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Cassini-Like Formula for Generalized Hyper-Fibonacci Numbers

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Abstract

In this paper, we give some properties of generalized hyper-Fibonacci numbers in order to obtain a Cassini-like formula for them.

1 Introduction

In this paper, we consider the generalized Fibonacci sequence $(w_n)_{n\geq 0}$ defined by

$$\begin{cases} w_0 = a, w_1 = b; \\ w_{n+2} = \alpha w_{n+1} + \beta w_n, \quad (n \geq 0) \end{cases}$$

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where $a, b, \alpha,$ and β are integers. Several authors [3, 7, 8, 12] studied properties of these numbers. If $\alpha^2 + 4\beta \neq 0$, then we have

$$w_n = as_1^n + (b - as_1)\left(\frac{s_1^n - s_2^n}{s_1 - s_2}\right), \qquad n \geq 0, \tag{1}$$

where $s_1 = (\alpha + \sqrt{\alpha + 4\beta})/2$ and $s_2 = (\alpha - \sqrt{\alpha + 4\beta})/2$ are the roots of $x^2 - \alpha x - \beta = 0$. If $A = b - as_2$ and $B = b - as_1$, then the identity (1) is equivalent to the well-known Binet-like formula

$$w_n = \frac{As_1^n - Bs_2^n}{s_1 - s_2}, \qquad n \geq 0.$$

If $\beta \neq 0$ and $\alpha^2 + 4\beta \neq 0$, then the generalized Fibonacci numbers for negative subscripts are defined, using Identity (1), by $w_{-n} = a(s_1^{-n} + s_2^{-n}) - (-\beta)^{-n} w_n$ for $n \geq 0$. If $\beta \neq 0$ and $\alpha^2 + 4\beta = 0$, then $\alpha = 2t$ and $\beta = -t^2$, where t is the double root of $x^2 - \alpha x - \beta = 0$. Thus, $w_n = ((2b\alpha^{-1} - a)n + a)t^n$ and $w_{-n} = ((a - 2b\alpha^{-1})n + a)t^{-n}$, for $n \geq 0$.

We introduce the generalized hyper-Fibonacci numbers associated with the sequence $(w_n)_{n \geq 0}$ as follows:

$$w_n^{(r+1)} = \sum_{k=0}^{n} \alpha^{n-k} w_k^{(r)}, \quad w_n^{(0)} = w_n, \quad w_0^{(r)} = a, \quad w_1^{(r)} = \alpha ar + b, \tag{2}$$

where r is a nonnegative integer.

The aim of this paper is to extend the well-known Cassini formula [5, 10, 14]

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1} \tag{3}$$

to the generalized hyper-Fibonacci numbers (2), where $(F_n)_n$ denotes the classical Fibonacci sequence. The identity (3) can be written in a determinant form as

$$\begin{vmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{vmatrix} = (-1)^{n+1}. \tag{4}$$

In [11], Stakhov generalized the Cassini formula (3) to the p-Fibonacci numbers and developed a new coding theory based on the Q_p-matrices. By analogy, one can use the compagnon matrices given bellow in the formula (19) instead of the Q_p-matrices. Halici [6] established the Cassini formula for the Fibonacci quaternions. Martinjak and Urbiha [9] extended the Cassini formula (4) to the hyper-Fibonacci numbers defined in [2, 4] as

$$F_n^{(r+1)} = \sum_{k=0}^{n} F_k^{(r)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1,$$

where r is a nonnegative integer. The number $F_n^{(r)}$ is called the nth hyper-Fibonacci number of the rth generation. The hyper-Fibonacci numbers satisfy many interesting number-theoretical and combinatorial properties, e.g., [2]. Martinjak and Urbiha [9] defined the matrix

$$A_{r,n} = \begin{pmatrix} F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+1}^{(r)} \\ F_{n+1}^{(r)} & F_{n+2}^{(r)} & \cdots & F_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r+2}^{(r)} & \cdots & F_{n+2r+2}^{(r)} \end{pmatrix}$$

and proved that

$$\det(A_{r,n}) = (-1)^{n+\lfloor (r+3)/2 \rfloor}, \tag{5}$$

where n and r are nonnegative integers. It is clear that for $r = 0$, we obtain (4). Recently, in [1], the authors generalized the identity (5) to the (a, b)-hyper-Fibonacci numbers.

