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On Landau's Inequality for the Prime Counting Function

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Abstract

We prove that the inequality $2\pi(n) - \pi(2n) \geq 2\omega(n)$ is valid for all $n \geq 71$. Here, $\pi(n)$ denotes the prime counting function and $\omega(n)$ denotes the number of distinct prime factors of n. Our inequality refines a recently published result by Zhang.

1 Introduction and statement of the main result

In 1909, Landau [4] conjectured that the inequality

$$\pi(2n) \leq 2\pi(n) \tag{1}$$

is valid for all integers $n \geq 2$. Here, $\pi(n)$ denotes the number of primes which are less than or equal to n. The first proof of (1) was given by Rosser and Schoenfeld [11] in 1966. Moreover, they showed that the sign of equality holds in (1) if and only if $n \in \{2, 4, 10\}$.

Landau's inequality attracted the attention of several mathematicians, who presented various extensions and counterparts of (1). For more information on this subject we refer to Ehrhart [3], Mitrinović, Sándor, Crstici [6, Chapter VII], Panaitopol [7, 8, 9], and Vlamos [12].

Our work was inspired by an interesting paper published by Zhang [13] in 2020. He obtained a positive lower bound for the difference $2\pi(n) - \pi(2n)$. More precisely, he proved that for $n \geq 59$,

$$2\pi(n) - \pi(2n) > \omega(2n), \tag{2}$$

where $\omega(n)$ denotes the number of distinct prime factors of n. Here, we offer the following improvement of (2) for $n \geq 71$.

Theorem 1. *Let $n \geq 71$ be an integer. Then*

$$2\pi(n) - \pi(2n) \geq 2\omega(n), \tag{3}$$

with equality if and only if $n \in \{78, 100, 102, 126\}$.

We note that $n = 70$ is the largest integer such that (3) is not true. In the next section, we present a proof of Theorem 1 and we show that (3) refines (2).

2 Proof of Theorem 1

Proof. Let

$$F(n) = 2\pi(n) - \pi(2n) - 2\omega(n). \tag{4}$$

We consider two cases.

Case 1. $71 \leq n \leq 30091$.

We used MAPLE 16 and the following computer program to verify (3).

```
with(NumberTheory):
w := n -> nops(PrimeFactors(n)):
F := n -> 2*pi(n)-pi(2*n)-2*w(n):
for k from 71 to 30091 do
   if F(n) <= 0 then print(k,F(n)) end if
end do;
```

For all n we obtain that $F(n)$ is positive with exactly four exceptions:

$$F(78) = F(100) = F(102) = F(126) = 0.$$

Case 2. $n \geq 30092$.

We apply the estimates

$$\frac{n}{\log(n)-1} \leq \pi(n) \quad (n \geq 5393), \tag{5}$$

$$\pi(n) \leq \frac{n}{\log(n)-1.1} \quad (n \geq 60184), \tag{6}$$

$$\omega(n) \leq c\frac{\log(n)}{\log(\log(n))} \quad (c = 1.3841;\ n \geq 3). \tag{7}$$

The inequalities (5) and (6) are due to Dusart [2], whereas (7) was proved by Robin [10]. Better bounds for $\pi(n)$ were given by Berkane and Dusart [1].

Let $F(n)$ be the function defined in (4). