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Linear Recurrences of Order at Most Two in Small Divisors

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Abstract

Given a positive integer n, the small divisors of n are defined as the positive divisors that do not exceed $\sqrt{n}$. Iannucci previously classified all n for which the small divisors of n form an arithmetic progression. In this paper, we classify all n for which the small divisors of n form a linear recurrence of order at most two.

1 Introduction

As usual, we say that a nonzero integer m divides another integer n whenever the quotient $\frac{n}{m}$ is itself an integer. When this occurs, we use the conventional notation $m \mid n$. Moreover, a *nontrivial* divisor of a natural number n is a divisor other than 1 and n (if such a divisor exists).

Consider a positive integer n, and let $\mathcal{S}_n$ be the set of *small* divisors of n, that is,

$$\mathcal{S}_n := \{d : 1 \leq d \leq \sqrt{n} \wedge d \mid n\}. \tag{1}$$

Large divisors, associated with $\mathcal{L}_n$, are defined with the inequality reversed. Iannucci's work classified all n for which the ordered elements of $\mathcal{S}_n$ form an arithmetic progression. For brevity, we let $s(n) = |\mathcal{S}_n|$. Moreover, for any tuple $(u, v, a, b) \in \mathbb{Z}^4$, there is an integral linear recurrence of order at most two, $(d_i)_{i=1}^{\infty}$, given by

$$d_i = \begin{cases} u, & \text{if } i = 1; \\ v, & \text{if } i = 2; \\ ad_{i-1} + bd_{i-2}, & \text{if } \ i \geq 3. \end{cases} \tag{2}$$

This recurrence is usually abbreviated as $U(u, v, a, b)$. The choice of beginning the index at 1, rather than 0, is due to the context of working with divisors.

As usual, we let $\tau(n)$ denote the number of positive divisors of n, that is,

$$\tau(n) = \sum_{d|n} 1.$$

Since there is a "coupling" between small and large divisors of n, linking d and $\frac{n}{d}$, we easily relate $\tau(n)$ to $s(n)$ as follows:

$$\tau(n) = \begin{cases} 2s(n) - 1, & \text{if } n \text{ is a perfect square;} \\ 2s(n), & \text{otherwise.} \end{cases} \tag{3}$$

In this paper, we classify all values of $n \in \mathbb{N}$ such that the *ordered* elements of $\mathcal{S}_n$ form an integral linear recurrence $(d_i)_{i=1}^{s(n)}$ of order at most two. A number n that satisfies this property is called a *recurrent* number.

Example 1. Notice that 60 is a recurrent number since the elements of $\mathcal{S}_{60} = \{1, 2, 3, 4, 5, 6\}$ satisfy the recurrence $d_i = 2d_{i-1} - d_{i-2}$. On the other hand, a simple computation or computer search would show that the smallest non-recurrent number is 36, as $\mathcal{S}_{36} = \{1, 2, 3, 4, 6\}$ does not satisfy a linear recurrence of order at most two.

Since 1 is the smallest divisor of n, we have $d_1 = 1$. To be able to easily determine whether $\mathcal{S}_n$ forms a linear recurrence, we impose the condition that $s(n) \geq 5$, although we revisit the cases $s(n) \leq 4$ at the end of the paper. Furthermore, since the smallest nontrivial divisor of any natural number is prime, we assign $d_2 = p$ throughout the paper. We also let $U(1, p, a, b)$ be the linear recurrence which produces the divisors of some recurrent number n. Notice that simply knowing p, a, b allows us to determine all elements in $\mathcal{S}_n$ whenever n is recurrent.

d_1	1	(4)
d_2	p	
d_3	$ap + b$	
d_4	$(a^2 + b)p + ab$	
d_5	$(a^3 + 2ab)p + b(a^2 + b)$	

Table 1: First 5 values of $(d_i)_{i=1}^{s(n)}$ in terms of a, b, p.