If $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ are two sequences satisfying the relation $a_{n+2} = \alpha a_{n+1} + \beta a_n$, then we have the identity [13]

$$b_n c_{n-1} - b_{n-1} c_n = (-\beta)^{n-1}(b_1 c_0 - b_0 c_1). \tag{6}$$

If we take $b_n = F_{n+2}$ and $c_n = F_{n+1}$, then the identity (6) reduces to (3). For $b_n = w_{n+2}$ and $c_n = w_{n+1}$, the identity (6) reduces to

$$w_n w_{n+2} - w_{n+1}^2 = (-\beta)^n(\beta a^2 + \alpha ab - b^2). \tag{7}$$

Cassini formula (7) can also be expressed as a determinant in the following way

$$\begin{vmatrix} w_n & w_{n+1} \\ w_{n+1} & w_{n+2} \end{vmatrix} = (-\beta)^{n-1}(\beta b^2 - \alpha\beta ab - \beta^2 a^2).$$

For $n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 0$, let us define the $(r + 2) \times (r + 2)$ matrix

$$W_{r,n} = \begin{pmatrix} w_n^{(r)} & w_{n+1}^{(r)} & \cdots & w_{n+r+1}^{(r)} \\ w_{n+1}^{(r)} & w_{n+2}^{(r)} & \cdots & w_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n+r+1}^{(r)} & w_{n+r+2}^{(r)} & \cdots & w_{n+2r+2}^{(r)} \end{pmatrix}.$$

In Section 2, we establish some combinatorial properties involving the generalized hyper-Fibonacci sequence $(w_n^{(r)})_{n\geq 0}$. In Section 3, we evaluate the determinant of the matrix $W_{r,n}$, which gives the identity (5) for $a = 0$ and $b = \alpha = \beta = 1$. For $a = 2$ and $b = \alpha = \beta = 1$, we deduce the determinant of the matrix

$$\begin{pmatrix} L_n^{(r)} & L_{n+1}^{(r)} & \cdots & L_{n+r+1}^{(r)} \\ L_{n+1}^{(r)} & L_{n+2}^{(r)} & \cdots & L_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+r+1}^{(r)} & L_{n+r+2}^{(r)} & \cdots & L_{n+2r+2}^{(r)} \end{pmatrix}$$

involving the hyper-Lucas numbers given by [2,4]

$$L_n^{(r+1)} = \sum_{k=0}^{n} L_k^{(r)}, \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 2r+1.$$

We let $\binom{n}{k}$ denote the binomial coefficient which is defined for a nonnegative integer n and an integer k by

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

and for a negative integer n and an integer k by

$$\binom{n}{k} = \begin{cases} (-1)^k \dbinom{-n+k-1}{k}, & \text{if } k \geq 0; \\ (-1)^{n-k} \dbinom{-k-1}{n-k}, & \text{if } k \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

In Sections 2 and 3 we assume that $\beta \neq 0$.

2 Properties of the sequences $(w_n^{(r)})_{n\geq 0}$

In this section, we give some properties satisfied by the generalized hyper-Fibonacci sequence $(w_n^{(r)})_{n\geq 0}$ given by the formula (2).

Lemma 1. *Let $n \geq 0$ be an integer. Then*

$$w_n^{(1)} = \frac{1}{\beta} w_{n+2} - \frac{\alpha^{n+1}}{\beta} b. \tag{8}$$

Proof. We prove the lemma by induction on $n \geq 0$. For $n = 0$, the identity (8) is trivially checked. Now assume that (8) holds for an integer $n \geq 0$. Then

$$\begin{aligned} w_{n+1}^{(1)} &= \sum_{k=0}^{n+1} \alpha^{n+1-k} w_k \\ &= \alpha \sum_{k=0}^{n} \alpha^{n-k} w_k + w_{n+1} \\ &= \alpha w_n^{(1)} + w_{n+1} \\ &= \alpha \left(\frac{1}{\beta} w_{n+2} - \frac{\alpha^{n+1}}{\beta} b \right) + w_{n+1} \\ &= \frac{1}{\beta} w_{n+3} - \frac{\alpha^{n+2}}{\beta} b. \end{aligned}$$

We conclude that (8) holds for all $n \geq 0$. □

The following proposition expresses a generalized hyper-Fibonacci number of any generation $r \geq 1$ in terms of the 0th generation.

Proposition 2. *Let $r \geq 1$ be an integer. Then*

$$w_n^{(r)} = \frac{1}{\beta^r} w_{n+2r} - \alpha^{n+1} \sum_{l=0}^{r-1} \binom{n+r-l-1}{r-l-1} \frac{w_{2l+1}}{\beta^{l+1}}, \quad n \geq 0. \tag{9}$$

Proof. We prove the proposition by induction on $r \geq 1$. We deduce from Lemma 1 that (9) holds for $r = 1$. Now assume that (9) holds for an integer $r \geq 1$. Then

