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Interesting Ramanujan-Like Series Associated with Powers of Central Binomial Coefficients

Hongwei Chen
Department of Mathematics
Christopher Newport University
Newport News, VA 23606
USA
hchen@cnu.edu

Abstract

We find various Ramanujan-like series that involve the central binomial coefficients. In contrast with the traditional hypergeometric function approach, our method uses Fourier-Legendre series expansions via specialization, inner product, and Parseval's identity. Several classical identities are recovered as particular cases.

1 Introduction

A Ramanujan-like series for $1/\pi$ is an identity of the following form:

$$\sum_{n=0}^{\infty} \frac{an+b}{c^n} s(n) = \frac{1}{\pi},$$

where a and b are algebraic numbers, $c \in \mathbb{N}$, and $s(n)$ is an integer sequence obeying a certain recurrence relation. More than 100 years ago, without proof, Ramanujan recorded 17 formulas for $1/\pi$ at the end of his first paper published in England [13]. The first two formulas (with slight modification), which surprisingly appeared in the Disney movie *High School Musical*, are

$$\sum_{n=0}^{\infty} \frac{6n+1}{16^{2n}} \binom{2n}{n}^3 = \frac{4}{\pi} \tag{1}$$

and

$$\sum_{n=0}^{\infty} \frac{42n+5}{64^{2n}} \binom{2n}{n}^3 = \frac{16}{\pi}, \tag{2}$$

where $\binom{2n}{n}$ is the central binomial coefficient defined by $(2n)!/(n!)^2$. Except for Chowla's proof of (1) in 1928 [7], Ramanujan's series were forgotten by the mathematical community until 1987. The Borwein brothers [3] finally succeeded in proving all 17 of Ramanujan's series for $1/\pi$ via modular forms. Moreover, as an application, they used one modified Ramanujan series to calculate the digits of π and were able to obtain roughly 50 digits of π per term. Since 2002, Guillera et al. [10, 11] have discovered many new Ramanujan-like series for $1/\pi^2$. An excellent survey on the work of Ramanujan-like series before 2009 can be found in [2].

Following the ideas of the Borweins and Guillera et al., exploring new classes of Ramanujan-like series has become an active area of contemporary research. In my previous work [6], the following Ramanujan-like series

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2} = \frac{12 - 16\ln 2}{\pi},$$

where H_n denotes the nth harmonic number, was established by applying a differential operator to a hypergeometric function ${}_2F_1$-identity. Along this line, using the Gauss summation theorem and an extended ${}_3F_2$-series of Watson and Whipple type, Wang and Chu [16] offered a systematic evaluation of series like

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n p(n)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n^{(2)}}{16^n p(n)},$$

where $p(n) = n + \lambda, 1 + 2n - 2\lambda$ or $(1 + 2n - 2\lambda)^2$ for $\lambda \in \mathbb{N}$, and $H_n^{(2)} = \sum_{k=1}^{n} 1/k^2$. In particular, they obtained the following interesting Ramanujan-like series:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n^{(2)}}{16^n (2n-1)} = 4 - \frac{\pi}{3} - \frac{8}{\pi};$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n^{(2)}}{16^n (2n-1)^2} = -12 + \frac{2\pi}{3} + \frac{32}{\pi}.$$

The aim of the present paper is to study the Ramanujan-like series involving powers of central binomial coefficients. In contrast to [4, 16], our approach is based on Fourier-Legendre series expansions. We will derive a variety of Ramanujan-like series including

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{16^n} \binom{2n}{n}^2 = \frac{\sqrt{2}\,\Gamma^2(1/4)}{4\pi^{3/2}}; \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1)}{64^n(n+1)(1/2-n)} \binom{2n}{n}^3 = \frac{8}{\pi}; \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{(4n+1)}{256^n(n+1)^2(1/2-n)^2} \binom{2n}{n}^4 = \frac{128}{3\pi^2}. \tag{5}$$

Here $\Gamma(x)$ indicates the gamma function.

