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Infinite Series Associated with the Ratio and Product of Central Binomial Coefficients

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Abstract

We establish several interesting series associated with the ratio and the product of central binomial coefficients, namely $\binom{2n}{n}$ and $\binom{4n}{2n}$. Series involving the product of central binomial coefficients can be found in the papers of Campbell, D'Aurizio, and Sondow. In this paper, through the application of integration methods, we address broad generalizations of both classes. The techniques involved in constructing the integrals for the corresponding series are based on the use of the ordinary generating functions of central binomial coefficients and Wallis' well-known integral formulas.

1 Introduction

The central binomial coefficients

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

for $n \geq 0$ are the positive integers that appear exactly in the middle of the even-indexed rows of the Pascal triangle. These numbers play an important role in various fields such as analysis, number theory, and combinatorics. A number of facts about central binomial coefficients were compiled by Gould [8]. Lehmer [10] gave diverse identities and interesting results via specialization, integration, and differentiation tricks. Surprisingly, the coefficients appear in the infinite binomial series expansion of the function $(1-4x)^{-1/2}$ for $|x|<1/4$, yielding the ordinary generating function as follows:

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}} = 1+2x+6x^2+20x^3+70x^4+\cdots. \tag{1}$$

On the right-hand side of (1), we can observe the coefficients $1, 6, 70, 924$, etc., are the coefficients appearing at the even positions in the series. Considering the coefficients, we notice that these numbers can be represented in binomial form, namely $\binom{4n}{2n}$, which mean that they can be obtained by replacing n with $2n$. It is easy to note that (1) allows us to build the generating function of the coefficients $\binom{4n}{2n}$, which is

$$\sum_{n=0}^{\infty}\binom{4n}{2n}x^{2n} = \frac{1}{2}\left(\frac{1}{\sqrt{1-4x}}+\frac{1}{\sqrt{1+4x}}\right). \tag{2}$$

However, further routine simplification of (2) produces

$$\sum_{n=0}^{\infty}\binom{4n}{2n}x^{n} = \frac{1}{\sqrt{2}}\sqrt{\frac{1+\sqrt{1-16x}}{1-16x}}, \quad |x|<1/16, \tag{3}$$

which can be found in [1], including several other generating functions and related identities. The study of these numbers has been of great interest for a long time. Many interesting series associated with the central binomial coefficients, harmonic numbers, and Catalan numbers can be found in [2, 4, 5, 6], and among those several interesting series, two beautiful identities involving the square of central binomial coefficients and the harmonic numbers are

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n(2n-1)^2}H_n = \frac{12-16\ln 2}{\pi}$$

and

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n\,(2n-1)^2}H_{2n} = \frac{4G-12\ln 2+12}{\pi}.$$

Here H_n is the nth harmonic number, and G is Catalan's constant. The identities mentioned above exhibit mathematical beauty where one can observe the important mathematical constants G, π, and $\ln 2$ in their closed forms. We encourage interested readers to refer to the papers given in [3, 7], and we suggest looking at the references given therein for more identities.

Now focusing on the present aim of this paper, we study the series associated with the product and the ratio of central binomial coefficients, $\binom{2n}{n}\binom{4n}{2n}$ and $\binom{2n}{n}\binom{4n}{n}^{-1}$, respectively. Some motivating examples of the product of these coefficients can be found in the works of Campbell et al. [5], where those identities are obtained via generating functions and the well-known Wallis integral formulas. For instance,

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}}{64^n(2n+1)} = \frac{4}{\pi}\log\left(1+\sqrt{2}\right), \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}}{64^n(3+2n)} = \frac{4\sqrt{2}}{15\pi} + \frac{16}{15\pi}\log\left(1+\sqrt{2}\right). \tag{5}$$

The techniques discussed in this paper are simple but powerful for evaluating non-trivial series related to the product and the ratio of central binomial coefficients. Our general results are expressed in terms of finite binomial sums, which include the incomplete beta function, while other general results are expressed explicitly in terms of the beta function and the incomplete beta function. When the obtained general results presented in this paper are executed by the use of Mathematica or by direct evaluation of the corresponding integrals, we find a large number of intriguing identities involving the ratio and the product of central binomial coefficients. A few of them are Ramanujan-like series for $1/\pi$. Some interesting series related to the ratio of central binomial coefficients are as follows:

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(2n-1)^2(4n+1)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{5\sqrt{2}-4}{9}, \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{n(2n+1)\,4^n}{(2n-1)^2(4n+1)(4n+3)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{121\sqrt{2}-104}{450}. \tag{7}$$

We state Wallis' well-known integral formulas, namely

$$\int_0^{\frac{\pi}{2}} \sin^{2n} y\,dy = \frac{\pi}{2}\binom{2n}{n}\frac{1}{4^n}, \tag{8}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} y\,dy = \frac{4^n}{(2n+1)\binom{2n}{n}}, \tag{9}$$

that will be helpful in the course of the analysis of the results of in this paper. We organize the remaining work into different sections. In Section 2, we introduce several series involving the ratio of the central binomial coefficients. Series involving the product of the coefficients are highlighted in Section 3, where we list a few series involving Ramanujan-like formulas for $1/\pi$. In Section 4, we highlight a number of miscellaneous series. As an additional check, all the formulas were numerically verified by the use of Mathematica.

2 Interesting series of the ratio of central binomial coefficients

This section contains a number of series related to the ratio of central binomial coefficients. We present our main results in terms of finite binomial sums and the incomplete beta function. First, we mention some lemmas along with their proofs, and then we proceed to construct our main results.

