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Some Observations on Alternating Power Sums

Laala Khaldi
LIMPAF Laboratory
Department of Mathematics
University of Bouira
10000 Bouira
Algeria
l.khaldi@univ-bouira.dz

Rachid Boumahdi
National Higher School of Mathematics
Sidi Abdellah, Algiers
Algeria
r_boumehdi@esi.dz

Abstract

Let $x \geq 1$ be a real number and $T_n(m) = -1^m + 2^m - \cdots + (-1)^n n^m$, where n and m are nonnegative integers with $n \geq 1$. In this note we obtain an explicit formula for $T_{\lfloor x \rfloor}(m)$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x, and we establish a new expression for alternating power sums $T_n(m)$ in terms of Stirling numbers of the second kind. Moreover, we give a congruence involving alternating sums of falling factorial.

1 Introduction and statement of results

Although they are important in many branches of mathematics and science, such as combinatorics, statistics and number theory, alternating power sums of integers

$$\sum_{k=1}^{n} (-1)^k k^m$$

where n and m are nonnegative integers with $n \geq 1$, have attracted less interest, of mathematicians, compared to the sums of powers of integers with positive signs

$$\sum_{k=1}^{n} k^m, \ (m \geq 0).$$

These last sums have been studied since antiquity to the Renaissance by Faulhaber, Fermat, Pascal and Bernoulli, to the modern era, where we find an abundant wealth of formulas in terms of Bernoulli, Stirling, Euler numbers and various other number sequences. One can find the expression $\sum\limits_{k=1}^{n}(-1)^k k^m$ in a manuscript of Euler [4, p. 499] entitled "Institutiones Calculi Differentialis", published in 1755, and over the years many formulas were also found in term of many polynomials and number sequences, among them the Euler polynomials.

Recall that Euler polynomials $(E_n(x))_{n\geq 0}$ can be defined via generating series by

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1}, \quad (|t| < \pi)$$

and Euler numbers (see A122045 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [8]) can be defined by $E_n = 2^n E_n(\frac{1}{2})$. It is well-known that the alternating power sum $T_n(m)$ can be written via Euler polynomials as follows [1, p. 804] and [6, Theorem 1]:

$$T_n(m) = \frac{(-1)^n E_m(n+1) + E_m(0)}{2}, \quad \text{for } n, m \geq 1.$$

For example, the first four alternating power sums are

$$\begin{aligned}
T_n(0) &= \sum_{k=1}^{n}(-1)^k = \frac{-1+(-1)^n}{2} \\
T_n(1) &= \sum_{k=1}^{n}(-1)^k k = -\frac{1}{4} + \frac{(-1)^n}{4}(2n+1) \\
T_n(2) &= \sum_{k=1}^{n}(-1)^k k^2 = \frac{(-1)^n}{2} n(n+1) \\
T_n(3) &= \sum_{k=1}^{n}(-1)^k k^3 = \frac{1}{8} + \frac{(-1)^n}{8}(2n+1)(2n^2+2n-1).
\end{aligned}$$

For nonnegative integers n, k, the binomial coefficient is defined by

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Named after James Stirling, who defined and studied them in 1730, the Stirling numbers of the second kind $S(n,k)$ (A008277) may be defined as follows: $S(n,k)$ is the number of partitions of the set $\{1,\dots,n\}$ into k non-empty subsets. For more details on Stirling numbers and their properties, the reader is referred to [3, 5].

The following result expresses $T_{\lfloor x\rfloor}(m)$ by means of Euler polynomials:

Theorem 1. *Let $x \in [1,\infty[$. For a nonnegative integer $m \geq 1$, we have*

$$\sum_{k=1}^{\lfloor x\rfloor}(-1)^k k^m = \frac{1}{2}\left(E_m(0) + (-1)^{\lfloor x\rfloor}\sum_{j=0}^{m}(-1)^j\binom{m}{j}E_j(\{x\})x^{m-j}\right)$$

where $\lfloor x\rfloor = \max\{k \in \mathbb{Z}/\ k \leq x\}$ and $\{x\} = x - \lfloor x\rfloor$ is the fractional part of x.

For x a rational number we have the following corollary:

Corollary 2. *Let p be an odd prime and m a nonnegative integer. Then*

$$\sum_{k=1}^{\frac{p-1}{2}}(-1)^k k^{2m} = \begin{cases} \frac{-1+(-1)^{\lfloor \frac{p}{2}\rfloor}}{2}, & \text{if } m = 0; \\ \frac{(-1)^{\lfloor \frac{p}{2}\rfloor}}{2^{2m+1}}\sum\limits_{j=0}^{2m}(-1)^j\binom{2m}{j}p^{2m-j}E_j, & \text{if } m \geq 1, \end{cases}$$

and

$$\sum_{k=1}^{\frac{p-1}{2}}(-1)^k k^{2m-1} = \frac{1-4^m}{2m}B_{2m} + \frac{(-1)^{\lfloor \frac{p}{2}\rfloor}}{2^{2m}}\sum_{j=0}^{2m-1}(-1)^j\binom{2m-1}{j}p^{2m-1-j}E_j,$$

where B_n is the nth Bernoulli number.