The rest of paper is organized as follows. In Section 2, three combinatorial identities are established via the Wilf-Zeilberger method. These identities are used to derive several Fourier-Legendre series expansions. By using specialization, inner product, and Parseval's identity, those series expansions are used to find various Ramanujan-like series in Section 3. The paper ends with two remarks. To ensure accuracy, all formulas appearing in this paper were verified numerically by Mathematica.

2 Some Fourier-Legendre series expansions

Recall the Legendre polynomial [9, pp. 983–985]

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2-1)^n.$$

It is well-known that

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = \frac{2}{2n+1}\, \delta_{mn},$$

where δ_{mn} denotes the Kronecker delta. This implies that $(P_n(x))_{n=0}^{\infty}$ are orthogonal on $[-1,1]$. Now, we define

$$\tilde{P}_n(x) = P_n(2x-1).$$

Since $x \to 2x-1$ is an affine transformation that bijectively maps $[0,1]$ to $[-1,1]$, we see that $(\tilde{P}_n(x))_{n=0}^{\infty}$ are orthogonal on $[0,1]$ with

$$\int_{0}^{1} \tilde{P}_n(x)\tilde{P}_m(x)\, dx = \frac{1}{2n+1}\, \delta_{mn}.$$

Moreover, we have

$$\tilde{P}_n(x) = \frac{1}{n!} \frac{d^n}{dx^n}(x^2-x)^n = (-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-x)^k. \tag{6}$$

Let $f(x) \in L[0,1]$. The *Fourier-Legendre series* is defined by

$$f(x) = \sum_{n=0}^{\infty} a_n \tilde{P}_n(x),$$

where

$$a_n = (2n+1) \int_0^1 f(x) \tilde{P}_n(x)\, dx. \tag{7}$$

The following three lemmas display some features of the coefficients of variant Fourier-Legendre series.

Lemma 1. *Let $f(x) \in L[0,1]$. If $f(x) = \sum_{n=0}^{\infty} a_n \tilde{P}_n(x)$, then*

$$f(1-x) = \sum_{n=0}^{\infty} (-1)^n a_n \tilde{P}_n(x).$$

Proof. Let a_n^* be the nth coefficient of the Fourier-Legendre series for $f(1-x)$. By (7), we have

$$\begin{aligned} a_n^* &= (2n+1) \int_0^1 f(1-x) \tilde{P}_n(x)\, dx \\ &= (2n+1) \int_0^1 f(t) \tilde{P}_n(1-t)\, dt \quad (\text{use } t = 1-x) \\ &= (2n+1) \int_0^1 f(t)(-1)^n \tilde{P}_n(t)\, dt = (-1)^n a_n, \end{aligned}$$

where we have used the fact that $\tilde{P}_n(1-t) = P_n(-(2t-1)) = (-1)^n P_n(2t-1) = (-1)^n \tilde{P}_n(t)$. □

Lemma 2. *Let $f(x) \in L[0,1]$. If $f(x) = f(1-x)$ for $x \in [0,1]$ and $f(x) = \sum_{n=0}^{\infty} a_n \tilde{P}_n(x)$, then $a_n = 0$ when n is odd.*

Proof. Using (7), we rewrite a_n as

$$\begin{aligned} a_n &= (2n+1) \left(\int_0^{1/2} f(x) \tilde{P}_n(x)\, dx + \int_{1/2}^1 f(x) \tilde{P}_n(x)\, dx \right) \\ &= (2n+1) \left(\int_0^{1/2} f(x) \tilde{P}_n(x)\, dx + \int_0^{1/2} f(1-t) \tilde{P}_n(1-t)\, dt \right) \ (\text{use } t = 1-x) \\ &= (2n+1) \int_0^{1/2} (1 + (-1)^n) f(x) \tilde{P}_n(x)\, dx \ (\text{use } f(t) = f(1-t)). \end{aligned}$$

This leads to $a_n = 0$ when n is odd. □

Recall Bonnet's recursion formula [9, Formula 8.914 No.1, p. 985]

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

This implies that

$$(2n+1)x\tilde{P}_n(x) = \frac{n+1}{2}\tilde{P}_{n+1}(x) + \frac{2n+1}{2}\tilde{P}_n(x) + \frac{n}{2}\tilde{P}_{n-1}(x),$$

which yields the following result.