Lemma 1. *If $x \in [-1, 1]$, then*

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2 4^n} \binom{2n}{n} x^{2n-1} = \frac{\sin^{-1} x}{2}.$$

Proof. We recall the generating function of central binomial coefficients (1), and rewrite it as follows:

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{2n-2}}{4^n} = \frac{1}{x^2\sqrt{1-x^2}} - \frac{1}{x^2}. \tag{10}$$

By integrating (10), we obtain

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{2n-1}}{(2n-1)4^n} = \frac{1-\sqrt{1-x^2}}{x}. \tag{11}$$

The primitive of the integral of the latter quantity of (10) exists; it is $x^{-1} - x^{-1}\sqrt{1-x^2} + C$, where C is the integration constant. If $x \to 0$, then $C = 0$ and hence (11) follows. Again, dividing both sides of (11) by x and integrating, we get

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{2n-1}}{(2n-1)^2 4^n} = \frac{\sqrt{1-x^2} + x\sin^{-1} x - 1}{x}. \tag{12}$$

We observe that $\int x^{-2}(1-\sqrt{1-x^2})\, dx = x^{-1}\sqrt{1-x^2} + \sin^{-1} x - x^{-1} + K$, which can be easily deduced by applying linearity of integral and integration by parts to the latter integral, $\int x^{-2}\sqrt{1-x^2}\, dx$, where K is the constant of integration. If $x \to 0$, then $K = 0$. Finally, we multiply both sides of (12) by x, and differentiating with respect to x, we obtain

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2 4^n} \binom{2n}{n} x^{2n-1} = \frac{1}{2}\frac{d}{dx}\left(\sqrt{1-x^2} + x\sin^{-1} x - 1\right) = \frac{\sin^{-1} x}{2}.$$

□

Lemma 2. *If $x \in [-1, 1]$, then*

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2 (2n+1) 4^n} \binom{2n}{n} x^{2n} = \frac{1}{8}\left(\sqrt{1-x^2} + 2x\sin^{-1} x - \frac{\sin^{-1} x}{x}\right). \tag{13}$$

Proof. Multiplying both sides of Lemma 1 by x, integrating, and some algebraic simplification leads to the proposed result (13). Details are left to the reader. □

Lemma 3. *If m is an odd positive integer, then*

$$\int_0^{\frac{\pi}{2}} \sin^{4n+m} y\, dy = \frac{2^{2n+m-1}}{(4n+m)\binom{4n+m-1}{2n+\frac{m-1}{2}}}.$$

Proof. Recall Wallis' integral formula (9). First, we replace n with $2n$, then n with $\frac{n}{4}$, and again, the obtained result on replacing n with $n-1$ gives us

$$\int_0^{\frac{\pi}{2}} \sin^n y\, dy = \frac{2^{n-1}}{n\binom{n-1}{\frac{n-1}{2}}}. \tag{14}$$

Further, replacing n with $4n+m$ in (14) yields the required result. □

Lemma 4. *For $x \in [-1,1]$, the following holds:*

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2(2n+1)4^n}\binom{2n}{n}x^{2n} = \frac{1}{16}\left(-\sqrt{1-x^2} + \frac{\sin^{-1} x}{x} + 2x\sin^{-1} x\right). \tag{15}$$

Proof. The proof relies on Lemma 2. We leave the details to the reader. □

Theorem 5. *For positive odd integers m, the following equality holds:*

$$\sum_{n=1}^{\infty} \frac{n\, 4^n}{(2n-1)^2(4n+m)} \frac{\binom{2n}{n}}{\binom{4n+m-1}{2n+\frac{m-1}{2}}} = \sqrt{2}\sum_{k=0}^{\frac{m+1}{2}} \frac{(-1)^k}{(2k+1)2^{m-k}}\binom{\frac{m+1}{2}}{k}\mathcal{B}(k), \tag{16}$$

where $\mathcal{B}(k) = B_{\frac{1}{2}}\left(k+1, \frac{1}{2}\right)$ and $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt$ is the incomplete beta function.

Proof. Rewrite Lemma 1 as follows:

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2 4^n}\binom{2n}{n}x^{4n+m} = \frac{x^{2+m}\sin^{-1}(x^2)}{2}. \tag{17}$$

Now set $x = \sin y$ in (17) and integrate from 0 to $\pi/2$. From Lemma 3 we get

$$\sum_{n=1}^{\infty} \frac{n\, 4^n}{(2n-1)^2(4n+m)} \frac{\binom{2n}{n}}{\binom{4n+m-1}{2n+\frac{m-1}{2}}} = \frac{1}{2^m}\int_0^{\frac{\pi}{2}} \sin^{2+m} y \sin^{-1}(\sin^2 y)\, dy \tag{18}$$

$$= \frac{1}{2^m}\int_0^{\frac{\pi}{2}} \left(1-\cos^2 y\right)^{\frac{m+1}{2}} \sin y \sin^{-1}\left(1-\cos^2 y\right)\, dy$$

$$= \frac{1}{2^m}\int_0^{1} \left(1-t^2\right)^{\frac{m+1}{2}} \sin^{-1}\left(1-t^2\right)\, dt \tag{19}$$

$$= \frac{1}{2^m}\sum_{k=0}^{\frac{m+1}{2}}(-1)^k\binom{\frac{m+1}{2}}{k}\int_0^1 t^{2k}\sin^{-1}\left(1-t^2\right)\, dt. \tag{20}$$

Employing Newton's binomial formula, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ in (19) leads to (20) and applying integration by parts in (20) gives

$$\begin{aligned}\int_0^1 t^{2k} \sin^{-1}\left(1-t^2\right)\, dt &= \left.\frac{\sin^{-1}(1-t^2)\; t^{2k+1}}{2k+1}\right|_0^1 + \frac{2}{2k+1}\int_0^1 \frac{t^{2k+2}}{\sqrt{1-(1-t^2)^2}}\, dt \\ &= \frac{2}{2k+1}\int_0^1 \frac{t^{2k+2}\, dt}{\sqrt{2t^2-t^4}} = \frac{2}{2k+1}\int_0^1 \frac{t^{2k+1}\, dt}{\sqrt{2-t^2}} \\ &= \frac{2^{k+1}}{\sqrt{2}(2k+1)}\int_0^{\frac{1}{2}} \frac{t^k\, dt}{\sqrt{1-t}} = \frac{\sqrt{2}}{2^{-k}(2k+1)} B_{\frac{1}{2}}\left(k+1, \frac{1}{2}\right). \qquad (21)\end{aligned}$$