$$\begin{aligned}
w_n^{(r+1)} &= \sum_{k=0}^{n} \alpha^{n-k} w_k^{(r)} \\
&= \sum_{k=0}^{n} \alpha^{n-k} \left(\frac{1}{\beta^r} w_{k+2r} - \alpha^{k+1} \sum_{l=0}^{r-1} \binom{k+r-l-1}{r-l-1} \frac{w_{2l+1}}{\beta^{l+1}} \right) \\
&= \frac{1}{\beta^r} \sum_{k=0}^{n} \alpha^{n-k} w_{k+2r} - \sum_{k=0}^{n} \sum_{l=0}^{r-1} \binom{k+r-l-1}{r-l-1} \alpha^{n+1} \frac{w_{2l+1}}{\beta^{l+1}} \\
&= \frac{1}{\beta^r} \sum_{l=0}^{n+2r} \alpha^{n+2r-l} w_l - \frac{\alpha^{n+1}}{\beta^r} \sum_{l=0}^{2r-1} \alpha^{2r-1-l} w_l - \alpha^{n+1} \sum_{l=0}^{r-1} \frac{w_{2l+1}}{\beta^{l+1}} \sum_{k=0}^{n} \binom{k+r-l-1}{r-l-1} \\
&= \frac{1}{\beta^r} w_{n+2r}^{(1)} - \frac{\alpha^{n+1}}{\beta^r} w_{2r-1}^{(1)} - \alpha^{n+1} \sum_{l=0}^{r-1} \binom{n+r-l}{r-l} \frac{w_{2l+1}}{\beta^{l+1}} \\
&= \frac{1}{\beta^{r+1}} w_{n+2r+2} - \frac{\alpha^{n+1}}{\beta^{r+1}} w_{2r+1} - \alpha^{n+1} \sum_{l=0}^{r-1} \binom{n+r-l}{r-l} \frac{w_{2l+1}}{\beta^{l+1}} \\
&= \frac{1}{\beta^{r+1}} w_{n+2r+2} - \alpha^{n+1} \sum_{l=0}^{r} \binom{n+r-l}{r-l} \frac{w_{2l+1}}{\beta^{l+1}}.
\end{aligned}$$

We deduce that (9) holds for all $r \geq 1$. □

To go further, we need to define the generalized hyper-Fibonacci numbers of negative subscripts. To do this, we first study the particular case $a = 0$ and $b = 1$ of the sequence $(w_n)_n \geq 0$. Thus, let $(u_n)_n$ denote this particular sequence. The definition given by (2) reduces to

$$u_n^{(r+1)} = \sum_{k=0}^{n} \alpha^{n-k} u_k^{(r)}, \quad u_n^{(0)} = u_n, \quad u_0^{(r)} = 0, \quad u_1^{(r)} = 1. \tag{10}$$

Proposition 3. *Let $r \geq 0$ be an integer. Then*

$$u_n^{(r+1)} = \frac{1}{\beta} u_{n+2}^{(r)} - \binom{n+r+1}{r} \frac{\alpha^{n+1}}{\beta}, \quad n \geq 0. \tag{11}$$

Proof. We prove the proposition by induction on $r \geq 0$. We deduce from Lemma 1 that (11) holds for $r = 0$. Now assume that (11) holds for an integer $r \geq 0$. Then

$$
\begin{aligned}
u_n^{(r+2)} &= \sum_{k=0}^{n} \alpha^{n-k} u_k^{(r+1)} \\
&= \sum_{k=0}^{n} \alpha^{n-k} \left(\frac{1}{\beta} u_{k+2}^{(r)} - \binom{k+r+1}{r} \frac{\alpha^{k+1}}{\beta} \right) \\
&= \frac{1}{\beta} \sum_{k=0}^{n} \alpha^{n-k} u_{k+2}^{(r)} - \frac{1}{\beta} \sum_{k=0}^{n} \binom{k+r+1}{r} \alpha^{n+1} \\
&= \frac{1}{\beta} \sum_{l=2}^{n+2} \alpha^{n+2-l} u_l^{(r)} - \frac{1}{\beta} \sum_{l=2}^{n+2} \binom{l+r-1}{r} \alpha^{n+1} \\
&= \frac{1}{\beta} \left(\sum_{l=0}^{n+2} \alpha^{n+2-l} u_l^{(r)} - \alpha^{n+1} \right) - \frac{1}{\beta} \left(\sum_{l=1}^{n+2} \binom{l+r-1}{r} \alpha^{n+1} - \alpha^{n+1} \right) \\
&= \frac{1}{\beta} u_{n+2}^{(r+1)} - \frac{1}{\beta} \sum_{l=1}^{n+2} \binom{l+r-1}{r} \alpha^{n+1} \\
&= \frac{1}{\beta} u_{n+2}^{(r+1)} - \binom{n+r+2}{r+1} \frac{\alpha^{n+1}}{\beta}.
\end{aligned}
$$

We deduce that (11) holds for all $r \geq 0$. □

The following corollary allows us to extend the sequence $(u_n)_{n \geq 0}$ to negative subscripts.