Lemma 3. *Let $f(x) \in L[0,1]$ and $f(x) = \sum_{n=0}^{\infty} a_n \tilde{P}_n(x)$. If*

$$xf(x) = \sum_{n=0}^{\infty} b_n \tilde{P}_n(x),$$

then for all $n \geq 1$,

$$b_n = \frac{n+1}{2(2n+3)}\, a_{n+1} + \frac{1}{2}\, a_n + \frac{n}{2(2n-1)}\, a_{n-1}. \tag{8}$$

The following two lemmas establish some identities that we will need later to simplify the coefficients of the Fourier-Legendre series.

Lemma 4. *For any non-negative integer n,*

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+k}{k}\binom{2k}{k} 4^{2n-k} = \binom{2n}{n}^2. \tag{9}$$

This identity (9) appears as (6.35) in [8], and is credited to E. T. Bell without providing a proof. For completeness, following Tauraso's suggestion [14], we demonstrate a proof based on the Wilf-Zeilberger method (WZ-method).

Proof. Rewrite (9) as

$$\sum_{k=0}^{2n} \frac{(-1)^k \binom{2n}{k}\binom{2n+k}{k}\binom{2k}{k} 4^{2n-k}}{\binom{2n}{n}^2} = 1. \tag{10}$$

Let

$$F(n,k) = \frac{(-1)^k \binom{2n}{k}\binom{2n+k}{k}\binom{2k}{k} 4^{2n-k}}{\binom{2n}{n}^2}.$$

Using the WZ-method, we find

$$G(n,k) = -\frac{2(4n+3)k^3}{(2n-k+1)(2n-k+2)(2n+1)^2}F(n,k).$$

Now it suffices to check

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{11}$$

Notice that

$$F(n+1,k) = \frac{4(2n+k+2)(2n+k+1)(n+1)^2}{(2n-k+2)(2n-k+1)(2n+1)^2}\,F(n,k),$$
$$F(n,k+1) = -\frac{(2k+1)(2n+k+1)(2n-k)}{2(k+1)^3}\,F(n,k),$$
$$G(n,k+1) = \frac{(4n+3)(2k+1)(2n+k+1)}{(2n-k+1)(2n+1)^2}\,F(n,k);$$

with these (11) becomes the alleged identity

$$\frac{4(2n+k+2)(2n+k+1)(n+1)^2}{(2n-k+2)(2n-k+1)(2n+1)^2} - 1$$
$$= \frac{(4n+3)(2k+1)(2n+k+1)}{(2n+1-k)(2n+1)^2} + \frac{2(4n+3)k^3}{(2n+1-k)(2n+2-k)(2n+1)^2}?$$

Combining the fractions shows that each side leads to

$$\frac{(4n+3)(2+5k+k^2+6n+12kn+4n^2+8kn^2)}{(2n-k+1)(2n-k+2)(2n+1)^2}.$$

So (11) does actually hold. Since G is telescoping, we finally obtain

$$\sum_{k=0}^{2n} F(n,k) = \text{const}.$$

This constant does not depend on n. Thus (10), and so (9), follows from setting $n=0$. □

Similarly, applying the WZ-method, we can establish

Lemma 5. *For any non-negative integer n,*

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+k}{k} \frac{3\cdot 7\cdots(4k-1)}{(2k+1)k!\binom{2k}{k}} = \frac{1}{(4n+1)4^n}\binom{2n}{n}; \tag{12}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+k}{k} \frac{5\cdot 9\cdots(4k+1)}{(2k+3)(2k+1)k!\binom{2k}{k}} = \frac{1}{(4n+3)(1-4n)4^n}\binom{2n}{n}. \tag{13}$$

Now we are ready to establish some Fourier-Legendre series in explicit form.

