{\infty} \frac{1}{2m+1} \frac{1}{2m} \sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+1+2m}\right) \\
&= \sum_{m=1}^{\infty} \frac{1}{2m(2m+1)} \left(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2m+1}\right) \\
&= \sum_{m=1}^{\infty} \frac{O_m - \frac{2m}{2m+1}}{2m(2m+1)} \\
&= \sum_{m=1}^{\infty} \frac{O_m}{2m(2m+1)} - \sum_{m=1}^{\infty} \frac{1}{(2m+1)^2} \\
&= \sum_{m=1}^{\infty} \frac{O_m}{2m(2m+1)} - \frac{\pi^2}{8} + 1.
\end{aligned}$$

Hence we obtain that

$$B = \sum_{m=1}^{\infty} \frac{O_m}{2m(2m+1)} + 1 - \frac{3}{4}\zeta(2). \tag{11}$$

Now let us evaluate A.

$$\begin{aligned} A &= \sum_{m,n=0}^{\infty} \left(\int_0^1 x^{2m} dx\right) \left(\int_0^1 y^{2n} dy\right) \left(\int_0^1 t^{2m+2n} dt\right) \\ &= \int_0^1 \left(\int_0^1 \sum_{m=0}^{\infty} (xt)^{2m} dx \int_0^1 \sum_{n=0}^{\infty} (yt)^{2n} dy\right) dt \\ &= \int_0^1 \left(\int_0^1 \frac{1}{1-(xt)^2} dx \int_0^1 \frac{1}{1-(yt)^2} dy\right) dt \\ &= \frac{1}{4} \int_0^1 \frac{1}{t^2} \ln^2 \left(\frac{1+t}{1-t}\right) dt. \end{aligned}$$

The substitution $\frac{1+t}{1-t} = u$ immediately leads to the following equality:

$$A = \frac{1}{2} \int_1^{\infty} \frac{1}{(1-u)^2} \ln^2 u du.$$

Integration by parts gives

$$\begin{aligned} A &= \int_1^{\infty} \frac{1}{u(u-1)} \ln u \, du = \int_1^{\infty} \frac{1}{1-\frac{1}{u}} \frac{\ln u}{u^2} du \\ &= \sum_{k=0}^{\infty} \int_1^{\infty} u^{-k-2} \ln u \, du = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \zeta(2) = \frac{\pi^2}{6}. \end{aligned} \tag{12}$$

Similarly, from (10), (11) and (12) we have

$$\frac{\pi^2}{6} = \frac{\pi^2}{4} - 1 + \sum_{m=1}^{\infty} \frac{O_m}{2m(2m+1)} - \frac{3}{4}\zeta(2) + 1$$

and as a result

$$\sum_{m=1}^{\infty} \frac{O_m}{2m(2m+1)} = \frac{1}{4}\zeta(2).$$

□

Remark 6. Considering (11) we have

$$B = \sum_{m=1}^{\infty} \frac{H_{2m+1} - 1 - \frac{1}{2}H_m}{2m(2m+1)}. \tag{13}$$

Now we use (10) and (12) to conclude that

$$\sum_{m=1}^{\infty} \frac{2H_{2m+1} - H_m}{2m(2m+1)} = 2(2 - \ln 2) - \zeta(2). \tag{14}$$

In the following theorem we obtain interesting relationships between some special double series and zeta values $\zeta(2)$ and $\zeta(3)$.

Theorem 7. *We have the following series evaluations:*

(a) $\sum_{m,n=0}^{\infty} \frac{1}{(m+\frac{1}{2})(n+\frac{1}{2})(m+n+\frac{1}{2})(m+n+1)} = 16\zeta(2) - 14\zeta(3).$

(b) $\sum_{m,n=0}^{\infty} \frac{1}{(m+\frac{1}{2})(n+\frac{1}{2})(m+n+1)(m+n+\frac{3}{2})} = 14\zeta(3) - 8\zeta(2)$.