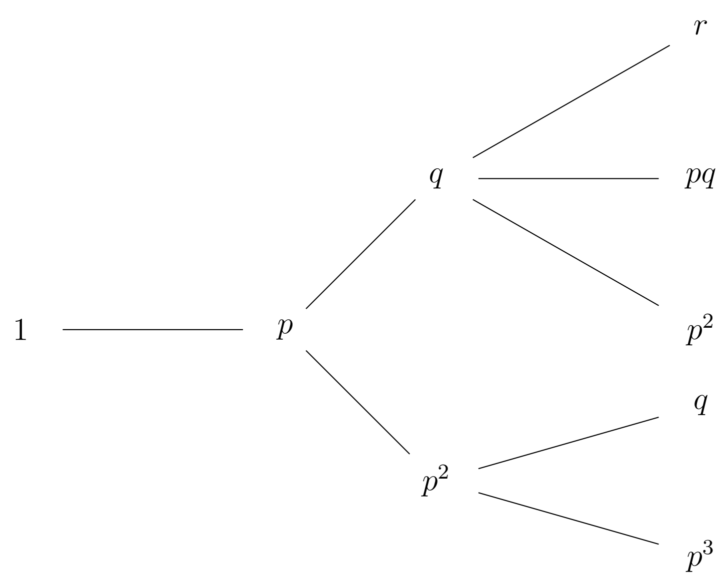

Figure 1: The above tree represents the possible "configurations" of the first four divisors of a natural number $n \geq 1$, where $p < q < r$ are distinct primes. For example, this says that the first divisor must be 1, the second divisor must be some prime p, and the third divisor can either be p^2 or a second prime q, etc ... Our proof makes use of this "tree" approach in determining the divisors d_i.

2 Preliminary results

Proposition 2. *Let a, b, n be positive integers such that $a \leq b$. If $ab \mid n$, then $a \in \mathcal{S}_n$.*

Proof. Since $ab \mid n$, it follows that a is a divisor of n. Moreover, we get that $ab \leq n$. However, $a^2 \leq ab \leq n$, and hence $a \leq \sqrt{n}$. □

Theorem 3. *Given a recurrent number n associated with $U(1, p, a, b)$, at least one of the following statements is true:*

1. $\gcd(a, b) = 1$.
2. *All nontrivial elements of $\mathcal{S}_n$ are divisible by p.*
3. $a = 0$.

Proof. Suppose $\gcd(a, b) \neq 1$, so some prime divides both a and b.

If $p \mid \gcd(a, b)$, then we see that all nontrivial elements in $\mathcal{S}_n$ are divisible by p, so the second possibility holds. If $p \nmid \gcd(a, b)$, then we obtain that $\gcd(a, b) | ap + b = d_3$, so $\gcd(a, b)$ is some prime q according to Figure 1, and $d_3 = q$. Similarly, $q \mid d_4$ and thus $aq + bp = d_4$ must be pq. From $ap + b = q$ and $aq + bp = bp$, we get $a(p^2 - q) = 0$, hence $a = 0$. □

Proposition 4. *If all nontrivial elements of $\mathcal{S}_n$ are divisible by some prime p, then $n = p^k$ or $n = p^k q$ for some prime $q > p^k$.*

Proof. We proceed by considering two cases.

Case I: If all nontrivial divisors of n are divisible by p, then $n = p^k$ for some $k \geq 1$.

Case II: Assume that n has a divisor $d > \sqrt{n}$ that is not divisible by p.

If d is composite, then one of its prime divisors must be less than or equal to $\sqrt{d} < \sqrt{n}$, and must hence be divisible by p. This contradicts our assumption, and d must thus be some prime q.

We now prove uniqueness. If there exists another prime r in the interval $(\sqrt{n}, n)$ that divides n, then qr must divide n but $qr > \sqrt{n} \cdot \sqrt{n} = n$, another contradiction.

Hence, q is the sole divisor of n that is not divisible by p, and thus we have $n = p^k q$. $\square$

Proposition 5. *In a recurrent number n, if $\mathcal{S}_n$ satisfies a recurrence $U(1, p, 0, b)$ (i.e. , $a = 0$), then exactly one of the following possibilities holds, where p, q, r are primes and $k \in \mathbb{N}$:*

1. *$n = p^k$.*
2. *$n = p^k q$ for $p^k < q$.*
3. *$n = pq^k$ for $p < q$.*
4. *$n = pq^k r$ for $p < q < pq^k < r$.*

Proof. If $a = 0$, then $\mathcal{S}_n$ forms a bifurcated series with $d_i = bd_{i-2}$, but taking $i = 3$ requires that $d_3 = b$ must be either p^2 or q (for a prime $q > p$). The former gives rise to $\mathcal{S}_n = \{1, p, p^2, \ldots, p^k\}$, while the latter implies that $\mathcal{S}_n = \{1, p, q, pq, q^2, \ldots, pq^k\}$ or $\{1, p, q, pq, q^2, \ldots, q^k\}$, depending on parity. Identical to the proof of Proposition 4, there is at most one prime divisor greater than $\sqrt{n}$ that divides n, and hence the possibilities follow directly. $\square$

3 The case $\gcd(a, b) = 1$

We now turn our attention to the major case where $\gcd(a, b) = 1$. We also assume, henceforth, that n has at least two distinct small prime factors—otherwise we are simply in the second and third cases of Theorem 3. In particular, $ab \neq 0$ since if one of the numbers is zero, then the other cannot have absolute value 1, as negative numbers do not appear in $\mathcal{S}_n$ and no number appears twice in the sequence of small divisors.

