Journal of Integer Sequences, Vol. 25 (2022),
Article 22.4.6

On The Complex-Type Catalan Transform of the k-Fibonacci Numbers

Ömür Deveci
Department of Mathematics
Faculty of Science and Letters
Kafkas University
36100, Turkey
odeveci36@hotmail.com

Anthony G. Shannon
Honorary Fellows
Warrane College
University of New South Wales
2033, Australia
tshannon38@gmail.com

Abstract

We define a type of complex Catalan number and find some its properties. We also produce a complex Catalan transform and its inverse, together with associated generating functions and related matrices. These lead to connections with complex Catalan transforms of the k-Fibonacci numbers and the determinants of their Hankel matrices. The paper finishes with a conjecture.

1 Introduction and preliminaries

For $n \geq 1$, the nth Catalan number C_n is described [13] by

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

with generating function given by

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Barry [13] gave the Catalan transform for a given sequence (a_n) and its inverse as follows, respectively:

$$b_n = \sum_{k=0}^{n} \frac{k}{2n-k}\binom{2n-k}{n-k} a_k$$

and

$$a_n = \sum_{k=0}^{n} \binom{k}{n-k} (-1)^{n-k}\, b_k.$$

For a positive real integer k, the k-Fibonacci sequence $(F_{k,n})_{n\in\mathbb{N}}$ is defined by the following homogeneous linear recurrence relation:

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \tag{1}$$

for $n \geq 1$, with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 0 = 1$ (cf. [19, 20, 21]). When $k = 1$ and $k = 2$ in (1), the sequence $(F_{k,n})_{n\in\mathbb{N}}$ reduces to the usual Fibonacci sequence and Pell sequence, respectively.

Number-theoretic properties such as these obtained from the Catalan transforms of the k-Fibonacci numbers relevant to this article have been studied by Falcon. Falcon [18] derived a number of closed-form formulas for the Catalan transform of the k-Fibonacci sequence $(F_{k,n})_{n\in\mathbb{N}}$ by the matrix method as follows:

$$\begin{pmatrix} CF_{k,1} \\ CF_{k,2} \\ CF_{k,3} \\ CF_{k,4} \\ CF_{k,5} \\ CF_{k,6} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 14 & 14 & 9 & 4 & 1 & 0 & 0 & \cdots \\ 42 & 42 & 28 & 14 & 5 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} F_{k,1} \\ F_{k,2} \\ F_{k,3} \\ F_{k,4} \\ F_{k,5} \\ F_{k,6} \\ \vdots \end{pmatrix},$$

where $CF_{k,n}$ is the nth Catalan transform of the k-Fibonacci numbers. We denote the above lower-triangular matrix by C. Note that the first column of the matrix C is the sequence of the Catalan numbers. For more information on the matrix C, see [13, 18].

The Hankel matrix H_n associated with a given sequence of real numbers (a_n) is defined as follows:

$$H_n = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The sequence $(Ha_n = \det(a_{i+j-2})_{1\le i,j\le n+1})$ is called the Hankel matrix transform of the sequence (a_n). Namely, the Hankel determinant of order n of the sequence (a_n) is the upper-left $n \times n$ subdeterminant of the matrix H_n (cf. [1, 7, 10, 15, 17]). Falcon [18] gave the relationship between the Hankel matrix transform of Catalan transform of the k-Fibonacci sequence and the Fibonacci sequence by

$$(HCF_{k,n}) = (F_{2n+1}),$$

where F_{2n+1} is the $(2n+1)$th Fibonacci number.

In the literature, many interesting properties and applications of the Catalan numbers relevant to this paper have been studied by many authors; for example, [2, 3, 4, 6, 8, 9, 16, 22]. In the first part of the paper, we consider a new generalization of the Catalan sequence that we call the complex-type Catalan sequence. Then we give some of its properties, such as the generating matrix, the generating function, the exponential representation, and some combinatorial representations.