Substituting the value of (21) into (20) proves (16). ◻

Theorem 6. *For positive odd integers, $m \geq 3$, we have*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n\, 4^n}{(2n-1)^2\,(2n+1)(4n+m)} \frac{\binom{2n}{n}}{\binom{4n+m-1}{2n+\frac{m-1}{2}}} &= \sum_{n=1}^{\infty} \frac{n\, 2^{2n-1}}{(2n-1)^2(4n+m)} \frac{\binom{2n}{n}}{\binom{4n+m-1}{2n+\frac{m-1}{2}}} + \frac{\varphi(m)}{2^{m+2}} \\ &\quad - \frac{1}{2^{3/2}} \sum_{k=0}^{\frac{m-3}{2}} \frac{(-1)^k}{(2k+1)2^{m-k}} \binom{\frac{m-3}{2}}{k} \mathcal{B}(k), \qquad (22)\end{aligned}$$

where $\varphi(m) = \int_0^1 t^m \sqrt{1+t^2}\, dt$.

Proof. Rewrite Lemma 2 in the following manner:

$$\sum_{n=1}^{\infty} \frac{n\, 4^n}{(2n-1)^2\,(2n+1)} \binom{2n}{n} x^{4n+m} = \frac{x^m}{8}\left(\sqrt{1-x^4} + 2x^2 \sin^{-1}(x^2) - \frac{\sin^{-1}(x^2)}{x^2}\right). \qquad (23)$$

Substituting $x = \sin y$ in (23), integrating from 0 to $\pi/2$, and invoking Lemma 3, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n\, 4^n}{(2n-1)^2\,(2n+1)(4n+m)} \frac{\binom{2n}{n}}{\binom{4n+m-1}{2n+\frac{m-1}{2}}} &= \frac{1}{2^{m+1}} \int_0^{\frac{\pi}{2}} \sin^{2+m} y \sin^{-1}\left(\sin^2 y\right)\, dy \qquad (24) \\ &+ \frac{1}{2^{m+2}} \int_0^{\frac{\pi}{2}} \sin^m y \sqrt{1-\sin^4 y}\, dy \qquad (25) \\ &- \frac{1}{2^{m+2}} \int_0^{\frac{\pi}{2}} \sin^{m-2} y \sin^{-1}\left(\sin^2 y\right)\, dy. \qquad (26)\end{aligned}$$

Since integral (24) is evaluated in (18), and the integral appears in (25), we write

$$\int_0^{\frac{\pi}{2}} \sin^m y \sqrt{1-\sin^4 y}\, dy = \int_0^{\frac{\pi}{2}} \sin^m y \cos y \sqrt{1+\sin^2 y}\, dy = \int_0^1 t^m \sqrt{1+t^2}\, dt = \varphi(m). \qquad (27)$$

Similarly, integral (26) can be expressed as

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{m-2} y \sin^{-1}\left(\sin^2 y\right) \, dy &= \int_0^1 \sin y \left(\sin^2 y\right)^{\frac{m-3}{2}} \sin^{-1}\left(\sin^2 y\right) \, dy \\ &= \int_0^1 \left(1-t^2\right)^{\frac{m-3}{2}} \sin^{-1}\left(1-t^2\right) \, dt \\ &= \sum_{k=0}^{\frac{m-3}{2}} (-1)^k \binom{\frac{m-3}{2}}{k} \int_0^1 t^{2k} \sin^{-1}\left(1-t^2\right) \, dt. \end{aligned} \tag{28}$$

Substituting the result (21) into (28) then back to (26) and combining (24), (25), (26), and (27), we conclude (22). □

2.1 Values of $\mathcal{B}(k)$ and $\varphi(m)$

We get $\mathcal{B}(k)$ and $\varphi(m)$ from Theorem 5 and Theorem 6, which have integral representations, $\int_0^{\frac{1}{2}} \frac{t^k}{\sqrt{1-t}} \, dt$ and $\int_0^1 t^m \sqrt{1+t^2} \, dt$, respectively. It is easy to deduce the primitives of the integrals for some particular values of k and m; however, we do not have compact closed forms in terms of elementary functions that can easily generate an infinite number of solutions. We now display a few values of $\mathcal{B}(k)$ and then $\varphi(m)$.

$$\mathcal{B}(0) = 2-\sqrt{2}, \quad \mathcal{B}(1) = \frac{8-5\sqrt{2}}{6}, \quad \mathcal{B}(2) = \frac{64-43\sqrt{2}}{60},$$
$$\mathcal{B}(3) = \frac{256-177\sqrt{2}}{280}, \quad \mathcal{B}(4) = \frac{4096-2867\sqrt{2}}{5040}.$$

Similarly, for positive odd integers $m \geq 3$, we have

$$\varphi(3) = \frac{2(1+\sqrt{2})}{15}, \quad \varphi(5) = \frac{2(11\sqrt{2}-4)}{105}, \quad \varphi(7) = \frac{2(8+13\sqrt{2})}{315}.$$

Example 7. Invoking Theorem 5 and using the values of $\mathcal{B}(k)$ from Subsection 2.1 yields (6), (7), and some more series as follows:

$$\sum_{n=1}^{\infty} \frac{n(n+1)(2n+1)\,4^n}{(2n-1)^2\,(4n+1)(4n+3)(4n+5)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{1559\sqrt{2}-1216}{29400}, \tag{29}$$

$$\sum_{n=1}^{\infty} \frac{n(n+1)(2n+1)(2n+3)\,4^n}{(2n-1)^2(4n+1)(4n+3)(4n+5)(4n+7)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{41161\sqrt{2}-33664}{1587600}, \tag{30}$$

$$\sum_{n=1}^{\infty} \frac{n(n+1)(n+2)(2n+1)(2n+3)\,4^n}{(2n-1)^2(4n+1)(4n+3)(4n+5)(4n+7)(4n+9)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{857977\sqrt{2}-659968}{153679680}. \tag{31}$$

Example 8. Similarly, applying Theorem 6 and values of $\varphi(m)$ yields the following series classes, in particular