Note that by using the extended Boole summation formula, Schumacher [7, Section 4] obtained the following expression

$$\sum_{k=1}^{\lfloor x\rfloor}(-1)^{k+1}k^m = \eta(-m) + \frac{(-1)^{x-\lfloor x\rfloor}}{2(m+1)}\sum_{k=0}^{m}(-1)^{k+1}(k+1)\binom{m+1}{k+1}E_k(\{x\})x^{m-k},$$

where $\eta(s) := \sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^s}$ is the Dirichlet eta function.

The following theorem provides an expression of $\sum\limits_{k=0}^{n-1}(-1)^k k^m$ in terms of Stirling numbers of second kind.

Theorem 3. *Let n and m be nonnegative integers with $n \geq 1$. Then*

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^k k^m &= (-1)^m\sum_{k=1}^{m+1}\frac{(-1)^{k-1}(k-1)!}{2^k}S(m+1,k) \\ &\quad + (-1)^{n+1}\sum_{k=0}^{m}\sum_{\ell=1}^{k+1}\binom{m}{k}\frac{(-1)^{k+\ell-1}(\ell-1)!}{2^\ell}n^{m-k}S(k+1,\ell),\end{aligned}$$

where $S(n,k)$ is the kth Stirling number of the second kind.

The following theorem gives us a congruence modulo p of the alternating sum of the falling factorials.

Recall that a p-integer is a rational number whose denominator is coprime with p. In the ring of p-integers, denoted by $\mathbb{Z}_p$, we write $\frac{a}{b} \equiv \frac{c}{d} \pmod{p}$ when $ad - bc \equiv 0 \pmod{p}$ in $\mathbb{Z}$.

Theorem 4. *Let $p \geq 3$ be a prime and m a nonnegative integer. Then*

$$\sum_{k=0}^{p-1} (-1)^k k^{\underline{m}} \equiv (-1)^m \frac{m!}{2^m} \pmod{p}.$$

Note that the falling factorial $x^{\underline{m}}$ (A068424) is the polynomial in x defined by $x^{\underline{0}} := 1$ and $x^{\underline{m}} := \prod_{k=0}^{m-1}(x-k)$ for integers $m \geq 1$.

2 Proofs

The proof of Theorem 1 is mainly based on the following two lemmas and some properties of Euler polynomials [1, p. 804].

Lemma 5. *For integers $n \geq 1$ and $m \geq 0$, we have*

$$T_n(m) = \frac{1}{2}(E_m(0) + (-1)^{n+m} E_m(-n)).$$

Proof. By using the relation $E_n(x+1) + E_n(x) = 2x^n$, we have

$$\begin{aligned}
\sum_{k=0}^{n-1} (-1)^k k^m &= \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k (E_m(k+1) + E_m(k)) \\
&= \frac{1}{2} \left(\sum_{k=0}^{n-1} (-1)^k E_m(k) - \sum_{k=1}^{n} (-1)^k E_m(k) \right) \\
&= \frac{1}{2}(E_m(0) - (-1)^n E_m(n)).
\end{aligned}$$

Then we deduce that

$$\sum_{k=1}^{n} (-1)^k k^m = \frac{1}{2}(E_m(0) + (-1)^n E_m(n+1)).$$

As $E_m(1-x) = (-1)^m E_m(x)$, for $x = -n$ we obtain $E_m(1+n) = (-1)^m E_m(-n)$, then it follows that

$$\sum_{k=1}^{n} (-1)^k k^m = \frac{1}{2}(E_m(0) + (-1)^{n+m} E_m(-n)).$$

$\square$

Lemma 6. *Let $m,\ j,\ k$ be non-negative integers. Then*

$$\binom{m}{k}\binom{m-k}{j}=\binom{m}{j}\binom{m-j}{k}.$$

Proof. A direct calculation gives

$$\begin{aligned}\binom{m}{k}\binom{m-k}{j} &= \frac{m!}{k!(m-k)!}\frac{(m-k)!}{j!(m-k-j)!}=\frac{m!}{k!j!(m-k-j)!}\\ &= \frac{m!}{j!(m-j)!}\frac{(m-j)!}{k!(m-j-k)!}=\binom{m}{j}\binom{m-j}{k}.\end{aligned}$$