Barry [13] defined the Catalan transform of a sequence and its inverse by the aid of the Riordan group and then gave many transformed sequences. Since then, the Catalan transforms of the linear recurrence sequences have been a topic of interest, and some new sequences have been derived by using these transforms; see [5, 11, 12, 14, 18]. Here we derive the complex-type Catalan transform and its inverse by the generating matrix of the complex-type Catalan sequence and then give the generating function of the complex-type Catalan transform of a given sequence. Finally, we obtain new sequences from the complex-type Catalan transforms of the k-Fibonacci numbers and the determinants of their Hankel matrices as applications of the results produced.

2 The main results

Define the complex-type Catalan numbers as shown, for $n \ge 0$

$$C_n^{(i)} = \frac{1}{n+1}\binom{2n}{n} i^n \tag{2}$$

where $\sqrt{-1} = i$.

From Eq. (2), we can write the following lower-triangular matrix:

$$C^{(i)} = \begin{pmatrix} i & & & & & & \\ i & i^2 & & & & & \\ 2i & 2i^2 & i^3 & & & & \\ 5i & 5i^2 & 3i^3 & i^4 & & & \\ 14i & 14i^2 & 9i^3 & 4i^4 & i^5 & & \\ 42i & 42i^2 & 28i^3 & 14i^4 & 5i^5 & i^6 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix $C^{(i)}$ is said to be the complex-type Catalan matrix. Then we can write the following matrix relation:

$$\begin{pmatrix} i^3 & i^4 & i^5 & i^6 & i^7 & i^8 & \cdots \end{pmatrix} \begin{pmatrix} i \\ i \\ 2i \\ 5i \\ 14i \\ 42i \\ \vdots \end{pmatrix} = \begin{pmatrix} C_0^{(i)} \\ C_1^{(i)} \\ C_2^{(i)} \\ C_3^{(i)} \\ C_4^{(i)} \\ C_5^{(i)} \\ \vdots \end{pmatrix}.$$

It can be readily established that

$$\left(C^{(i)}\right)^{-1} = \begin{pmatrix} -i & & & & & \\ 1 & -1 & & & & \\ 0 & -2i & i & & & \\ 0 & 1 & -3 & 1 & & \\ 0 & 0 & -3i & 4i & -i & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let

$$M = \begin{pmatrix} i & & & & & \\ i & i^2 & & & & \\ i & i^2 & i^3 & & & \\ i & i^2 & i^3 & i^4 & & \\ i & i^2 & i^3 & i^4 & i^5 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is clear that $C^{(i)} = C \circ M$, where $C \circ M$ denotes the Hadamard product C and M. Suppose that the terms of the matrices $(C^{(i)})^{-1}$ and $C^{-1} \circ M$ are indicated by $(c^{(i)})^{-1}_{k,j}$ and $(c^{-1} \circ m)^{-1}_{k,j}$, respectively. Then, by a simple calculation, we obtain

$$((C^{(i)})^{-1}_{k,j}) i^{k+j} = (c^{-1} \circ m)^{-1}_{k,j}.$$

It is easy to see that the generating function of the complex-type Catalan sequence $(C_n^{(i)})$ is

$$\begin{aligned} c^{(i)}(x) &= \frac{1 - \sqrt{1 - 4ix}}{2ix} \\ &= 1 + ix - 2x^2 - 5ix^3 + 14x^4 \cdots . \end{aligned}$$

Now we give an exponential representation for the complex-type Catalan numbers by the aid of the generating function with the following Proposition.

Proposition 1. *The complex-type Catalan sequence* $(C_n^{(i)})$ *has the following exponential representation:*

$$c^{(i)}(x) = \frac{1}{2ix \exp\left(\sum\limits_{n=1}^{\infty} \frac{1}{n}(\sqrt{1-4ix})^n\right)}.$$

Proof. By a simple calculation, we may write

$$\begin{aligned}
\ln(2ix \cdot c^{(i)}(x)) &= \ln(1 - \sqrt{1-4ix}\ ) \\
&= -(\sqrt{1-4ix} + \frac{1}{2}(\sqrt{1-4ix}\ )^2 + \frac{1}{3}(\sqrt{1-4ix}\ )^3 + \cdots) \\
&= -\left(\sum_{n=1}^{\infty} \frac{1}{n}(\sqrt{1-4ix}\ )^n\right).
\end{aligned}$$

So we have the conclusion. □

The following proposition gives an alternative version for the above expression.