Lemma 6. *For all small divisors d_i of a recurrent number we have $\gcd(b, d_i) = 1$.*

Proof. Suppose that $q \mid \gcd(b, d_i)$ for some prime q and some index i. Notice that this $i \neq 1$. Similarly, if $i = 2$, then $p \mid b$ and inductively, all small divisors are divisible by p which contradicts the assumption of having two distinct small prime factors.

Suppose $i \geq 3$. Since $d_i = ad_{i-1} + bd_{i-2}$, we have $q \mid ad_{i-1}$. The coprimality of a and b implies that $q \mid d_{i-1}$. We can inductively show that smaller divisors, including $d_1 = 1$, are divisible by q, which is absurd. Hence the coprimality follows. $\square$

Corollary 7. *For all i such that d_i, d_{i+1} are small divisors of a recurrent number, we have* $\gcd(d_i, d_{i+1}) = 1$.

Proof. We proceed by induction. The base case clearly holds. For the inductive step, observe that

$$\gcd(d_i, d_{i+1}) = \gcd(d_i, ad_i + bd_{i-1}) = \gcd(d_i, bd_{i-1}).$$

The inductive hypothesis implies that $\gcd(d_i, d_{i-1}) = 1$, while Lemma 6 implies that $\gcd(d_i, b) = 1$, and hence the coprimality follows. $\square$

As a direct consequence of this, we conclude that d_3 cannot be p^2 in a recurrent number, and hence d_3 must be prime. Henceforth, we set $d_3 = q$.

Lemma 8. *All configurations of the first four nontrivial divisors* (d_2, d_3, d_4, d_5) *except possibly* $(p, q, p^2, r), (p, q, r, p^2)$*, and* (p, q, r, s)*, where p, q, r, s are distinct primes, are impossible in a recurrent number.*

Proof. We have already determined d_2, d_3. Now, d_4 can be p^2, q^2, pq, or r (where r is another prime). By Corollary 7, d_4 cannot be pq or q^2. Hence, we have either $d_4 = p^2$ or $d_4 = r$.

If $d_4 = p^2$, then we can similarly see that the possibilities for the next divisor d_5 are p^3, q^2, pq, r for a prime r. However, Corollary 7 once again implies that p^3, pq are impossible. If $d_5 = q^2$, then since $d_5 = ap^2 + bq$ we would get that $q \mid ap^2$, and so, $q \mid a$. Yet, since $q = ap + b, q \mid a$ would imply that $q \mid b$ and thus contradicts Lemma 6. Hence, d_5 ought to be another prime, r.

If $d_4 = r$, then we see that our possibilities for the next divisor d_5 are p^2, pq, s, for a new prime s. A reasoning similar to above shows that $q \nmid d_5$, so the only possibilities left are those of $d_5 = p^2$ and $d_5 = s$. $\square$

Lemma 9. *The only pair* $(u, x) \in \mathbb{N} \times \mathbb{N}_{\geq 2}$ *that satisfies*

$$-ux^5 + (u+3)x^4 - \frac{2u+3}{u}x^3 + \frac{3u+1}{u^2}x^2 - \frac{2}{u^2}x + \frac{1}{u^2} \geq x^2 + 1$$

is $(1, 2)$.

Proof. Define

$$P(x) := -ux^5 + (u+3)x^4 - \frac{2u+3}{u}x^3 + \frac{3u+1}{u^2}x^2 - \frac{2}{u^2}x + \frac{1}{u^2} - (x^2+1).$$

Notice that

$$\begin{aligned}P'(x) &= -5ux^4 + 4(u+3)x^3 - \frac{3(2u+3)}{u}x^2 + \frac{2(3u+1)}{u^2}x - \frac{2}{u^2} - 2x \\ &= -5ux^4 + 4(u+3)x^3 - \frac{3(2u+3)}{u}x^2 + \frac{-2u^2+6u+2}{u^2}x - \frac{2}{u^2}.\end{aligned}$$