Theorem 6. *For $x \in (0,1)$,*

$$\frac{1}{\sqrt{x(1-x)}} = \sum_{n=0}^{\infty} \frac{(4n+1)\pi}{16^n}\binom{2n}{n}^2 \tilde{P}_{2n}(x). \tag{14}$$

Proof. Let $f(x) = 1/\sqrt{x(1-x)}$. Then $f(x) = f(1-x)$ for all $x \in (0,1)$. Using Lemma 2, $a_n = 0$ for odd n. Moreover,

$$a_{2n} = (4n+1)\int_0^1 f(x)\tilde{P}_{2n}(x)\,dx.$$

Invoking (6) leads to

$$\begin{aligned} a_{2n} &= (4n+1)\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+k}{k}\int_0^1 x^{k-1/2}(1-x)^{-1/2}\,dx \\ &= (4n+1)\sum_{k=0}^{2n}\binom{2n}{k}\binom{2n+k}{k}(-1)^k B(k+1/2,1/2), \end{aligned}$$

where $B(x,y)$ is the Euler Beta function. Using

$$\Gamma(1/2) = \sqrt{\pi} \quad \text{and} \quad \Gamma(k+1/2) = \frac{(2k)!}{4^k\,k!}\sqrt{\pi}, \tag{15}$$

it follows that

$$B(k+1/2,1/2) = \frac{\Gamma(k+1/2)\Gamma(1/2)}{\Gamma(k+1)} = \frac{(2k)!}{4^k\,k!^2}\,\pi = \frac{\pi}{4^k}\binom{2k}{k}.$$

This, together with (9), implies

$$a_{2n} = (4n+1)\pi\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+k}{k}\binom{2k}{k}4^{-k} = \frac{(4n+1)\pi}{16^n}\binom{2n}{n}^2,$$

which proves (14). □

Applying (8) to (14) gives

$$\begin{aligned} \frac{x}{\sqrt{x(1-x)}} &= \sum_{n=0}^{\infty}\frac{(4n+1)\pi}{2\cdot 16^n}\binom{2n}{n}^2\tilde{P}_{2n}(x) \\ &\quad + \sum_{n=0}^{\infty}\frac{(4n+3)(2n+1)\pi}{4\cdot 16^n(n+1)}\binom{2n}{n}^2\tilde{P}_{2n+1}(x). \end{aligned} \tag{16}$$

While applying (8) to (16) yields

$$\begin{aligned} \frac{x^2}{\sqrt{x(1-x)}} &= \sum_{n=0}^{\infty}\frac{(4n+1)(8n^2+4n-3)\pi}{8\cdot 16^n(n+1)(2n-1)}\binom{2n}{n}^2\tilde{P}_{2n}(x) \\ &\quad + \sum_{n=0}^{\infty}\frac{(4n+3)(2n+1)\pi}{4\cdot 16^n(n+1)}\binom{2n}{n}^2\tilde{P}_{2n+1}(x). \end{aligned} \tag{17}$$

Notice that

$$\sqrt{x(1-x)} = \frac{x}{\sqrt{x(1-x)}} - \frac{x^2}{\sqrt{x(1-x)}}.$$

Combining (16) and (17) leads to

Theorem 7. *For $x \in (0,1)$,*

$$\sqrt{x(1-x)} = \frac{1}{8}\sum_{n=0}^{\infty} \frac{(4n+1)\pi}{16^n(n+1)(1-2n)} \binom{2n}{n}^2 \tilde{P}_{2n}(x). \tag{18}$$

Applying the above process to (18) yields

Theorem 8. *For $x \in (0,1)$,*

$$\left(\sqrt{x(1-x)}\right)^3 = \frac{9}{8^2}\sum_{n=0}^{\infty} \frac{(4n+1)\pi}{16^n(n+1)(n+2)(1-2n)(3-2n)} \binom{2n}{n}^2 \tilde{P}_{2n}(x). \tag{19}$$

By mathematical induction, we conclude

Theorem 9. *Let m be a positive odd integer, and let $p = (m+1)/2$. Then*

$$\left(\sqrt{x(1-x)}\right)^m$$
$$= \frac{(m!!)^2}{8^p}\sum_{n=0}^{\infty} \frac{(4n+1)\pi}{16^n(n+1)\cdots(n+p-1)(1-2n)\cdots(2p-1-2n)} \binom{2n}{n}^2 \tilde{P}_{2n}(x).$$