Proposition 2. *The complex-type Catalan sequence* $(C_n^{(i)})$ *has the following combinatorial representation:*

$$c^{(i)}(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1} i^n.$$

Proof. By the generalized binomial theorem, it is readily seen that

$$\sqrt{1-4ix} = \sum_{n=0}^{\infty} \frac{-1}{(2n-1)} \binom{2n}{n} x^n i^n.$$

Then we have

$$\begin{aligned}
\frac{1-\sqrt{1-4ix}}{2ix} &= \frac{1-(1-\binom{2}{1}xi - \frac{1}{3}\binom{4}{2}x^2i^2 - \frac{1}{5}\binom{6}{3}x^3i^3 - \cdots)}{2ix} \\
&= 1 + \frac{1}{6}\binom{4}{2}xi + \frac{1}{10}\binom{6}{3}x^2i^2 + \cdots \\
&= \sum_{n=1}^{\infty} \frac{1}{2\,(2n-1)} \binom{2n}{n} x^{n-1} i^{n-1}.
\end{aligned}$$

Substituting m for $n-1$, in the equation, we may write

$$\begin{aligned}
c^{(i)}(x) &= \sum_{m=0}^{\infty} \frac{1}{2(2m+1)} \binom{2m+2}{m+1} x^m i^m \\
&= \sum_{n=0}^{\infty} \binom{2m}{m} \frac{x^m}{m+1} i^m.
\end{aligned}$$

Thus the proof is complete. □

Now we define a new sequence that we call the complex-type Catalan transform of a given sequence (a_n) as follows:

$$c^{(i)}[a_n] = \sum_{k=0}^{n} \frac{k}{2n-k}\binom{2n-k}{n-k} i^k a_k. \tag{3}$$

By Eq. (3), we can easily derive

$$\begin{pmatrix} c^{(i)}[a_1] \\ c^{(i)}[a_2] \\ c^{(i)}[a_3] \\ c^{(i)}[a_4] \\ c^{(i)}[a_5] \\ c^{(i)}[a_6] \\ \vdots \end{pmatrix} = \begin{pmatrix} i & & & & & & \\ i & i^2 & & & & & \\ 2i & 2i^2 & i^3 & & & & \\ 5i & 5i^2 & 3i^3 & i^4 & & & \\ 14i & 14i^2 & 9i^3 & 4i^4 & i^5 & & \\ 42i & 42i^2 & 28i^3 & 14i^4 & 5i^5 & i^6 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ \vdots \end{pmatrix}$$

From the definition of the matrix $(C^{(i)})^{-1}$, we see that the inverse of the complex-type Catalan transform is

$$a_n = \sum_{k=0}^{n} \binom{k}{n-k} (-1)^k i^n c^{(i)}[a_k].$$

Let $c(x)$ be the generating function of the Catalan sequence (C_n) and let $g(x)$ be the generating function of the sequence (a_n). In [13], it is proved that the generating function of the Catalan transform of the sequence (a_n) is $g(xc(x))$. Since $c^{(i)}(x) = c(xi)$, it can be clearly seen that $g(xc^{(i)}(x))$ is the generating function of the complex-type Catalan transform of the sequence (a_n).

Now we concentrate on the complex-type Catalan transform of the k-Fibonacci sequence $(F_{k,n})_{n\in\mathbb{N}}$ Using Eq. (3), we define the following sequence:

$$c^{(i)}[F_{k,n}] = \sum_{j=1}^{n} \frac{j}{2n-j}\binom{2n-j}{n-j} i^j F_{k,j}$$

with $c^{(i)}[F_{k,0}] = 0$.