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(2n-1)^2(4n+1)(4n+3)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{4\left(1+\sqrt{2}\right)}{225}, \tag{32}$$

$$\sum_{n=1}^{\infty} \frac{n(n+1)\,4^n}{(2n-1)^2(4n+1)(4n+3)(4n+5)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{137\sqrt{2}-88}{11025}, \tag{33}$$

$$\sum_{n=1}^{\infty} \frac{n(n+1)(2n+3)\,4^n}{(2n-1)^2(4n+1)(4n+3)(4n+5)(4n+7)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{1033\sqrt{2}-592}{198450}. \tag{34}$$

In addition, in view of Theorem 5 and Theorem 6, we have

Theorem 9. *For positive odd integers, $m \geq 3$, we have*

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(4n^2-1)(4n+m)} \frac{\binom{2n}{n}}{\binom{4n+m-1}{2n+\frac{m-1}{2}}} = \frac{1}{\sqrt{2}} \sum_{k=0}^{\frac{m-3}{2}} \frac{(-1)^k}{(2k+1)2^{m-k}} \binom{\frac{m-3}{2}}{k} \mathcal{B}(k) - \frac{\varphi(m)}{2^{m+1}}.$$

Proof. The proof directly follows from (22). □

Example 10. The immediate consequences of Theorem 9 are

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(2n+1)(2n-1)(4n+3)} \frac{\binom{2n}{n}}{\binom{4n+2}{2n+1}} = \frac{7\sqrt{2}-8}{60},$$

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(2n+1)(2n-1)(4n+5)} \frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}} = \frac{71\sqrt{2}-64}{5040}.$$

2.2 Propositions and their proofs

We mention some integral identities that play a crucial role when evaluating series based on the ratio of central binomial coefficients. They are as follows:

Proposition 11. *The following equality holds:*

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{-1}\left(\sin^2 x\right)}{\sin^2 x}\, dx = 2\log\left(1+\sqrt{2}\right). \tag{35}$$

Proof. Applying integration by parts, the above intergral boils down to

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin^{-1}(\sin^2 x)}{\sin^2 x}\, dx &= 2\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sqrt{1-\sin^4 x}}\, dx = 2\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sqrt{(1-\sin^2 x)(1+\sin^2 x)}}\, dx \\ &= 2\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\, dx = 2\int_0^1 \frac{dt}{\sqrt{1+t^2}} = 2\sinh^{-1}(1).\end{aligned}$$

The latter integral in the second line is obtained by substituting $\sin x = t$. Furthermore, we can easily obtain (35) by noting that $\frac{d}{dt}\sinh^{-1} t = \frac{1}{\sqrt{1+t^2}} = \log(t+\sqrt{1+t^2})$. □

Proposition 12. *If G denotes Catalan's constant defined by $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$ and $\psi_1(z) = \sum_{n=0}^{\infty} 1/(n+z)^2$ is the trigamma function for $z > 0$, then the following integral equality holds:*

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{-1}\left(\sin^2 x\right)}{\sin x}\, dx = -2G - \frac{\pi^2}{2\sqrt{2}} + \frac{1}{8\sqrt{2}}\left(\psi_1\left(\frac{1}{8}\right) + \psi_1\left(\frac{3}{8}\right)\right). \tag{36}$$

Proof. Substituting $\sin^2 x = y$ and employing integration by parts leads to

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin^{-1}\left(\sin^2 x\right)}{\sin x}\, dx &= \frac{1}{2}\int_0^1 \frac{\sin^{-1}(y)}{y\sqrt{1-y}}\, dy = \frac{1}{2}\int_0^1 \frac{\log\left(\frac{1+\sqrt{1-y}}{1-\sqrt{1-y}}\right)}{\sqrt{1-y^2}}\, dy = -\int_0^1 \frac{\log\left(\frac{1-y}{1+y}\right)}{\sqrt{2-y^2}}\, dy \\ &= -\int_0^{\frac{\pi}{4}} \log\left(\frac{1-\sqrt{2}\sin u}{1+\sqrt{2}\sin u}\right) du = -\int_0^{\frac{\pi}{4}} \log\left(\frac{\frac{1}{\sqrt{2}} - \sin u}{\frac{1}{\sqrt{2}} + \sin u}\right) du = I.\end{aligned}$$

The last integral is obtained by substituting $y = \sqrt{2}\sin u$. Furthermore, the last obtained integral result can be expressed as

$$I = -\int_0^{\frac{\pi}{4}} \log\left(\tan\left(\frac{u}{2}\right)\frac{1+\tan\left(\frac{u}{2}\right)}{1-\tan\left(\frac{u}{2}\right)}\right) du = -2\int_0^{\frac{\pi}{8}} \log\left(\tan u\frac{1+\tan u}{1-\tan u}\right) du.$$

The key notion for the former integral is the identity, $\frac{1-\sqrt{2}\sin u}{1+\sqrt{2}\sin u} = \tan\left(\frac{\pi-4u}{8}\right)\cot\left(\frac{\pi+4u}{8}\right)$, which is due to Kamel Benaicha (personal communication), and the latter is obtained by replacing $u/2$ with u. Again, putting $\tan u = z$ and substituting $\frac{1-z}{1+z} = t$, we yield

$$I = -4\int_0^{\sqrt{2}-1} \frac{\log(t)}{1+t^2}\, dt + 2\int_0^1 \frac{\log(t)}{1+t^2}\, dt = -4\int_0^{\frac{\pi}{8}} \log(\tan t)\, dt + 2\int_0^{\frac{\pi}{4}} \log(\tan t)\, dt.$$

It is well-known that the integral $\int_0^{\frac{\pi}{4}} \log(\tan t)\, dt = -G$ (see [9, Entry 4.227.2]). To evaluate the former integral, we use the Fourier series [9, Entry 1.442.2] of

$$\log(\tan t) = -2\sum_{n=0}^{\infty} \frac{\cos((2n+1)2t)}{2n+1}, \ 0 < t < \pi/2.$$