Corollary 4. *Let $r \geq 0$ and $n \geq 0$ be integers. Then*

$$
u_{n+2}^{(r+1)} = \alpha u_{n+1}^{(r+1)} + \beta u_n^{(r+1)} + \binom{n+r+1}{r} \alpha^{n+1}. \tag{12}
$$

Proof. According to (10), we have

$$
u_{n+1}^{(r)} = u_{n+1}^{(r+1)} - \alpha u_n^{(r+1)}, \tag{13}
$$

where $n \geq 0$ and $r \geq 0$. From (11) and (13) we get

$$
u_n^{(r+1)} = \frac{1}{\beta} u_{n+2}^{(r+1)} - \frac{\alpha}{\beta} u_{n+1}^{(r+1)} - \binom{n+r+1}{r} \frac{\alpha^{n+1}}{\beta}.
$$

We deduce that

$$
u_{n+2}^{(r+1)} = \alpha u_{n+1}^{(r+1)} + \beta u_n^{(r+1)} + \binom{n+r+1}{r} \alpha^{n+1}.
$$

□

Now we give a linear relation between the sequences $(w_n)_{n\geq 0}$ and $(u_n)_{n\geq 0}$.

Lemma 5. *Let $n \geq 0$ be an integer. Then*

$$w_{n+1} = bu_{n+1} + \beta a u_n. \tag{14}$$

Proof. We prove the lemma by induction on $n \geq 0$. For $n = 0, 1$ and 2, the identity (14) is trivially checked. Now assume that (14) holds up to some order $n \geq 1$. Then

$$\begin{aligned} w_{n+2} &= \alpha w_{n+1} + \beta w_n \\ &= \alpha\left(bu_{n+1} + \beta a u_n\right) + \beta\left(bu_n + \beta a u_{n-1}\right) \\ &= b\left(\alpha u_{n+1} + \beta u_n\right) + \beta a\left(\alpha u_n + \beta u_{n-1}\right) \\ &= bu_{n+2} + \beta a u_{n+1}. \end{aligned}$$

We conclude that (14) holds for all $n \geq 0$. □

In the following lemma, we give a relation between the sequences $(w_n^{(1)})_{n\geq 0}$ and $(u_n^{(1)})_{n\geq 0}$.

Lemma 6. *Let $n \geq 0$ be an integer. Then*

$$w_n^{(1)} = bu_n^{(1)} + \beta a u_{n-1}^{(1)} + a\alpha^n. \tag{15}$$

Proof. We prove the lemma by induction on $n \geq 0$. Since $u_0^{(1)} = u_{-1}^{(1)} = 0$, the identity (15) holds for $n = 0$. Now assume that (15) holds for an integer $n \geq 0$. Then

$$\begin{aligned} w_{n+1}^{(1)} &= \sum_{k=0}^{n+1} \alpha^{n+1-k} w_k \\ &= w_{n+1} + \alpha \sum_{k=0}^{n} \alpha^{n-k} w_k \\ &= w_{n+1} + \alpha w_n^{(1)} \\ &= bu_{n+1} + \beta a u_n + \alpha(bu_n^{(1)} + \beta a u_{n-1}^{(1)} + a\alpha^n) \\ &= b(u_{n+1} + \alpha u_n^{(1)}) + \beta a(u_n + \alpha u_{n-1}^{(1)}) + a\alpha^{n+1} \\ &= bu_{n+1}^{(1)} + \beta a u_n^{(1)} + a\alpha^{n+1}. \end{aligned}$$

We conclude that (15) holds for all $n \geq 0$. □

Now we derive a relation between the sequences $(w_n^{(r)})_{n\geq 0}$ and $(u_n^{(r)})_{n\geq 0}$, for any generation $r \geq 1$.

Proposition 7. *Let $r \geq 1$ be an integer. Then*

$$w_n^{(r)} = bu_n^{(r)} + \beta a u_{n-1}^{(r)} + a\binom{n+r-1}{r-1}\alpha^n, \qquad n \geq 0. \tag{16}$$

Proof. We prove the proposition by induction on $n \geq 0$. We deduce from Lemma 6 that Identity (16) holds for $r = 1$. Now assume that (16) holds for an integer $r \geq 1$. Then

$$\begin{aligned}
w_n^{(r+1)} &= \sum_{k=0}^{n} \alpha^{n-k} w_k \\
&= \sum_{k=0}^{n} \alpha^{n-k} \left(bu_k^{(r)} + \beta a u_{k-1}^{(r)} + a \binom{k+r-1}{r-1} \alpha^k \right) \\
&= \sum_{k=0}^{n} \alpha^{n-k} u_k^{(r)} + \beta a \sum_{k=0}^{n} \alpha^{n-k} u_{k-1}^{(r)} + a\alpha^n \sum_{k=0}^{n} \binom{k+r-1}{r-1} \\
&= bu_n^{(r+1)} + \beta a \sum_{l=0}^{n-1} \alpha^{(n-1)-l} u_l^{(r)} + \binom{n+r}{r} a\alpha^n \\
&= bu_n^{(r+1)} + \beta a u_{n-1}^{(r+1)} + \binom{n+r}{r} a\alpha^n.
\end{aligned}$$