Since the generating functions of the k-Fibonacci sequence $(F_{k,n})_{n\in\mathbb{N}}$ and the complex-type Catalan sequence $(C_n^{(i)})$ are

$$g(x) = \frac{x}{1-kx-x^2}$$

and

$$c^{(i)}(x) = \frac{1-\sqrt{1-4ix}}{2ix},$$

respectively, we can give the generating function of the sequence $(c^{(i)}[F_{k,n}])$ as

$$g(xc^{(i)}(x)) = \frac{i - i\sqrt{1-4ix}}{-3 + i(2x-k) + (ik+1)\sqrt{1-4ix}}.$$

The complex-type Catalan transform of the first k-Fibonacci numbers, that is, the first members of the sequence $(c^{(i)}[F_{k,n}])$ are the following polynomials in k:

$$\begin{aligned}
c^{(i)}[F_{k,1}] &= \sum_{j=1}^{1} \frac{j}{2-j}\binom{2-j}{1-j} i^j F_{k,j} = i,\\
c^{(i)}[F_{k,2}] &= \sum_{j=1}^{2} \frac{j}{4-j}\binom{4-j}{2-j} i^j F_{k,j} = -k + i,\\
c^{(i)}[F_{k,3}] &= \sum_{j=1}^{3} \frac{j}{6-j}\binom{6-j}{3-j} i^j F_{k,j} = -ik^2 - 2k + i,\\
c^{(i)}[F_{k,4}] &= \sum_{j=1}^{4} \frac{j}{8-j}\binom{8-j}{4-j} i^j F_{k,j} = k^3 - 3ik^2 - 3k + 2i,\\
c^{(i)}[F_{k,5}] &= ik^4 + 4k^3 - 6ik^2 - 6k + 6i,\\
c^{(i)}[F_{k,6}] &= -k^5 + 5ik^4 + 10k^3 - 13ik^2 - 17k + 19i,\\
c^{(i)}[F_{k,7}] &= -ik^6 - 6k^5 + 15ik^4 + 24k^3 - 36ik^2 - 54k + 61i,\\
c^{(i)}[F_{k,8}] &= k^7 - 7ik^6 - 21k^5 + 40ik^4 + 67k^3 - 114ik^2 - 176k + 200i,\\
c^{(i)}[F_{k,9}] &= ik^8 + 8k^7 - 28ik^6 - 62k^5 + 115ik^4 + 212k^3 - 376ik^2 - 584k + 670i,\\
c^{(i)}[F_{k,10}] &= -k^9 + 9ik^8 + 36k^7 - 91ik^6 - 186k^5 + 366ik^4 + 706k^3 - 1263ik^2\\
&\quad - 1974k + 2286i.
\end{aligned}$$

From the coefficients of the complex-type Catalan transform of the k-Fibonacci sequence $(F_{k,n})_{n\in\mathbb{N}}$ we can produce the an infinite triangle, where the following are the first few rows:

$c^{(i)}[F_{k,1}]$	i									
$c^{(i)}[F_{k,2}]$	-1	i								
$c^{(i)}[F_{k,3}]$	$-i$	-2	i							
$c^{(i)}[F_{k,4}]$	1	$-3i$	-3	$2i$						
$c^{(i)}[F_{k,5}]$	i	4	$-6i$	-6	$6i$					
$c^{(i)}[F_{k,6}]$	-1	$5i$	10	$-13i$	-17	$19i$				
$c^{(i)}[F_{k,7}]$	$-i$	-6	$15i$	24	$-36i$	-54	$61i$			
$c^{(i)}[F_{k,8}]$	1	$-7i$	-21	$40i$	67	$-114i$	-176	$200i$		
$c^{(i)}[F_{k,9}]$	i	8	$-28i$	-62	$115i$	212	$-376i$	-584	$670i$	
$c^{(i)}[F_{k,10}]$	-1	$9i$	36	$-91i$	-186	$366i$	706	$-1263i$	-1974	$2286i$

Table 1: The complex-type Catalan triangle of the k-Fibonacci sequence.