This leads to our original integral being

$$\begin{aligned} I &= -2G - 4\int_0^{\frac{\pi}{8}} \log(\tan t)\, dt = -2G + 8\sum_{n=0}^{\infty} \frac{1}{2n+1}\int_0^{\frac{\pi}{8}} \cos((2n+1)2t)\, dt \\ &= -2G + 8\sum_{n=0}^{\infty} \frac{\sin\left(\frac{\pi}{4}(2n+1)\right)}{(2n+1)^2} = -2G + \frac{4}{\sqrt{2}}\sum_{n=0}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right)}{(2n+1)^2} \\ &= -2G + \frac{4}{\sqrt{2}}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(4n+3)^2} + \frac{(-1)^n}{(4n+1)^2}\right) = -2G + \frac{4}{64\sqrt{2}}\left(\psi_1\left(\frac{3}{8}\right) - \psi_1\left(\frac{7}{8}\right)\right) \\ &+ \frac{4}{64\sqrt{2}}\left(\psi_1\left(\frac{1}{8}\right) - \psi_1\left(\frac{5}{8}\right)\right) = -2G + \frac{1}{8\sqrt{2}}\left(\psi_1\left(\frac{3}{8}\right) + \psi_1\left(\frac{5}{8}\right)\right) - \frac{\pi^2}{2\sqrt{2}}. \end{aligned}$$

In the last line, we make use of the reflection formula of the trigamma function, namely $\psi_1(1-z)+\psi_1(z)=\frac{\pi^2}{\sin^2 \pi z}$. Putting $z=1/8$ and $z=3/8$ in the reflection formula and simplifying proves (36). □

Proposition 13. *The following relation holds:*

$$\int_0^{\frac{\pi}{2}} \sin^{-1}\left(\sin^2 x\right)\,dx = \frac{7\pi^2}{24} - \frac{\log^2(2)}{4} - 2\operatorname{Li}_2\left(\frac{1}{\sqrt{2}}\right), \tag{37}$$

$$\int_0^{\frac{\pi}{2}} \frac{x\sin x}{\sqrt{1+\sin^2 x}}\,dx = \operatorname{Li}_2\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi^2}{48} + \frac{\log^2(2)}{8}. \tag{38}$$

Here, $\operatorname{Li}_2(z)=\sum_{n=1}^{\infty} z^n/n^2$ *is a dilogarithm function with* $|z|\le 1$.

Proof. Integration by parts results in

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{-1}(\sin^2 x)\,dx &= \frac{\pi^2}{4} - 2\int_0^{\frac{\pi}{2}} \frac{x\sin x}{\sqrt{1+\sin^2 x}}\,dx = \frac{\pi^2}{4} - 2\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{\sqrt{1-2x^2}}\,dx \\ &= \frac{\pi^2}{4} - \int_0^{\frac{1}{\sqrt{2}}}\int_0^1 \frac{2x}{\sqrt{(1/2-x^2)(1-y^2x^2)}}\,dy\,dx \\ &= \frac{\pi^2}{4} - \int_0^1\int_0^{\frac{1}{\sqrt{2}}} \frac{2x}{\sqrt{(1/2-x^2)(1-y^2x^2)}}\,dx\,dy \\ &= \frac{\pi^2}{4} - \int_0^1 \frac{\log\left(y+\sqrt{2}\right)-\log\left(\sqrt{2}-y\right)}{y}\,dy.\end{aligned}$$

Now, by using the definition of the dilogarithm integral [11, p. 1] and the dilogarithm identity, $\operatorname{Li}_2(z)+\operatorname{Li}_2(-z)=\frac{\operatorname{Li}_2(z^2)}{2}$ (see [11, p. 6, Eqn. (1.15)]), we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{-1}(\sin^2 x)\,dx &= \frac{\pi^2}{4} - \left(\operatorname{Li}_2\left(\frac{1}{\sqrt{2}}\right) - \operatorname{Li}_2\left(-\frac{1}{\sqrt{2}}\right)\right) \\ &= \frac{\pi^2}{4} - 2\operatorname{Li}_2\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\operatorname{Li}_2\left(\frac{1}{2}\right).\end{aligned}$$

Using the well-known classical result [11, p. 6, Eqn. (1.16)]

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}$$

in the last equality yields the announced results (37) and (38), respectively. □

Clearly, Theorems 5, 6, and 9 hold good only for positive odd integers. With the exception of Examples 7, Example 8, and Example 10, the obtained corresponding integrals can produce exotic and slightly different identities.

So, using propositions from Subsection 2.2 and integrals (18), (24), (25), and (26), we get the following result.

Theorem 14. *The following relations hold:*

$$\sum_{n=1}^{\infty} \frac{4^n}{(2n-1)^2} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = 4\left(\sqrt{2}-1\right), \tag{39}$$

$$\sum_{n=1}^{\infty} \frac{(4n-1)\,4^n}{(2n-1)^3} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{1}{2\sqrt{2}}\left(\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{3}{8}\right)\right) - 8G - \sqrt{2}\pi^2, \tag{40}$$

$$\sum_{n=1}^{\infty} \frac{4^n}{(2n-1)^3} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{1}{2\sqrt{2}}\left(\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{3}{8}\right)\right) - 8G - \sqrt{2}\pi^2 - \frac{8}{1+\sqrt{2}}. \tag{41}$$

Proof. Putting $m=-1$ in (18), and calculating the corresponding integral proves (39). In a similar fashion, putting $m=-3$ in (18) and using (36) proves (40). Thus, (41) follows from (39) and (40). □

For some particular cases, even integers like $m=0$, $m=2$, and $m=4$ in (18) give

$$\sum_{n=1}^{\infty} \frac{4^n}{(2n-1)^2} \frac{\binom{2n}{n}}{\binom{4n-1}{2n-\frac{1}{2}}} = \frac{7\pi^2}{12} - \frac{\log^2(2)}{2} - 4\,\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) + 2\sqrt{2} - 2\sinh^{-1}(1),$$