For $u \geq 4$ and $x \geq 2$ we have

$$5ux^4 \geq 10ux^3 \geq 4(u+3)x^3,$$

and in addition, we have $-\frac{3(2u+3)}{u}x^2, \frac{-2u^2+6u+2}{u^2}x, -\frac{2}{u^2}$ are each negative. Notice that $P'(x)$ is negative. Hence, for $u \geq 4$, the function $P(x)$ is strictly decreasing in the interval $[2, \infty)$. Moreover, it is easy to check that $P(2) < 0$. Hence, it suffices to simply consider $u \in \{1, 2, 3\}$. One may easily check that the only possible value is $u = 1$. Another simple computation reveals that the inequality only holds for $x = 2$. □

Theorem 10. *The only recurrent number with configuration* $(d_2, d_3, d_4, d_5) = (p, q, p^2, r)$ *is* 60*.*

Proof. In this case, we have

$$p^2 = d_4 = aq + bp = a(ap + b) + bp. \tag{5}$$

Since $p \mid ab$ and $p \nmid b$ by Lemma 6, we have $p \mid a$. Let $a = kp$ for some integer k. We can rewrite (5) as

$$k^2p^2 + b(k+1) = p,$$

which is equivalent to

$$b = \frac{p(1 - k^2p)}{k+1}.$$

However, since we know that $p \nmid b$, we obtain that $p \mid k+1$. In other words, there exists a positive integer u such that $k + 1 = up$. From here, we are able to express a, b in terms of u, p, and use that to obtain r in terms of u, p only. We end up obtaining

$$a = up^2 - p,$$

$$b = \frac{1-p}{u} - up^3 + 2p^2,$$

and

$$r = ap^2 + bq = -up^5 + (u+3)p^4 - \frac{2u+3}{u}p^3 + \frac{3u+1}{u^2}p^2 - \frac{2}{u^2}p + \frac{1}{u^2}.$$

Since $r = d_5 \geq d_4 + 1 = p^2 + 1$, Lemma 9 gives that $(u, p) = (1, 2)$ and substituting reveals $a = 2, b = -1$. Hence the first five divisors would be $1, 2, 3, 4, 5$. This reduces to the case of an arithmetic sequence with common difference 1, which, using the results of Iannucci [1, Lemma 2], implies that $s(n) = 5$ or $s(n) = 6$. If $s(n) = 5$, we have $n \geq \text{lcm}(1, 2, 3, 4, 5) = 60$, but 6 is a small divisor that does not appear in $\mathcal{S}_n$, a contradiction. When $s(n) = 6$, we see that n is a multiple of 60. However, if $n \geq 180$ then $12 \leq \sqrt{n}$ does not appear in the list of small divisors, contradicting the recurrency. Clearly, $n = 120$ is not recurrent neither as $8 \notin \{1, 2, 3, 4, 5, 6\}$. Hence, the only solution is $\mathcal{S}_n = \{1, 2, 3, 4, 5, 6\}$ for when $n = 60$, which is indeed recurrent. □

Corollary 11. *For an initial configuration (p, q, r, p^2) in a recurrent number, we have that $p \mid d_j$ if and only if $j \equiv 2 \pmod 3$.*

Proof. Firstly, the statement holds for $d_2 = p$. Since $p^2 = d_5 = (a^3 + 2ab)p + b(a^2 + b)$, Lemma 6 implies that $p \mid (a^2 + b)$. Hence,

$$d_{i+3} = ad_{i+2} + bd_{i+1} = a(ad_{i+1} + bd_i) + bd_{i+1} = (a^2 + b)d_{i+1} + abd_i.$$

Lemma 6 once again gives us that $p \mid d_{i+3}$ if and only if $p \mid d_i$, which is the inductive step. □

Theorem 12. *The configuration of divisors (p, q, r, p^2) cannot occur in a recurrent number.*

Proof. Assume otherwise, that is, there exists a recurrent number n of this configuration. We prove, by induction, that $s(n) \geq 3i + 2$ for all $i \in \mathbb{N}$, which is absurd.

The base case is already assumed. Assume that $s(n) \geq 3i + 2$ for some $i \in \mathbb{N}$. By Corollary 11, $p \nmid d_{3i}d_{3i+1}$ and since $\gcd(d_{3i}, d_{3i+1}) = 1$ by Corollary 7, we have $p^2 d_{3i} d_{3i+1}$ divides n. By Proposition 2, pd_{3i} is a small divisor. We consider two cases.