In terms of the Pochhammer symbol, $(a)_k := a(a+1)\cdots(a+k-1)$ for $k \geq 1$, the above expression can be rewritten compactly as

$$\left(\sqrt{x(1-x)}\right)^m = \frac{(m!!)^2}{16^p}\sum_{n=0}^{\infty} \frac{(4n+1)\pi}{16^n(n+1)_p(1/2-n)_p} \binom{2n}{n}^2 \tilde{P}_{2n}(x). \tag{20}$$

Next, let $f(x) = [x(1-x)]^{-1/4}$. Using Lemma 2, $a_n = 0$ for odd n. Moreover, in view of (6),

$$\begin{aligned} a_{2n} &= (4n+1)\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+k}{k} \int_0^1 x^{k-1/4}(1-x)^{-1/4}\, dx \\ &= (4n+1)\sum_{k=0}^{2n} \binom{2n}{k}\binom{2n+k}{k} (-1)^k B(k+3/4, 3/4). \end{aligned} \tag{21}$$

Using (15) and

$$\Gamma(k+3/4) = \frac{3\cdot 7\cdots(4k-1)}{4^k}\Gamma(3/4),$$

we have

$$\begin{aligned}
B(k+3/4,3/4) &= \frac{\Gamma(k+3/4)\Gamma(3/4)}{\Gamma(k+3/2)} \\
&= \frac{4\Gamma^2(3/4)}{\sqrt{\pi}} \frac{3\cdot 7\cdots(4k-1)(k+1)!}{(2(k+1))!} \\
&= \frac{4\Gamma^2(3/4)}{\sqrt{\pi}} \frac{3\cdot 7\cdots(4k-1)}{2(2k+1)k!\binom{2n}{n}}.
\end{aligned}$$

Substituting this into (21) and then applying (12) yields

$$a_{2n} = \frac{2\Gamma^2(3/4)}{\sqrt{\pi}} \frac{1}{4^n}\binom{2n}{n}.$$

Notice that

$$\Gamma(3/4)\Gamma(1/4) = \frac{\pi}{\sin(\pi/4)} = \sqrt{2}\,\pi. \tag{22}$$

Theorem 10. *For $x \in (0,1)$,*

$$\frac{1}{\sqrt[4]{x(1-x)}} = \frac{4\pi^{3/2}}{\Gamma^2(1/4)} \sum_{n=0}^{\infty} \frac{1}{4^n}\binom{2n}{n}\tilde{P}_{2n}(x). \tag{23}$$

Finally, applying (6), (13) and (22) gives

Theorem 11. *For $x \in (0,1)$,*

$$\sqrt[4]{x(1-x)} = \frac{\pi^{3/2}}{2\Gamma^2(3/4)} \sum_{n=0}^{\infty} \frac{4n+1}{4^n(4n+3)(1-4n)}\binom{2n}{n}\tilde{P}_{2n}(x). \tag{24}$$

3 Some new Ramanujan-like series

The Fourier-Legendre series expansions (14), (18), (19), (20), (23) and (24), along with their specialization, inner product and Parseval's identity, will enable us to find many Ramanujan-like series involving powers of central binomial coefficients.

First, replacing $2x-1$ by t in (14), (16) and (18),

Theorem 12. *For $|t| < 1$,*

$$\begin{aligned}
\frac{1}{\sqrt{1-t^2}} &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(4n+1)\pi}{16^n}\binom{2n}{n}^2 P_{2n}(t); \\
\frac{t}{\sqrt{1-t^2}} &= \frac{1}{4}\sum_{n=0}^{\infty} \frac{(4n+3)(2n+1)\pi}{16^n(n+1)}\binom{2n}{n}^2 P_{2n+1}(t); \\
\sqrt{1-t^2} &= \frac{1}{8}\sum_{n=0}^{\infty} \frac{(4n+1)\pi}{16^n(n+1)(1/2-n)}\binom{2n}{n}^2 P_{2n}(t).
\end{aligned}$$

These recover Formulas 8.922 No. 3–5 in [9, p. 988], for which I am unable to find references to their proofs. It is interesting to compare these identities with the well-known generating function of the central binomial coefficients:

$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{1}{4^n}\binom{2n}{n} t^{2n}.$$

Furthermore, let $x = \cos^2(\theta/2)$ in (14), (18)–(20), respectively. Then $2x-1 = \cos\theta$, and $\tilde{P}_{2n}(x) = P_{2n}(\cos\theta)$.