We conclude that (16) holds for all $n \geq 0$. □

Remark 8. Using Identity (12), we extend, for $r \geq 1$ and $\alpha \neq 0$, the sequence $(u_n)_{n \geq 0}$ to negative subscripts as follows:

$$u_{-n}^{(r)} = \frac{1}{\beta} u_{-n+2}^{(r)} - \frac{\alpha}{\beta} u_{-n+1}^{(r)} - \binom{-n+r}{r-1} \frac{\alpha^{-n+1}}{\beta}, \quad n \geq 0.$$

It is easy to see that

$$u_{-n}^{(r)} = 0, \quad \text{for } 0 \leq n \leq r \quad \text{and} \quad u_{-r-1}^{(r)} = \frac{1}{\beta} \left(\frac{-1}{\alpha} \right)^r \neq 0.$$

Now, using Identity (16), we define, for $r \geq 1$ and $\alpha \neq 0$, the generalized hyper-Fibonacci numbers for negative subscripts by

$$w_{-n}^{(r)} = bu_{-n}^{(r)} + \beta a u_{-n+1}^{(r)} + a \binom{-n+r-1}{r-1} \alpha^{-n}, \qquad n \geq 0.$$

We deduce that

$$w_{-n}^{(r)} = 0 \qquad \text{for } 1 \leq n \leq r.$$

If $\alpha = 0$, we have respectively from (2) and (10) that $w_n^{(r)} = w_n$ and $u_n^{(r)} = u_n$ for $n \geq 0$. Thus,

$$w_{-n}^{(r)} = w_{-n} \quad \text{and} \quad u_{-n}^{(r)} = u_{-n}, \quad n \geq 0.$$

The following theorem is the key assertion behind the computation of the generalized Cassini formula.

Theorem 9. *Assume that $\alpha \neq 0$ and let $r \geq 0$ be an integer. Then*

$$w_{n+r+2}^{(r)} = \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k} \right) w_{n+k}^{(r)}, \qquad n \geq -r. \tag{17}$$

Proof. Let us proceed by induction on r. For $r = 0$, the identity (17) holds by definition of the sequence $(w_n)_n$. Now assume that (17) holds for an integer $r \geq 0$. Since $n+1 \geq n \geq -r$, we have

$$w_{n+r+3}^{(r)} = \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k} \right) w_{n+k+1}^{(r)}. \tag{18}$$

Since $n+r+3 \geq n+r+2 \geq 0$, we have $w_{n+r+3}^{(r)} = w_{n+r+3}^{(r+1)} - \alpha w_{n+r+2}^{(r+1)}$. For $k = 0, 1 \ldots, r+1$, we have $w_{n+k+1}^{(r)} = w_{n+k+1}^{(r+1)} - \alpha w_{n+k}^{(r+1)}$ because

- If $n+k+1 < 0$ then we obtain $0 = 0 - 0$.
- If $n+k+1 \geq 0$ and $n+k < 0$ then $n+k = -1$, we obtain $a = a - 0$.
- If $n+k \geq 0$ then $n+k+1 > 0$ and we obtain $w_{n+k+1}^{(r)} = w_{n+k+1}^{(r+1)} - \alpha w_{n+k}^{(r+1)}$.

Thus, from (18) we get

$$w_{n+r+3}^{(r+1)} - \alpha w_{n+r+2}^{(r+1)} = \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k} \right) (w_{n+k+1}^{(r+1)} - \alpha w_{n+k}^{(r+1)}).$$

We deduce that

$$\begin{aligned} w_{n+r+3}^{(r+1)} &= \alpha w_{n+r+2}^{(r+1)} + \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k} \right) w_{n+k+1}^{(r+1)} \\ &\quad + \sum_{k=0}^{r+1} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k} \right) w_{n+k}^{(r+1)} \\ &= (\alpha r + 2\alpha) w_{n+r+2}^{(r+1)} + \sum_{k=0}^{r} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k} \right) w_{n+k+1}^{(r+1)} \\ &\quad + (-1)^{r+1} \alpha^{r+1} \beta w_n^{(r+1)} + \sum_{k=1}^{r+1} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k} \right) w_{n+k}^{(r+1)} \\ &= (\alpha r + 2\alpha) w_{n+r+2}^{(r+1)} + \sum_{l=1}^{r+1} (-1)^{r-l} \alpha^{r+1-l} \left(\alpha^2 \binom{r+1}{l-2} - \beta \binom{r}{l-1} \right) w_{n+l}^{(r+1)} \\ &\quad + (-1)^{r+1} \alpha^{r+1} \beta w_n^{(r+1)} + \sum_{k=1}^{r+1} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k} \right) w_{n+k}^{(r+1)} \end{aligned}$$

$$= \sum_{k=0}^{r+2} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 \binom{r+1}{k-2} + \alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k-1} - \beta \binom{r}{k} \right) w_{n+k}^{(r+1)}$$

$$= \sum_{k=0}^{r+2} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 \binom{r+2}{k-1} - \beta \binom{r+1}{k} \right) w_{n+k}^{(r+1)}.$$