Case I: $pd_{3i} = d_{3i+2}$. Hence, we obtain that $d_{3i} \mid ad_{3i+1}$, and by Corollary 7, we obtain that $d_{3i} \mid a$. Observe that the sequence $(d_i \bmod a)$ takes values congruent to $1, p, b, bp, b^2, \ldots$, and thus, d_{3i} divides an element of the form $b^k p$ where $k \in \mathbb{N}$. However, Corollary 11 and Lemma 6 yield

$$\gcd(p, d_{3i+1}) = 1 = \gcd(b, d_{3i+1}),$$

which is a contradiction.

Case II: $pd_{3i} > d_{3i+2}$, hence, $pd_{3i} \geq d_{3i+5}$ by Corollary 11. Therefore, $s(n) \geq 3i + 5$, and the induction is complete. □

Lemma 13. *In a recurrent number of configuration (p, q, r, s), the small divisors d_i, d_{i+2} are coprime.*

Proof. Assume a prime t divides $\gcd(d_i, d_{i+2})$. Thus $t \mid ad_{i+1}$. Clearly, $t \nmid d_{i+1}$ by Lemma 7, and therefore, $t \mid a$. As such, the sequence $(d_i \bmod t)$ takes values congruent to $1, p, b, bp, b^2, \ldots$ and hence $t \mid b^k p$ for some $k \geq 0$. Yet Lemma 6 implies that $\gcd(b, t) = 1$, hence $t \mid p$ and so $t = p$. However, this would imply that $t \mid r = (a^2 + b)p + ab$, which is a contradiction as $p < r$ are distinct primes. □

Theorem 14. *The configuration of divisors (p, q, r, s) cannot occur in a recurrent number.*

Proof. Assume that the configuration does occur in some recurrent number n. Since $pq < pr$ are both small by Proposition (2), there exist at least three small divisors that are multiples of p. Let $d_j = pv$ be the greatest small divisor that is divisible by p. In particular, $v > p$. By Lemma 13, $p \nmid d_{j-2}d_{j-1}$, and we conclude that $d_{j-2}d_{j-1}pv \mid n$, so Proposition 2 implies that $pd_{j-2} \in \mathcal{S}_n$. Moreover, $pd_{j-2} \neq d_j$ by Lemma 13, and so $pd_{j-2} > d_j$ is a small divisor greater than d_j, a contradiction. $\square$

4 Concluding remarks

In conclusion, imposing the condition $s(n) \geq 5$ returns certain infinite families of recurrent integers n with at most 3 distinct prime divisors, in addition to the sole "sporadic" case of 60. We now revisit the condition we imposed and relax it to find all recurrent numbers by relating $s(n)$ to $\tau(n)$ using Equation (3).

(i) $s(n) = 1$. Hence, $\tau(n) \in \{1, 2\}$. That is, $n = 1$ or $n = p$ for some prime p, and in both cases, n is vacuously recurrent.

(ii) $s(n) = 2$. Thus, $\tau(n) \in \{3, 4\}$, and we have the possibilities that $n = p^2, p^3$, or pq for primes $p < q$. In all cases, n is vacuously recurrent.

(iii) $s(n) = 3$. This implies $\tau(n) \in \{5, 6\}$, and hence the possibilities are $n = p^4$, $n = p^5$, $n = p^2q$, or $n = pq^2$ for primes $p < q$. These are again vacuously recurrent.

(iv) $s(n) = 4$. Consequently, $\tau(n) \in \{7, 8\}$.

  (I) If $\tau(n) = 7$, we get that $n = p^6$ and $\mathcal{S}_n = \{1, p, p^2, p^3\}$ indeed forms a linear recurrence of order at most two.

  (II) If $\tau(n) = 8$, the possibilities are p^7, p^3q, pq^3, and pqr for primes $p < q < r$.

    i. $n = p^7$: Here n is recurrent by the aforementioned geometric recurrences.

    ii. $n = p^3q$: Here Lemma 7 implies that the only possibility is $\mathcal{S}_n = \{1, p, q, p^2\}$. Setting up a linear system of equations to solve for a, b, we deduce that it is both necessary and sufficient for $\frac{q-p}{p^2-q}$ to be an integer for primes $p < q < p^2$. Based on numerical evidence, we conjecture that this occurs for infinitely many pairs of primes (p, q), but we defer analyzing this conjecture to another time.

    iii. $n = pq^3$: Clearly, we have $\mathcal{S}_n = \{1, p, q, pq\}$, but this is not possible by Lemma 7.

    iv. $n = pqr$: If $pq < r$, then $\mathcal{S}_n = \{1, p, q, pq\}$ and the same argument above holds. If $pq > r$, then $\mathcal{S}_n = \{1, p, q, r\}$. Setting up a linear system of equations, we see that it is both necessary and sufficient for $\frac{pq-r}{p^2-q}$ to be an integer.