Theorem 13. *For $\theta \in (0, \pi)$,*

$$\sum_{n=0}^{\infty} \frac{4n+1}{16^n}\binom{2n}{n}^2 P_{2n}(\cos\theta) = \frac{2}{\pi\sin\theta}, \tag{25}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{16^n(n+1)(1/2-n)}\binom{2n}{n}^2 P_{2n}(\cos\theta) = \frac{8\sin\theta}{\pi}, \tag{26}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{64^n(n+1)(n+2)(1/2-n)(3/2-n)}\binom{2n}{n}^2 P_{2n}(\cos\theta) = \frac{32\sin^3\theta}{9\pi}, \tag{27}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{64^n(n+1)_p(1/2-n)_p}\binom{2n}{n}^2 P_{2n}(\cos\theta) = \frac{16^p\sin^m\theta}{2^m(m!!)^2\pi}, \tag{28}$$

where m is a positive odd integer and $p = (m+1)/2$.

Identity (26) appeared in [9, p. 989] as Formula 8.925 No. 2. Next, let $x = \cos^2(\theta/2)$ in (23) and (24), respectively.

Theorem 14. *For $\theta \in (0, \pi)$,*

$$\sum_{n=0}^{\infty} \frac{1}{4^n}\binom{2n}{n} P_{2n}(\cos\theta) = \frac{\sqrt{2}\,\Gamma^2(1/4)}{4\sqrt{\sin\theta}\pi^{3/2}}, \tag{29}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{4^n(4n+3)(1-4n)}\binom{2n}{n} P_{2n}(\cos\theta) = \frac{\sqrt{2\sin\theta}\,\Gamma^2(3/4)}{\pi^{3/2}}. \tag{30}$$

Based on Theorems 13 and 14, via specialization, we find some Ramanujan-like series as follows.

Example 1. Recall that $P_{2n}(0) = (-1)^n\binom{2n}{n}/4^n$. Letting $\theta = \pi/2$ in Theorem 13 yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1)}{64^n}\binom{2n}{n}^3 = \frac{2}{\pi}, \tag{31}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1)}{64^n(n+1)(1/2-n)}\binom{2n}{n}^3 = \frac{8}{\pi}, \tag{32}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1)}{64^n(n+1)(n+2)(1/2-n)(3/2-n)}\binom{2n}{n}^3 = \frac{32}{9\pi}, \tag{33}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1)}{64^n(n+1)_p(1/2-n)_p}\binom{2n}{n}^3 = \frac{16^p}{2^m(m!!)^2\pi}, \tag{34}$$

where m is a positive odd integer and $p = (m+1)/2$. (31) reveals Bailey's classical formula (See [1], or [3, Formula 5.5.24, p. 184]), which was originally obtained via hypergeometric functions.

Example 2. Let $\theta = \pi/2$ in Theorem 14. Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{16^n}\binom{2n}{n}^2 = \frac{\sqrt{2}\Gamma^2(1/4)}{4\pi^{3/2}}, \tag{35}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1)}{16^n(4n+3)(1-4n)}\binom{2n}{n}^2 = \frac{\sqrt{2}\Gamma^2(3/4)}{\pi^{3/2}}. \tag{36}$$

We also obtain some interesting series following in another direction:

Example 3. Since $P_n(1) = 1$ for all $n \geq 0$, letting $\theta = 0$ in (30), (26)–(28), respectively, then splitting the $n = 0$ term, we find

$$\sum_{n=1}^{\infty} \frac{4n+1}{4^n(4n+3)(4n-1)}\binom{2n}{n} = \frac{1}{3}, \tag{37}$$

$$\sum_{n=1}^{\infty} \frac{4n+1}{16^n(n+1)(1/2-n)}\binom{2n}{n}^2 = -2, \tag{38}$$

$$\sum_{n=1}^{\infty} \frac{4n+1}{16^n(n+1)(n+2)(1/2-n)(3/2-n)}\binom{2n}{n}^2 = -\frac{2}{3}, \tag{39}$$

$$\sum_{n=1}^{\infty} \frac{4n+1}{16^n(n+1)_p(1/2-n)_p}\binom{2n}{n}^2 = -\frac{1}{p!(1/2)_p}, \tag{40}$$

where $p \in \mathbb{N}$.