We conclude that (17) holds for all $r \geq 0$. □

3 A generalization of the Cassini formula

Cassini formula (7) can be expressed as a 2×2 determinant in the following way

$$\begin{vmatrix} w_n & w_{n+1} \\ w_{n+1} & w_{n+2} \end{vmatrix} = (-\beta)^{n-1} (\beta b^2 - \alpha \beta a b - \beta^2 a^2).$$

For $n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 0$, let us define the $(r+2) \times (r+2)$ matrix

$$W_{r,n} = \begin{pmatrix} w_n^{(r)} & w_{n+1}^{(r)} & \cdots & w_{n+r+1}^{(r)} \\ w_{n+1}^{(r)} & w_{n+2}^{(r)} & \cdots & w_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n+r+1}^{(r)} & w_{n+r+2}^{(r)} & \cdots & w_{n+2r+2}^{(r)} \end{pmatrix}.$$

Our purpose is to evaluate the determinant of the matrix $W_{r,n}$. Note that

$$W_{0,n} = \begin{pmatrix} w_n & w_{n+1} \\ w_{n+1} & w_{n+2} \end{pmatrix}.$$

From Theorem 9, we can write

$$w_{n+r+2}^{(r)} = \sum_{k=0}^{r+1} q_k w_{n+k}^{(r)}, \qquad n \geq -r,$$

where

$$q_k = (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 \binom{r+1}{k-1} - \beta \binom{r}{k} \right), \quad 0 \leq k \leq r+1.$$

Let

$$V_{r+2} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ q_0 & q_1 & q_2 & \cdots & q_r & q_{r+1} \end{pmatrix} \tag{19}$$

denote the companion matrix of the generalized hyper-Fibonacci sequence $\left(w_n^{(r)}\right)_n$. We deduce from Theorem 9 that the generalized hyper-Fibonacci sequence $(w_n^{(r)})_n$ can be defined by the vector recurrence relation

$$\begin{pmatrix} w_{n+1}^{(r)} \\ w_{n+2}^{(r)} \\ \vdots \\ w_{n+r+2}^{(r)} \end{pmatrix} = V_{r+2} \begin{pmatrix} w_n^{(r)} \\ w_{n+1}^{(r)} \\ \vdots \\ w_{n+r+1}^{(r)} \end{pmatrix}, \tag{20}$$

where $n + r \geq 0$.

Lemma 10. *Let $n \geq 0$ and $r \geq 0$ be integers. Then*

$$W_{r,n} = V_{r+2}^n W_{r,0}.$$

Proof. From the relation (20) we can write $W_{r,n} = V_{r+2} W_{r,n-1}$. It follows that

$$W_{r,n} = V_{r+2} W_{r,n-1} = V_{r+2}^2 W_{r,n-2} = \cdots = V_{r+2}^n W_{r,0}.$$

□

Lemma 11. *Let $r \geq 0$ be an integer. Then*

$$\det(V_{r+2}) = -\alpha^r \beta.$$

Proof. It is clear that

$$\det(V_{r+2}) = (-1)^{r+3} q_0 = (-1)^{r+3}(-1)^{r+2}\alpha^r\beta = -\alpha^r\beta.$$

□

Theorem 12. *Let $n \geq 0$ and $r \geq 0$ be integers. Then*

$$\det(W_{r,n}) = (-1)^{n+\lfloor (r+3)/2 \rfloor} \alpha^{nr+r^2} \beta^{n+r} b^r (b^2 - \alpha ab - \beta a^2). \tag{21}$$

Proof. For $r = 0$, the result follows from Identity (7). Assume that $r \geq 1$. For $\alpha \neq 0$, we deduce from (20) that multiplication by V_{r+2}^{-1} decreases by 1 the subscript of each component, i.e.,

$$V_{r+2}^{-1} W_{r,0} = \begin{pmatrix} w_{-1}^{(r)} & w_0^{(r)} & \cdots & w_r^{(r)} \\ w_0^{(r)} & w_1^{(r)} & \cdots & w_{r+1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ w_r^{(r)} & w_{r+1}^{(r)} & \cdots & w_{2r+1}^{(r)} \end{pmatrix}.$$

Thus,

$$V_{r+2}^{-r}W_{r,0} = \begin{pmatrix} w_{-r}^{(r)} & w_{1-r}^{(r)} & \cdots & w_{1}^{(r)} \\ w_{1-r}^{(r)} & w_{2-r}^{(r)} & \cdots & w_{2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1}^{(r)} & w_{2}^{(r)} & \cdots & w_{r+2}^{(r)} \end{pmatrix}.$$