We summarize our findings in the following result.

Theorem 15. *All recurrent numbers n fall into one of the following categories.*

1. $n = p^k$, *for some prime p and a natural number k, hence* $\mathcal{S}_n = \{1, p, \dots, p^{\lfloor \frac{k}{2} \rfloor}\}$.
2. $n = p^k q$, *for some primes p, q and a natural number k, such that $q > p^k$. Hence,* $\mathcal{S}_n = \{1, p, \dots, p^k\}$.
3. $n = pq^k$, *for some primes $p < q$ and k an odd (resp. , even) natural number with* $\mathcal{S}_n = \{1, p, q, pq, q^2, \dots, pq^{\frac{k-1}{2}}\}$ *(resp. ,* $\mathcal{S}_n = \{1, p, q, pq, q^2, \dots, q^{\frac{k}{2}}\}$*).*
4. $n = pq^k r$, *for some primes p, q, r and a natural number k such that $p < q$ and $r > pq^k$. Hence,* $\mathcal{S}_n = \{1, p, q, pq, \dots, pq^k\}$.
5. $n = 60$, *with* $\mathcal{S}_n = \{1, 2, 3, 4, 5, 6\}$.
6. $n = p^3 q$ *for some primes p, q such that $p < q < p^2$ and $p^2 - q \mid q - p$. Hence,* $\mathcal{S}_n = \{1, p, q, p^2\}$.
7. $n = pqr$, *for some primes p, q, r such that $p < q < r$ and $p^2 - q \mid pq - r$. Hence,* $\mathcal{S}_n = \{1, p, q, r\}$.

Notice that $\mathcal{S}_n$ may contain an arbitrarily long subset of divisors that form a second-order linear recurrence, without all of $\mathcal{S}_n$ forming such a recurrence. Consider numbers of the form $pq^k r^2$ where $p < q < r$ are primes and $k \in \mathbb{N}$ such that $pq^k < r$. If r is large enough, we see that the divisors $1, p, q, pq, q^2, \dots$ may form an arbitrarily long linear recurrence that is interrupted by r, but n is not itself recurrent.

Finally, we briefly study the analytic distribution of recurrent numbers. Define $f(x)$ to be the number of recurrent integers in the interval $[1, x]$. The fact that recurrent numbers have at most 3 distinct prime divisors implies that they have null density among the natural numbers. For $k \in \mathbb{N}$, let $\pi_k(x)$ be the number of integers in $[1, x]$ having exactly k distinct prime factors. Hardy and Ramanujan [2, Lemma A, p. 265] showed that there exist constants A, B such that

$$\pi_k(x) < \frac{Ax(\log\log x + B)^{k-1}}{(k-1)!\log x}.$$

In fact, Landau [3] proved that that

$$\pi_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}.$$

Since $f(x) \le \pi_1(x) + \pi_2(x) + \pi_3(x)$, we obtain that there exists some constant C such that for all $x \ge 1$,

$$f(x) \le C\left(\frac{x}{\log x} + \frac{x\log\log x}{\log x} + \frac{x(\log\log x)^2}{2\log x}\right).$$

Chu previously extended the results of Iannucci by classifying all natural numbers whose large divisors [4] or nontrivial, small divisors [5] form an arithmetic progression. Based on our results, the interested reader may naturally ask whether the aforementioned results can be generalized from the case of arithmetic progressions to that of linear recurrences of order at most two.

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References

[1] D. E. Iannucci, When the small divisors of a natural number are in arithmetic progression, *Integers* **18** (2018), Paper #A77.

[2] S. Ramanujan, *Collected Papers of Srinivasa Ramanujan,* American Mathematical Soc., 2000.

[3] E. Landau, *Handbuch der Lehre von der Vorteilung der Primzahlen,* B. G. Teubner, 1909.

[4] H. V. Chu, When the large divisors of a natural number are in arithmetic progression, *J. Integer Sequences* **23** (2020), Article 20.6.1.

[5] H. V. Chu, When the nontrivial, small divisors of a natural number are in arithmetic progression, preprint, 2020. Available at `http://arxiv.org/abs/2001.08634`.

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