Next, let $f(x) = \sum_{n=0}^{\infty} a_n\tilde{P}_n(x)$ and $g(x) = \sum_{n=0}^{\infty} b_n\tilde{P}_n(x)$. If $f(x)g(x) \in L^2[0,1]$, then the inner product of f and g leads to

$$\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2n+1}}a_n\right)\cdot\left(\frac{1}{\sqrt{2n+1}}b_n\right) = \int_0^1 f(x)g(x)\,dx. \tag{41}$$

The inner products of (14) with (18), (14) with (19), (18) with (19) and (14) with (20) lead to the following

Theorem 15. *Let m be a positive odd integer and $p = (m+1)/2$. Then*

$$\sum_{n=0}^{\infty} \frac{4n+1}{256^n (n+1)(1/2-n)} \binom{2n}{n}^4 = \frac{16}{\pi^2}, \tag{42}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{256^n (n+1)(n+2)(1/2-n)(3/2-n)} \binom{2n}{n}^4 = \frac{128}{27\pi^2}, \tag{43}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{256^n (n+1)^2(n+2)(1/2-n)^2(3/2-n)} \binom{2n}{n}^4 = \frac{2048}{135\pi^2}, \tag{44}$$

$$\sum_{n=0}^{\infty} \frac{(4n+1)}{256^n (n+1)_p(1/2-n)_p} \binom{2n}{n}^4 = \frac{16^p[(p-1)!]^2}{(m!!)^2 m! \pi^2}. \tag{45}$$

Similarly, the inner products of (14) with (23), (14) with (24), (18) with (23), (18) with (24), (19) with (23), and (19) with (24) lead to

Theorem 16.

$$\sum_{n=0}^{\infty} \frac{1}{64^n} \binom{2n}{n}^3 = \frac{\Gamma^4(1/4)}{4\pi^3}, \tag{46}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{64^n (4n+3)(1-4n)} \binom{2n}{n}^3 = \frac{4\Gamma^4(3/4)}{\pi^3}, \tag{47}$$

$$\sum_{n=0}^{\infty} \frac{1}{64^n (n+1)(1/2-n)} \binom{2n}{n}^3 = \frac{\Gamma^4(1/4)}{3\pi^3}, \tag{48}$$

$$\sum_{n=0}^{\infty} \frac{(4n+1)}{64^n (n+1)(1/2-n)(4n+3)(1-4n)} \binom{2n}{n}^3 = \frac{48\Gamma^4(3/4)}{5\pi^3}, \tag{49}$$

$$\sum_{n=0}^{\infty} \frac{1}{64^n (n+1)(n+2)(1/2-n)(3/2-n)} \binom{2n}{n}^3 = \frac{20\Gamma^4(1/4)}{189\pi^3}, \tag{50}$$

$$\sum_{n=0}^{\infty} \frac{(4n+1)}{64^n (n+1)(n+2)(1/2-n)(3/2-n)(4n+3)(1-4n)} \binom{2n}{n}^3 = \frac{448\Gamma^4(3/4)}{135\pi^3}. \tag{51}$$

Furthermore, the inner product of (23) and (24) leads to a simple Ramanujan-like series:

$$\sum_{n=0}^{\infty} \frac{1}{16^n (4n+3)(1-4n)} \binom{2n}{n}^2 = \frac{1}{\pi}. \tag{52}$$

In general, taking the inner product (20) with (23) and (24), respectively, yields