Since $w_{-n}^{(r)} = 0$ for $1 \leq n \leq r$, then

$$V_{r+2}^{-r}W_{r,0} = \begin{pmatrix} 0 & 0 & \cdots & 0 & w_{0}^{(r)} & w_{1}^{(r)} \\ 0 & 0 & \cdots & w_{0}^{(r)} & w_{1}^{(r)} & w_{2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & w_{0}^{(r)} & \cdots & w_{r-2}^{(r)} & w_{r-1}^{(r)} & w_{r}^{(r)} \\ w_{0}^{(r)} & w_{1}^{(r)} & \cdots & w_{r-1}^{(r)} & w_{r}^{(r)} & w_{r+1}^{(r)} \\ w_{1}^{(r)} & w_{2}^{(r)} & \cdots & w_{r}^{(r)} & w_{r+1}^{(r)} & w_{r+2}^{(r)} \end{pmatrix}.$$

Thus,

$$\det(W_{r,0}) = \det(V_{r+2})^{r} \cdot \Delta, \tag{22}$$

where

$$\Delta = \begin{vmatrix} 0 & 0 & \cdots & 0 & w_{0}^{(r)} & w_{1}^{(r)} \\ 0 & 0 & \cdots & w_{0}^{(r)} & w_{1}^{(r)} & w_{2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & w_{0}^{(r)} & \cdots & w_{r-2}^{(r)} & w_{r-1}^{(r)} & w_{r}^{(r)} \\ w_{0}^{(r)} & w_{1}^{(r)} & \cdots & w_{r-1}^{(r)} & w_{r}^{(r)} & w_{r+1}^{(r)} \\ w_{1}^{(r)} & w_{2}^{(r)} & \cdots & w_{r}^{(r)} & w_{r+1}^{(r)} & w_{r+2}^{(r)} \end{vmatrix}.$$

Let L_j denote the jth line of Δ, where $j = 1, 2, \ldots, r+2$. First, we replace L_{i+1} by $L_{i+1} - \alpha L_i$ for $i = r+1, r, \ldots, 1$. Since $w_{i+1}^{(r-1)} = w_{i+1}^{(r)} - \alpha w_{i}^{(r)}$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & \cdots & w_{0}^{(r-1)} & w_{1}^{(r-1)} + 2\alpha \\ 0 & 0 & \cdots & w_{1}^{(r-1)} & w_{2}^{(r-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{0}^{(r-1)} & w_{1}^{(r-1)} & \cdots & w_{r}^{(r-1)} & w_{r+1}^{(r-1)} \\ w_{1}^{(r-1)} & w_{2}^{(r-1)} & \cdots & w_{r+1}^{(r-1)} & w_{r+2}^{(r-1)} \end{vmatrix}.$$

Using the same method again $(r-1)$ times, we obtain

$$\Delta = \begin{vmatrix} 0 & 0 & \cdots & 0 & w_0 & w_1 + d_1 \\ 0 & 0 & \cdots & w_0 & w_1 & w_2 + d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & w_0 & \cdots & w_{r-2} & w_{r-1} & w_r + d_r \\ w_0 & w_1 & \cdots & w_{r-1} & w_r & w_{r+1} \\ w_1 & w_2 & \cdots & w_r & w_{r+1} & w_{r+2} \end{vmatrix},$$

where $d_i = a(-1)^{i-1}\binom{r}{i}\alpha^i$ for $1 \le i \le r$. Now let C_j denote the jth column of this last determinant, where $j = 1, \dots, r+2$. Replacing the column C_i by $C_i - (\alpha C_{i-1} + \beta C_{i-2})$ for $i = r+2, r+1, \dots, 3$ and using the fact that $w_i = \alpha w_{i-1} + \beta w_{i-2}$ we get

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & w_0 & w_1 - \alpha w_0 + d_1 \\ 0 & 0 & 0 & \cdots & w_0 & w_1 - \alpha w_0 & d_2 \\ 0 & 0 & 0 & \cdots & w_1 - \alpha w_0 & 0 & d_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & w_0 & w_1 - \alpha w_0 & \cdots & 0 & 0 & d_r \\ w_0 & w_1 & 0 & \cdots & 0 & 0 & 0 \\ w_1 & w_2 & 0 & \cdots & 0 & 0 & 0 \end{vmatrix}.$$

Now we permute the column C_i with column C_{r+3-i} for $1 \le i \le \lfloor (r+2)/2 \rfloor$, and obtain

$$\Delta = (-1)^{\lfloor \frac{r+2}{2} \rfloor} \begin{vmatrix} w_1 - \alpha w_0 + d_1 & w_0 & 0 & \cdots & 0 & 0 & 0 \\ d_2 & w_1 - \alpha w_0 & w_0 & \cdots & 0 & 0 & 0 \\ d_3 & 0 & w_1 - \alpha w_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_r & 0 & 0 & \cdots & w_1 - \alpha w_0 & w_0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & w_1 & w_0 \\ 0 & 0 & 0 & \cdots & 0 & w_2 & w_1 \end{vmatrix}.$$