□

Proof of Theorem 1. With the help of the relation

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}$$

and Lemma 5, we have

$$\sum_{k=1}^{n}(-1)^k k^m=\frac{1}{2}\left(E_m(0)+(-1)^{n+m}\sum_{k=0}^{m}\binom{m}{k}\frac{E_k}{2^k}\left(-n-\frac{1}{2}\right)^{m-k}\right). \tag{1}$$

Taking $n=\lfloor x\rfloor = x-\{x\}$ for $x\geq 1$ in Eq. (1), we obtain the following:

$$\sum_{k=1}^{\lfloor x\rfloor}(-1)^k k^m=\frac{1}{2}E_m(0)+\frac{(-1)^{\lfloor x\rfloor}}{2^{m+1}}\sum_{k=0}^{m}\binom{m}{k}(-1)^k E_k(2x+1-\{x\})^{m-k}. \tag{2}$$

Putting $a=2x$, $b=1-2\{x\}$ and $n=m-k$ in Newton's binomial formula

$$(a+b)^n=\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j},$$

we obtain

$$(2x+1-2\{x\})^{m-k}=\sum_{j=0}^{m-k}\binom{m-k}{j}(2x)^j(1-2\{x\})^{m-k-j}. \tag{3}$$

Then by Relations (2) and (3), we deduce

$$T_{\lfloor x\rfloor}(m)=\frac{1}{2}E_m(0)+\frac{(-1)^{\lfloor x\rfloor}}{2}\sum_{j=0}^{m}(-1)^{m-j}\frac{E_k}{2^k}x^j\sum_{k=0}^{m-j}\binom{m}{k}\binom{m-k}{j}\left(\{x\}-\frac{1}{2}\right)^{m-j-k}.$$

Finally, an appeal to Lemma 6, gives

$$\begin{aligned}T_{\lfloor x\rfloor}(m) &= \frac{1}{2}E_m(0) + \frac{(-1)^{\lfloor x\rfloor}}{2}\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}x^j\sum_{k=0}^{m-j}\binom{m-j}{k}\frac{E_k}{2^k}\left(\{x\}-\frac{1}{2}\right)^{m-j-k}\\ &= \frac{1}{2}E_m(0) + \frac{(-1)^{\lfloor x\rfloor}}{2}\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}E_{m-j}(\{x\})x^j.\end{aligned}$$

This completes the proof. □

Proof of Theorem 3. We consider the function $\Phi_n(x)=\sum_{k=0}^{n-1}(-1)^k e^{kx}$. Since $\Phi_n(x)$ is the sum of the geometric sequence of common ratio $-e^x$, then we have

$$\Phi_n(x)=\sum_{k=0}^{n-1}(-1)^k e^{kx} = (-1)^{n+1}\frac{e^{nx}}{e^x+1}+\frac{1}{e^x+1}. \tag{4}$$

Differentiating the members of Eq. (4) m times, we obtain

$$\Phi_n^{(m)}(x)=\sum_{k=0}^{n-1}(-1)^k k^m e^{kx} = (-1)^{n+1}\varphi(x)+\psi(x), \tag{5}$$

where $\varphi(x)=\left(\frac{e^{nx}}{e^x+1}\right)^{(m)}$ and $\psi(x)=\left(\frac{1}{e^x+1}\right)^{(m)}$.

Note that $g^{(m)} := \frac{d^m g}{dx^m}$ is the mth derivative of g with respect to x. By applying Leibniz's formula of derivation, we can write $\varphi(x)$ as follows:

$$\varphi(x)=\left(\frac{e^{nx}}{e^x+1}\right)^{(m)} = \sum_{k=0}^{m}\binom{m}{k}n^{m-k}e^{nx}\left(\frac{1}{e^x+1}\right)^{(k)}.$$

Taking $\lambda=-1$ and $\alpha=1$ in the Formula (3.1) of [9, Theorem 3.1]

$$\left(\frac{1}{1-\lambda e^{\alpha x}}\right)^{(m)} = (-\alpha)^m\sum_{k=1}^{m+1}\frac{(-1)^{k-1}(k-1)!}{(1-\lambda e^{\alpha x})^k}S(m+1,k), \tag{6}$$

allows us to write $\varphi(x)$ and $\psi(x)$ as follows:

$$\varphi(x)=n^m\sum_{k=0}^{m}\sum_{\ell=1}^{k+1}\binom{m}{k}\frac{1}{n^k}(-1)^{k+\ell-1}(\ell-1)!S(k+1,\ell)\frac{e^{nx}}{(e^x+1)^\ell}, \tag{7}$$

and

$$\psi(x)=(-1)^m\sum_{k=1}^{m+1}\frac{(-1)^{k-1}(k-1)!}{(1+e^x)^k}S(m+1,k). \tag{8}$$

Replacing x by 0 in Eq. (5), and using the expressions of $\varphi(x)$ and $\psi(x)$ given by (7) and (8) gives us the wanted result. □