Now we give the following useful results by the aid of the above triangle:

- The first diagonal sequence $(i, i, i, 2i, 6i, 19i, 61i, 200i, 670i, 2286i, \ldots)$ is the complex-type Catalan transform of the sequence $(1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots)$.
- The second diagonal sequence $(-1, -2, -3, -6, -17, -54, -176, -584, -1974 \ldots)$ is the complex-type Catalan transform of the sequence $(i, 1, i, 1, 2i, 2, 2i, 2, 3i, \ldots)$, and so on.

It is well-known that the iteration of a function f is denoted by superscript; f^n means the nth iteration of function f, i.e.,

$$f^n(x) = f(f \ldots f(x) \ldots).$$

Clearly,

$$f^0(x) = x \text{ and } f^n(x) = f^{n-1}(f(x)) \text{ for } n \geq 1.$$

We denote $c^{(i)}[(c^{(i)})^{n-1}[a_n]]$ by $(c^{(i)})^n[a_n]$. The sequence $(c^{(i)})^n[a_n]$ is said to be the nth iteration of the complex-type Catalan transform of the sequence (a_n).

The second iteration of the complex-type Catalan transform of the first k-Fibonacci numbers, that is, the first members of the sequence $((c^{(i)})^2[F_{k,n}])$ are the following polynomials in k:

$$\begin{aligned}
(c^{(i)})^2\left[F_{k,1}\right] &= \sum_{j=1}^{1} \frac{j}{2-j}\binom{2-j}{1-j} i^j c^{(i)}\left[F_{k,j}\right] = -1, \\
(c^{(i)})^2[F_{k,j}] &= \sum_{j=1}^{2} \frac{j}{4-j}\binom{4-j}{2-j} i^j c^{(i)}[F_{k,j}] = k - i - 1, \\
(c^{(i)})^2[F_{k,j}] &= \sum_{j=1}^{3} \frac{j}{6-j}\binom{6-j}{3-j} i^j c^{(i)}[F_{k,j}] = -k^2 + k(2+2i) - 2i - 1, \\
(c^{(i)})^2[F_{k,4}] &= \sum_{j=1}^{4} \frac{j}{8-j}\binom{8-j}{4-j} i^j c^{(i)}[F_{k,j}] = k^3 + k^2(-3i-3) + k(6i+2) - 3i - 2, \\
(c^{(i)})^2[F_{k,5}] &= -k^4 + k^3(4i+4) + k^2(-12i-3) + k(12i+2) - 6i - 11, \\
(c^{(i)})^2[F_{k,6}] &= k^5 + k^4(-5i-5) + k^3(20i+4) + k^2(-29i+2) + k(26i+17) + -33i - 44, \\
(c^{(i)})^2[F_{k,7}] &= -k^6 + k^5(6i+6) + k^4(-30i-5) + k^3(56i-12) + k^2(-66i-6) \\
&\quad + k(114i+90) - 150i - 101.
\end{aligned}$$

From the coefficients of the second iteration of the complex-type Catalan transform of the k-Fibonacci sequence $(F_{k,n})_{n\in\mathbb{N}}$, we can produce an infinite triangle, of which the following are the first few rows:

$(c^{(i)})^2[F_{k,1}]$	-1						
$(c^{(i)})^2[F_{k,2}]$	-1	$-1-i$					
$(c^{(i)})^2[F_{k,3}]$	-1	$2+2i$	$-2i-1$				
$(c^{(i)})^2[F_{k,4}]$	1	$-3i-3$	$6i+2$	$-3i-2$			
$(c^{(i)})^2[F_{k,5}]$	-1	$4i+4$	$-12i-3$	$12i+2$	$-6i-11$		
$(c^{(i)})^2[F_{k,6}]$	1	$-5i-5$	$20i+4$	$-29i+2$	$26i+17$	$-33i-44$	
$(c^{(i)})^2[F_{k,7}]$	-1	$6i+6$	$-30i-5$	$56i-12$	$-66i-6$	$114i+90$	$-150i-101$

Table 2: The triangle of the sequence $((c^{(i)})^2[F_{k,n}])$.