(c) $\sum_{m,n=0}^{\infty} \frac{1}{(m+\frac{1}{2})(n+\frac{1}{2})(m+n+\frac{1}{2})(m+n+1)(m+n+\frac{3}{2})} = 24\zeta(2) - 28\zeta(3).$

Proof. (a) We have

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+2)} \\
&= \sum_{m,n=0}^{\infty} \left(\int_0^1 x^{2m} dx\right) \left(\int_0^1 y^{2n} dy\right) \left(\int_0^1 t^{2m+2n+1} dt\right) \\
&= \int_0^1 \left(\int_0^1 \sum_{m=0}^{\infty} (xt)^{2m} dx \int_0^1 \sum_{n=0}^{\infty} (yt)^{2n} dy\right) t dt \\
&= \frac{1}{4} \int_0^1 \frac{1}{t} \ln^2\left(\frac{1+t}{1-t}\right) dt.
\end{aligned}$$

Here the substitution $\frac{1+t}{1-t} = u$ leads to the following equality:

$$\begin{aligned}
\frac{1}{4} \int_0^1 \frac{1}{t} \ln^2\left(\frac{1+t}{1-t}\right) dt &= \frac{1}{2} \int_1^{\infty} \frac{1}{(u^2-1)} \ln^2 u du = \frac{1}{2} \int_1^{\infty} \frac{1}{u^2} \frac{1}{(1-\frac{1}{u^2})} \ln^2 u du \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \int_1^{\infty} u^{-2k-2} \ln u \, du.
\end{aligned}$$

After integration by parts we get

$$\frac{1}{2} \sum_{k=0}^{\infty} \int_1^{\infty} u^{-2k-2} \ln u \, du = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \frac{7}{8} \zeta(3).$$

Hence

$$\sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+2)} = \frac{7}{8} \zeta(3). \tag{15}$$

On the other hand, according to the formula (12),

$$A = \sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+1)} = \zeta(2). \tag{16}$$

Now, from (15) and (16) it follows that

$$\begin{aligned}\zeta(2)-\frac{7}{8}\zeta(3)&=\sum_{m,n=0}^{\infty}\frac{1}{(2m+1)(2n+1)(2m+2n+1)}\\&\quad-\sum_{m,n=0}^{\infty}\frac{1}{(2m+1)(2n+1)(2m+2n+2)}\\&=\sum_{m,n=0}^{\infty}\frac{1}{(2m+1)(2n+1)(2m+2n+1)(2m+2n+2)}\end{aligned}$$

and this proves (a).

(b) By the same method in the proof of (a), we have

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{1}{(2m+1)(2n+1)(2m+2n+3)}\\&=\sum_{m,n=0}^{\infty}\left(\int_0^1 x^{2m}dx\right)\left(\int_0^1 y^{2n}dy\right)\left(\int_0^1 t^{2m+2n+2}dt\right)\\&=\int_0^1\left(\int_0^1\frac{1}{1-(xt)^2}dx\int_0^1\frac{1}{1-(yt)^2}dy\right)t^2dt\\&=\frac{1}{4}\int_0^1\ln^2\left(\frac{1+t}{1-t}\right)dt\\&=\frac{1}{2}\int_1^{\infty}\frac{1}{(u+1)^2}\ln^2 u du=-\frac{1}{2}\int_1^{\infty}\ln^2 u d(\frac{1}{u+1})\\&=\int_1^{\infty}\frac{1}{u(u+1)}\ln u\,du=\sum_{k=2}^{\infty}(-1)^k\int_1^{\infty}u^{-k}\ln u\,du\\&=\sum_{k=2}^{\infty}(-1)^k\frac{1}{(k-1)^2}=\frac{\pi^2}{12}.\end{aligned}$$

Thus

$$\sum_{m,n=0}^{\infty}\frac{1}{(2m+1)(2n+1)(2m+2n+3)}=\frac{\pi^2}{12}=\frac{1}{2}\zeta(2). \tag{17}$$

Now, from (15) and (17) we have

$$\frac{7}{8}\zeta(3)-\frac{1}{2}\zeta(2)=\sum_{m,n=0}^{\infty}\frac{1}{(2m+1)(2n+1)(2m+2n+2)(2m+2n+3)}$$

and this proves (b).

(c) Finally, formula (c) can be obtained by subtracting the formula (b) from the formula (a).

□

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