We deduce that

$$\Delta = (-1)^{\lfloor \frac{r+2}{2} \rfloor} \Delta' \begin{vmatrix} w_1 & w_0 \\ w_2 & w_1 \end{vmatrix}, \tag{23}$$

where

$$\Delta' = \begin{vmatrix} d_1 - (\alpha a - b) & a & 0 & \cdots & 0 & 0 \\ d_2 & -(\alpha a - b) & a & \cdots & 0 & 0 \\ d_3 & 0 & -(\alpha a - b) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{r-1} & 0 & 0 & \cdots & -(\alpha a - b) & a \\ d_r & 0 & 0 & \cdots & 0 & -(\alpha a - b) \end{vmatrix}.$$

We distinguish two cases for the calculation of Δ'.

- If $\alpha a = b \neq 0$, the expansion with respect to the first column gives

$$\Delta' = \begin{vmatrix} d_1 & a & 0 & \cdots & 0 & 0 \\ d_2 & 0 & a & \cdots & 0 & 0 \\ d_3 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{r-1} & 0 & 0 & \cdots & 0 & a \\ d_r & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{r+1} d_r a^{r-1} = (\alpha a)^r.$$

• If $b-\alpha a \neq 0$, let L_j denote the jth line of Δ', where $j=1,\ldots,r$. Replacing the line L_i by $L_i + \dfrac{a}{\alpha a - b} L_{i+1}$ for $i = r-1, \ldots, 1$ gives

$$\Delta' = \begin{vmatrix} (b-\alpha a) + \sum_{i=1}^{r} \left(\frac{a}{\alpha a - b}\right)^{i-1} d_i & 0 & \cdots & 0 & 0 \\ & b-\alpha a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{r-1} + \frac{a}{\alpha a - b} d_r & 0 & \cdots & b-\alpha a & 0 \\ d_r & 0 & \cdots & 0 & b-\alpha a \end{vmatrix}.$$

We deduce that

$$\begin{aligned} \Delta' &= (b-\alpha a)^{r-1} \left((b-\alpha a) + \sum_{i=1}^{r} \left(\frac{a}{\alpha a - b}\right)^{i-1} d_i \right) \\ &= (b-\alpha a)^{r-1} \left((b-\alpha a) + a \sum_{i=1}^{r} \left(\frac{a}{b-\alpha a}\right)^{i-1} \binom{r}{i} \alpha^i \right) \\ &= (b-\alpha a)^{r-1} \left((b-\alpha a) + (b-\alpha a) \sum_{i=1}^{r} \left(\frac{\alpha a}{b-\alpha a}\right)^{i} \binom{r}{i} \right) \\ &= (b-\alpha a)^{r-1} \left((b-\alpha a) \sum_{i=0}^{r} \left(\frac{\alpha a}{b-\alpha a}\right)^{i} \binom{r}{i} \right) \\ &= (b-\alpha a)^{r} \left(\frac{\alpha a}{b-\alpha a} + 1 \right)^r \\ &= (b-\alpha a)^{r} \left(\frac{b}{b-\alpha a} \right)^r \\ &= b^r, \end{aligned}$$

which coincides with the case $\alpha a = b \neq 0$. Since $\begin{vmatrix} w_1 & w_0 \\ w_2 & w_1 \end{vmatrix} = b^2 - \alpha ab - \beta a^2$, we deduce from Identities (22), (23) and Lemmas 10, 11 that

$$\det(W_{r,n}) = (-1)^{n+\lfloor (r+3)/2 \rfloor} \alpha^{nr+r^2} \beta^{n+r} b^r (b^2 - \alpha ab - \beta a^2).$$

□

Corollary 13. *Let $n \geq 0$ and $r \geq 0$ be integers. Then*

$$\begin{vmatrix} F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+1}^{(r)} \\ F_{n+1}^{(r)} & F_{n+2}^{(r)} & \cdots & F_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r+2}^{(r)} & \cdots & F_{n+2r+2}^{(r)} \end{vmatrix} = (-1)^{n+\lfloor (r+3)/2 \rfloor}.$$

Proof. Follows from Identity (21) for $a = 0$ and $b = \alpha = \beta = 1$. □

Corollary 14. *Let $n \geq 0$ and $r \geq 0$ be integers. Then*

$$\begin{vmatrix} L_n^{(r)} & L_{n+1}^{(r)} & \cdots & L_{n+r+1}^{(r)} \\ L_{n+1}^{(r)} & L_{n+2}^{(r)} & \cdots & L_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+r+1}^{(r)} & L_{n+r+2}^{(r)} & \cdots & L_{n+2r+2}^{(r)} \end{vmatrix} = 5(-1)^{n+\lfloor (r+1)/2 \rfloor}.$$

Proof. Follows from Identity (21) for $a = 2$ and $b = \alpha = \beta = 1$. □

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