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(2n-1)^2(2n+1)} \frac{\binom{2n}{n}}{\binom{4n+1}{2n+\frac{1}{2}}} = \frac{7\pi^2}{128} - \frac{3\log^2(2)}{64} - \frac{3}{8}\,\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) + \frac{5}{16\sqrt{2}} - \frac{3\sinh^{-1}(1)}{32},$$

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(2n-1)^2(n+1)} \frac{\binom{2n}{n}}{\binom{4n+3}{2n+\frac{3}{2}}} = \frac{35\pi^2}{1536} - \frac{5\log^2(2)}{256} - \frac{5}{32}\,\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) + \frac{101}{576\sqrt{2}} - \frac{25\sinh^{-1}(1)}{384}.$$

Similarly, by inserting the same even integers in (24), (25), (26), and utilizing propositions from Subsection 2.2, we end up having the conclusions:

$$\sum_{n=1}^{\infty} \frac{4^n}{(2n-1)(2n+1)} \frac{\binom{2n}{n}}{\binom{4n-1}{2n-\frac{1}{2}}} = 3\log\left(1+\sqrt{2}\right) - \sqrt{2},$$

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(2n-1)^2(2n+1)^2} \frac{\binom{2n}{n}}{\binom{4n+1}{2n+\frac{1}{2}}} = \frac{1}{4\sqrt{2}} - \frac{7\pi^2}{768} + \frac{\log^2(2)}{128} + \frac{1}{16}\,\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) - \frac{\sinh^{-1}(1)}{16},$$

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(2n-1)^2(2n+1)(4n+4)} \frac{\binom{2n}{n}}{\binom{4n+3}{2n+\frac{3}{2}}} = \frac{25}{2304\sqrt{2}} + \frac{7\pi^2}{12288} - \frac{\log^2(2)}{2048} - \frac{1}{256}\,\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) + \frac{\sinh^{-1}(1)}{1536}.$$

Also, we have

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{(2n-1)^2(2n+1)(4n+1)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{G}{4} + \frac{\pi^2}{16\sqrt{2}} + \frac{13}{18\sqrt{2}} - \frac{19}{72} - \frac{1}{64\sqrt{2}}\left(\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{3}{8}\right)\right),$$

which can easily be achieved by letting $m = 1$ in (24), (25), (26), and applying (36).

3 Some interesting series of the product of central binomial coefficients and some Ramanujan-like series

In this section, we establish various series involving the product of central binomial coefficients and associated Ramanujan-like series for $1/\pi$ via simple techniques. Most of the interesting series are deduced via general formulas provided here. Some other sorts of series involving the product of the coefficients are obtained by computing their corresponding elementary integrals. Using Lemma 1, Lemma 2, and some other intermediate results we have acquired, we can produce

Theorem 15. *If $p \in \mathbb{Z}$, then we have*

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)\,64^n} \binom{2n}{n} \binom{4n+4p}{2n+2p} = \binom{4p}{2p} + \frac{i \cdot 16^p}{\pi} B_{-1}\left(2p + \frac{1}{2}, \frac{3}{2}\right),$$

where $i = \sqrt{-1}$ and $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$ is the incomplete beta function.

Proof. We start with (11) where we replace x by $\sin^2 y$, multiply both sides by $\sin^{4p} y$, and integrating from 0 to $\pi/2$ gives

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)\,64^n} \binom{2n}{n} \binom{4n+4p}{2n+2p} = \frac{2 \cdot 16^p}{\pi} \int_0^{\frac{\pi}{2}} \sin^{4p} y \left(1 - \cos y \sqrt{1 + \sin^2 y}\right) dy. \tag{42}$$

Apply linearity and (8), the latter integral boils down to

$$\begin{aligned} \binom{4p}{2p} - \frac{2 \cdot 16^p}{\pi} \int_0^{\frac{\pi}{2}} &= \binom{4p}{2p} - \frac{2 \cdot 16^p}{\pi} \int_0^{\frac{\pi}{2}} \sin y \cos y (\sin y)^{4p-1} \sqrt{1 + \sin^2 y}\, dy \\ &= \binom{4p}{2p} - \frac{16^p}{\pi} \int_0^1 t^{2p-1/2} \sqrt{1+t}\, dt. \end{aligned} \tag{43}$$

Replacing t with $-t$ in (43) and, by definition of the incomplete beta function and $\sqrt{-1} = i$, gives the value of integral equal to $-iB_{-1}(2p + 1/2, 3/2)$, and hence putting the final result of (43) back to (42) completes the proof. □

Example 16. As stated, Theorem 15 enables us to construct an infinite number of solutions.

Some immediate consequences are as follows:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,64^n}\binom{2n}{n}\binom{4n}{2n}=1-\frac{\sqrt{2}}{\pi}-\frac{\sinh^{-1}(1)}{\pi},$$

$$\sum_{n=1}^{\infty}\frac{n}{(4n-3)(4n-1)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{\sqrt{2}}{6\pi},$$

$$\sum_{n=1}^{\infty}\frac{(4n+1)(4n+3)}{(2n-1)(n+1)(2n+1)\,64^n}\binom{2n}{n}\binom{4n}{2n}=3-\frac{7\sqrt{2}}{3\pi}-\frac{\sinh^{-1}(1)}{\pi}.$$

Taking rational numbers p into account in (42) allows us to construct a variety of other types of identities. For example,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,64^n}\binom{2n}{n}\binom{4n-1}{2n-\frac{1}{2}}=\frac{1}{\pi}-\frac{\sqrt{2}}{\pi}+\frac{\sinh^{-1}(1)}{\pi},$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,64^n}\binom{2n}{n}\binom{4n+1}{2n+\frac{1}{2}}=\frac{16}{3\pi}-\frac{8\sqrt{2}}{3\pi},$$

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)(4n-1)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{1}{\sqrt{2}\pi}-\frac{\sinh^{-1}(1)}{2\pi},$$

$$\sum_{n=1}^{\infty}\frac{4n+1}{(2n-1)(2n+1)\,64^n}\binom{2n}{n}\binom{4n}{2n}=1-\frac{3}{\sqrt{2}\pi}+\frac{\sinh^{-1}(1)}{2\pi}.$$

Putting $p=-1/8$ in (42) produces an exotic series

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,64^n}\binom{2n}{n}\binom{4n-\frac{1}{2}}{2n-\frac{1}{4}}=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi^{3/2}}-\frac{4}{3\pi}-\frac{\sqrt{2\pi}}{3\Gamma^2\left(\frac{3}{4}\right)}.$$

Here, $\Gamma(z)$ is the gamma function defined by $\int_0^\infty x^{z-1}e^{-x}\,dx$ for $\Re(z)>0$.