Thus, it can be easily seen that first diagonal sequence $(-1,-1-i,-2i-1,-3i-2,-6i-11,-33i-44,-150i-101,\ldots)$ is the complex-type Catalan transform of the sequence $(i,i,i,2i,6i,19i,61i,200i,670i,2286i,\ldots)$, which is first diagonal sequence of the complex-type Catalan triangle of the k-Fibonacci sequence.

We will now address the Hankel matrix transform of the sequence $(c^{(i)}[F_{k,n}])$. Consider the following recursively defined sequence:

$$x_{n+2} = i \cdot x_{n+1} + x_n$$

for $n \geq 0$, with initial conditions $x_0 = 0$ and $x_1 = 1$.

Let the Hankel determinant of the complex-type Catalan transform of the mth term of the k-Fibonacci sequence $(F_{k,n})_{n\in\mathbb{N}}$ denoted by $Hc^{(i)}[F_{k,m}]$. Then we obtain the early part of the sequence (x_n) as follows:

$$Hc^{(i)}\left[F_{k,0}\right] = \det(0) = 0 = x_0,$$

$$Hc^{(i)}\left[F_{k,1}\right] = \begin{vmatrix} 0 & i \\ i & -k+i \end{vmatrix} = 1 = x_1,$$

$$Hc^{(i)}\left[F_{k,2}\right] = \begin{vmatrix} 0 & i & -k+i \\ i & -k+i & -ik^2-2k+i \\ -k+i & -ik^2-2k+i & k^3-3ik^2-3k+2i \end{vmatrix} = i = x_2,$$

$$Hc^{(i)}\left[F_{k,3}\right] = \begin{vmatrix} 0 & i & -k+i & -ik^2-2k+i \\ i & -k+i & -ik^2-2k+i & k^3-3ik^2-3k+2i \\ -k+i & -ik^2-2k+i & k^3-3ik^2-3k+2i & ik^4+4k^3-6ik^2-6k+6i \\ -ik^2-2k+i & k^3-3ik^2-3k+2i & ik^4+4k^3-6ik^2-6k+6i & -k^5+5k^4i+10k^3-13k^2i-17k+19i \end{vmatrix}$$
$$= 0 = x_3,$$

$$Hc^{i}\left[F_{k,4}\right] = \begin{vmatrix} 0 & i & -k+i & -ik^2-2k+i & k^3-3ik^2-3k+2i \\ i & -k+i & -ik^2-2k+i & k^3-3ik^2-3k+2i & ik^4+4k^3-6ik^2-6k+6i \\ -k+i & -ik^2-2k+i & k^3-3ik^2-3k+2i & ik^4+4k^3-6ik^2-6k+6i & -k^5+5k^4i+10k^3-13k^2i-17k+19i \\ -ik^2-2k+i & k^3-3ik^2-3k+2i & ik^4+4k^3-6ik^2-6k+6i & -k^5+5k^4i+10k^3-13k^2i-17k+19i & -k^6i-6k^5+15k^4i+24k^3-36k^2i-54k+61i \\ k^3-3ik^2-3k+2i & ik^4+4k^3-6ik^2-6k+6i & -k^5+5k^4i+10k^3-13k^2i-17k+19i & -k^6i-6k^5+15k^4i+24k^3-36k^2i-54k+61i & k^7-7ik^6-21k^5+40ik^4+67k^3-114ik^2-176k+200i \end{vmatrix} = i = x_4\,.$$

Thus we get immediately:

Conjecture 3. For $n \geq 0$,

$$Hc^{(i)}[F_{k,n}] = x_n.$$

3 Acknowledgment

We would like to thank the editor and the referee for their valuable comments and suggestions.

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2020 *Mathematics Subject Classification*: Primary 11K31; Secondary 60E10, 39B32, 15A15, 11C20.

Keywords: Catalan number, Catalan transform, k-Fibonacci number, Hankel matrix, Hankel determinant.

(Concerned with sequence A000108.)

Received February 21 2022; revised versions received March 10 2022; April 4 2022. Published in *Journal of Integer Sequences*, April 11 2022.

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