Corollary 7. *Let n and m be nonnegative integers with $n \geq 1$. Then*

$$\sum_{k=1}^{n}(-1)^k k^m = \sum_{k=1}^{m+1} \frac{(-1)^k (k-1)!}{2^k} S(m+1,k) + (-1)^n \sum_{k=0}^{m} \sum_{\ell=1}^{k+1} \binom{m}{k} \frac{(-1)^{\ell-1}(\ell-1)!}{2^\ell} n^{m-k} S(k+1,\ell).$$

Proof. Here, using the function $\Psi_n(x) = \sum\limits_{k=1}^{n}(-1)^k e^{kx}$, differentiating m times, and then according to Eq. (6) for $\lambda = -1$ and $\alpha = -1$, the result follows. □

Proof of Theorem 4. In a similar way to the proof of Theorem 3, one considers the geometric sequence of common ratio $-x$, then the sum of the n first terms is

$$\sum_{k=0}^{n-1}(-1)^k x^k = \frac{1}{1+x} + (-1)^{n+1}\frac{x^n}{1+x}. \tag{9}$$

Differentiating the members of Eq. (9) m times provides

$$\sum_{k=0}^{n-1}(-1)^k k^{\underline{m}} x^{k-m} = (-1)^m \frac{m!}{(1+x)^{m+1}} + (-1)^{n+1}\phi(x), \tag{10}$$

where $\phi(x) = \left(\frac{x^n}{x+1}\right)^{(m)}$. Applying the Leibniz derivation formula, $\phi(x)$ may be written as follows:

$$\begin{aligned}\phi(x) &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} n(n-1)\cdots(n-k+1)\frac{(m-k)!}{(1+x)^{m-k+1}}x^{n-k} \\ &= m! \sum_{k=0}^{m}\binom{n}{k}(-1)^{m-k}\frac{x^{m-k}}{(1+x)^{m-k+1}}.\end{aligned} \tag{11}$$

Replacing (11) in (10), we get

$$\sum_{k=0}^{n-1}(-1)^k k^{\underline{m}} x^{k-m} = (-1)^m \frac{m!}{(1+x)^{m+1}} + (-1)^{n+1} m! \sum_{k=0}^{m}\binom{n}{k}(-1)^{m-k}\frac{x^{m-k}}{(1+x)^{m-k+1}}. \tag{12}$$

Now, replacing x by 1 in Eq. (12), we get

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^k k^{\underline{m}} &= (-1)^m \frac{m!}{2^{m+1}} + (-1)^{n+1} m! \sum_{k=0}^{m}\binom{n}{k}(-1)^{m-k}\frac{1}{2^{m-k+1}} \\ &= (-1)^m \frac{m!}{2^{m+1}}\left(1 + (-1)^{n+1}\sum_{k=0}^{m}\binom{n}{k}(-2)^k\right).\end{aligned} \tag{13}$$

Finally, taking $n = p$ in Eq. (13) and by the trivial congruence $\binom{p}{k} \equiv 0 \pmod{p}$ for $k = 1, \ldots, p-1$, we have

$$\sum_{k=0}^{p-1} (-1)^k k^{\underline{m}} \equiv (-1)^m \frac{m!}{2^m} \pmod{p}.$$

This evidently completes the proof. □

Note that Eq. (13) has been proven by Bazsó and Mező [2] by using Cauchy's residue theorem.

Remark 8. Taking $m = p-1$ in Theorem 4 and using Fermat little theorem for $a = 2$ with Wilson theorem, we obtain the following congruence:

$$\sum_{k=0}^{p-1} (-1)^k k^{\underline{p-1}} \equiv -1 \pmod{p}.$$

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, 1964.

[2] A. Bazsó and I. Mező, Some notes on alternating power sums of arithmetic progressions, *J. Integer Sequences* **21** (2018), Article 18.7.8.

[3] C. A. Charalambides, *Enumerative Combinatorics*, CRC Press, 2002.

[4] L. Euler, *Institutiones Calculi Differentialis*, Academie Imperialis Scientiareum Petropolitanae, 1755.

[5] H. W. Gould, *Combinatorial Identities for Stirling Numbers: The Unpublished Notes of H W Gould*, World Scientific, 2015.

[6] T. Kim, A note on the alternating sums of powers of consecutive integers, preprint, 2005. Available at `http://arxiv.org/abs/math/0508233`.

[7] R. Schumacher, Extension of summation formulas involving Stirling series, preprint, 2016. Available at `http://arxiv.org/abs/1605.09204`.

[8] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, 2022. Available at https://oeis.org.

[9] A.-M. Xu and Z.-D. Cen, Some identities involving exponential functions and Stirling numbers and applications, *J. Comput. Appl. Math.* **260** (2014), 201–207.

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Return to Journal of Integer Sequences home page.