Theorem 17. *If p is a real number, then we have*

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2\,64^n}\binom{2n}{n}\binom{4n+4p}{2n+2p}=\frac{16^p}{\pi}\int_0^{\frac{\pi}{2}}(\sin y)^{4p+2}\sin^{-1}\left(\sin^2 y\right)\,dy.$$

Proof. The result can be easily verified by using the Lemma 1 and (8). □

Example 18. Inserting $p=0$, $p=-3/4$ in Theorem 17, and using propositions from Subsection 2.2, we arrive at

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{7\pi}{48}+\frac{1}{\sqrt{2}\pi}-\frac{\log^2(2)}{8\pi}-\frac{1}{\pi}\operatorname{Li}_2\left(\frac{1}{\sqrt{2}}\right)-\frac{\sinh^{-1}(1)}{2\pi},$$

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2\,64^n}\binom{2n}{n}\binom{4n-3}{2n-\frac{3}{2}}=-\frac{G}{4\pi}-\frac{\pi}{16\sqrt{2}}+\frac{1}{64\pi\sqrt{2}}\left(\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{3}{8}\right)\right).$$

Similarly, for $p=-1/2$, $p=1$

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)^2(4n-1)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{7\pi}{96}-\frac{1}{2\pi}\operatorname{Li}_2\left(\frac{1}{\sqrt{2}}\right)-\frac{\log^2(2)}{16\pi},$$

$$\sum_{n=1}^{\infty}\frac{n(4n+3)(4n+1)}{(2n-1)^2(n+1)(2n+1)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{35\pi}{48}+\frac{101}{18\sqrt{2}\pi}-\frac{5\log^2(2)}{8\pi}-\frac{5}{\pi}\operatorname{Li}_2\left(\frac{1}{\sqrt{2}}\right)-\frac{25\sinh^{-1}(1)}{12\pi},$$

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{7\pi}{96}-\frac{\log^2(2)}{16\pi}+\frac{3}{4\sqrt{2}\pi}-\frac{1}{2\pi}\operatorname{Li}_2\left(\frac{1}{\sqrt{2}}\right)-\frac{5\sinh^{-1}(1)}{8\pi}.$$

The last identity is a particular case of Lemma 2. Via specialization, several more series involving the product of central binomial coefficients can be deduced. Next, we investigate more series involving the product of central binomial coefficients, yielding the Ramanujan-type series for $1/\pi$.

Lemma 19. *For $x\in[-1,1]$, the following holds:*

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)\,16^n}\binom{4n}{2n}x^{2n}=\frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$

Proof. The proof directly follows from the generating function (2), which on integration yields the desired result. □

Theorem 20. *For all integers $k>0$, we have*

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)\,64^n}\binom{2n+2k}{n+k}\binom{4n}{2n}=\frac{2\cdot 4^k}{\pi}\left(B\left(2k,\frac{1}{2}\right)-(-1)^{2k}B_{-1}\left(2k,\frac{1}{2}\right)\right),\qquad(44)$$

where $B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}\,dt$ is the incomplete beta function and at $x=1$, it is beta function.

Proof. We multiply both sides of Lemma 19 by x^{2k}. Assigning $x=\sin y$, integrating from 0 to $\pi/2$, and using (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(2n+1)\,64^n}\binom{2n+2k}{n+k}\binom{4n}{2n} &= \frac{2\cdot 4^k}{\pi}\int_0^{\frac{\pi}{2}} (\sin y)^{2k-1}\left(\sqrt{1+\sin y}-\sqrt{1-\sin y}\right) dy \\ &= \frac{2\cdot 4^k}{\pi}\int_0^1 \frac{t^{2k-1}}{\sqrt{1-t^2}}\left(\sqrt{1+t}-\sqrt{1-t}\right) dt \\ &= \frac{2\cdot 4^k}{\pi}\int_0^1 t^{2k-1}\left(\frac{1}{\sqrt{1-t}}-\frac{1}{\sqrt{1+t}}\right) dt \qquad (45) \\ &= \frac{2\cdot 4^k}{\pi}\left(B\left(2k,\frac{1}{2}\right)-(-1)^{2k}\int_0^{-1}\frac{t^{2k-1}}{\sqrt{1-t}}\,dt\right).\end{aligned}$$

Using the definition of the beta and the incomplete beta function, the proof is completed. □

In particular, Theorem 20 gives us Ramanujan-type series for $1/\pi$.

Theorem 21. *The following relations hold:*

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{8}{3}\cdot\frac{\sqrt{2}}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{2n+3}{(n+1)(n+2)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{144}{35}\cdot\frac{\sqrt{2}}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{(2n+5)(2n+3)}{(n+1)(n+2)(n+3)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{4832}{693}\cdot\frac{\sqrt{2}}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{(2n+7)(2n+5)(2n+3)}{(n+1)(n+2)(n+3)(n+4)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{79424}{6435}\cdot\frac{\sqrt{2}}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{(2n+9)(2n+7)(2n+5)(2n+3)}{(n+1)(n+2)(n+3)(n+4)(n+5)\,64^n}\binom{2n}{n}\binom{4n}{2n}=\frac{5174656}{230945}\cdot\frac{\sqrt{2}}{\pi}.$$

In terms of the Pochhammer symbol, $k^{(n)} = k(k+1)(k+2)\ldots(k-n+1)$ for $n \geq 1$, we notice $(2n+1)(2n+3)(2n+5)\cdots(2n+2k-1) = 2^k(n+1/2)_k$ and $(n+1)(n+2)(n+3)\cdots(n+k) = (n+1)_k$, then in view of Theorem 21, Theorem 20 can be compactly written as follows:

$$\sum_{n=0}^{\infty}\frac{2^k\,(n+1/2)_k}{(2n+1)(n+1)_k\,64^n}\binom{2n}{n}\binom{4n}{2n}=b_k\cdot\frac{\sqrt{2}}{\pi},$$

where $b_k = i/j$, i, and j are positive integers.