Theorem 17. *Let m be a positive odd integer and $p=(m+1)/2$. Then*

$$\sum_{n=0}^{\infty} \frac{1}{64^n (n+1)_p (1/2-n)_p} \binom{2n}{n}^3 = \frac{16^p (1\cdot 5\cdot (4p-3))^2 (2p)!\Gamma^4(1/4)}{4(m!!)^2 (4p)!\pi^3}, \tag{53}$$

$$\sum_{n=0}^{\infty} \frac{(4n+1)}{64^n (n+1)_p (1/2-n)_p (4n+3)(1-4n)} \binom{2n}{n}^3 = \frac{16^{p+1} (3\cdot 7\cdot (4p-1))^2 (2p+1)!\Gamma^4(3/4)}{2(m!!)^2 (4p+2)!\pi^3}. \tag{54}$$

Finally, recall Parseval's identity: If $f(x) \in L^2[0,1]$, then

$$\sum_{n=0}^{\infty} \left[\frac{1}{\sqrt{2n+1}} a_n \right]^2 = \int_0^1 f^2(x)\, dx. \tag{55}$$

Applying (55) to (23), (24) and (18)–(20), respectively, yields

Theorem 18. *Let m be a positive odd integer and $p=(m+1)/2$. Then*

$$\sum_{n=0}^{\infty} \frac{1}{16^n (4n+1)} \binom{2n}{n}^2 = \frac{\Gamma^4(1/4)}{16\pi^2}, \tag{56}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{16^n (4n+3)^2 (1-4n)^2} \binom{2n}{n}^2 = \frac{\Gamma^4(3/4)}{2\pi^2}, \tag{57}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{256^n (n+1)^2 (1/2-n)^2} \binom{2n}{n}^4 = \frac{128}{3\pi^2}, \tag{58}$$

$$\sum_{n=0}^{\infty} \frac{4n+1}{256^n (n+1)^2 (n+2)^2 (1/2-n)^2 (3/2-n)^2} \binom{2n}{n}^4 = \frac{16384}{2835\pi^2}, \tag{59}$$

$$\sum_{n=0}^{\infty} \frac{(4n+1)}{256^n (n+1)_p^2 (1/2-n)_p^2} \binom{2n}{n}^4 = \frac{256^p (m!)^2}{(m!!)^4 (2m+1)!\pi^2}. \tag{60}$$

4 Remarks

We now conclude this paper with two remarks.

1. Recall the complete elliptical integral of the first kind defined by

$$K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}}.$$

Invoking the generating function of the central binomial coefficients

$$\sum_{n=0}^{\infty} \frac{1}{4^n}\binom{2n}{n} x^{2n} = \frac{1}{\sqrt{1-x^2}} \quad \text{for } |x|<1$$

and Wallis' formula

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2^{2n+1}}\binom{2n}{n},$$

we have

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{1}{16^n}\binom{2n}{n}^2 k^{2n}.$$

Moreover, Watson's elliptic function identity [3, p. 188] claims

$$\frac{\pi^2}{4}\sum_{n=0}^{\infty} \frac{1}{64^n}\binom{2n}{n}^3 = K^2(1/\sqrt{2}).$$

Thus, together with (46), we obtain another proof of the classical result [3, Theorem 1.7, p. 25]:

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}.$$

2. The neat forms of Fourier-Legendre series in Section 3 rely on the combinatorial identities (9), (12) and (13). Similar to (9), by the WZ-method, Tauraso [15, Lemma 4.2] proved

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+k}{k}\binom{2k}{k} 4^{2n-k} H_k = \binom{2n}{n}^2 H_{2n}.$$

   This enables us to find Ramanujan-like series and other well-known constants involving the product of the central binomial coefficients and the harmonic numbers like

$$\sum_{n=1}^{\infty} \frac{1}{16^n(2n-1)}\binom{2n}{n}^2 H_{2n} = \frac{6\ln 2 - 2}{\pi}$$

   and

$$\sum_{n=0}^{\infty} \frac{1}{16^n(2n+1)}\binom{2n}{n}\left(3H_{2n+1} + \frac{4}{2n+1}\right) = 8G,$$

   where G is the Catalan constant defined by $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$. The interested reader is encouraged to pursue results in this direction.

Addendum. After this paper was submitted for publication, the referee brought my attention to the papers [12] and [5]. The identities (33), (34), and (58)–(60) are found in [12], in which Levrie used a different approach to derive the Fourier-Legendre series (20). The identities (58)–(60) were also rediscovered by Cantarini and D'Aurizio [5].

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