The general form of i/j is easily deducible by comparing the left- and right-hand quantities of Theorem 20. The aforementioned Ramanujan-like series are merely the consequences of Theorem 20, which are obtained by putting $k = 1$, $k = 2$, $k = 3$, $k = 4$, and $k = 5$ respectively. Details are left to the reader.

Example 22. In a similar fashion, one can easily find several interesting series by evaluating the integrals obtained for any rational number k in (45).

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\,64^n} \binom{2n}{n} \binom{4n}{2n} = \frac{4}{\pi} \sinh^{-1}(1),$$
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\,64^n} \binom{2n+1}{n+\frac{1}{2}} \binom{4n}{2n} = \frac{8}{\pi} \left(2 - \sqrt{2}\right),$$
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\,64^n} \binom{2n+\frac{1}{2}}{n+\frac{1}{4}} \binom{4n}{2n} = 2\sqrt{2} \left(1 - \frac{2\sinh^{-1}(1)}{\pi}\right),$$
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\,64^n} \binom{2n+\frac{3}{2}}{n+\frac{3}{4}} \binom{4n}{2n} = \frac{4}{\pi} \left(-2 + \frac{\pi}{\sqrt{2}} + \sqrt{2}\sinh^{-1}(1)\right).$$

Details are left to the reader.

4 Some additional series

Employing the identity (13) and the identity (21) given in [2, pp. 5–6], respectively, we arrive at slightly different conclusions:

$$\sum_{n=1}^{\infty} \frac{1}{n\,64^n} \binom{2n}{n} \binom{4n}{2n} = 6\log(2) - \sqrt{2}\pi + \frac{1}{2\sqrt{2}\pi} \left(\psi_1\left(\frac{5}{8}\right) + \psi_1\left(\frac{7}{8}\right)\right), \tag{46}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(n+1)^2\,64^n} \binom{2n}{n} \binom{4n+4}{2n+2} &= 58 - \frac{64\sqrt{2}}{\pi} + 32\sqrt{2}\pi - \frac{64\sinh^{-1}(1)}{\pi} - 192\log(2) \\ &\quad - \frac{8\sqrt{2}}{\pi} \left(\psi_1\left(\frac{5}{8}\right) + \psi_1\left(\frac{7}{8}\right)\right),\end{aligned} \tag{47}$$

where $\psi_1(z) = \sum_{n=0}^{\infty} 1/(n+z)^2$ for $z > 0$ is the trigamma function. Identity (46) can also be proved using [1, Thm. 3.1, p. 3]. Using the proposition established in [2, p. 9] and calculating the corresponding integrals obtained for the particular values $m = 1$, $m = 2$, and $m = 3$, we conclude some fascinating Ramanujan-like series as follows:

$$\sum_{n=0}^{\infty} \frac{1}{(n+2)\,64^n} \binom{2n}{n} \binom{4n}{2n} = \frac{152}{105} \cdot \frac{\sqrt{2}}{\pi}, \tag{48}$$
$$\sum_{n=0}^{\infty} \frac{1}{(n+3)\,64^n} \binom{2n}{n} \binom{4n}{2n} = \frac{10568}{10395} \cdot \frac{\sqrt{2}}{\pi}, \tag{49}$$
$$\sum_{n=0}^{\infty} \frac{1}{(n+4)\,64^n} \binom{2n}{n} \binom{4n}{2n} = \frac{178328}{225225} \cdot \frac{\sqrt{2}}{\pi}. \tag{50}$$

These series have quite simple appearances in their closed form, which include rational number p/q, and also $\sqrt{2}$ and $1/\pi$. Based on the patterns observed in the first equation of Theorem 21, and equations (48), (49), and (50), for $m \in \mathbb{N} \cup \{0\}$, we conjecture that

$$\sum_{n=0}^{\infty} \frac{1}{64^n} \binom{2n}{n} \binom{4n}{2n} \frac{1}{n+m+1} = a_m \cdot \frac{\sqrt{2}}{\pi},$$

where $a_m = p/q$, p, and q are positive integers. The problem of determining the general expression of a_m remains open. Furthermore, utilizing Lemma 4, we have

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2(2n+1)\,64^n} \binom{2n}{n} \binom{4n}{2n} = \frac{7\pi}{192} + \frac{1}{8\pi\sqrt{2}} - \frac{\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right)}{4\pi} - \frac{\log^2(2)}{32\pi} + \frac{\sinh^{-1}(1)}{16\pi},$$

$$\sum_{n=1}^{\infty} \frac{n^2 4^n}{(2n-1)^2(2n+1)(4n+1)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = -\frac{G}{8} - \frac{13}{144} + \frac{7}{36\sqrt{2}} - \frac{\pi^2}{32\sqrt{2}} + \frac{1}{128\sqrt{2}} \left(\psi_1\left(\frac{1}{8}\right) + \psi_1\left(\frac{3}{8}\right) \right).$$

Moreover, applying the lemmas introduced in the recent paper, we can easily deduce some more variety of series involving the square of the central binomial coefficients, yielding exotic results such as

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2(2n+1)\,16^n} \binom{2n}{n}^2 = \frac{G}{4\pi} + \frac{1}{8\pi},$$

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(n+1)\,16^n} \binom{2n}{n}^2 = \frac{4}{9\pi},$$

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(2n+1)(2n+3)\,16^n} \binom{2n}{n}^2 = \frac{31}{128\pi} - \frac{13G}{64\pi}.$$

We encourage interested readers to pursue the results and to investigate more series.

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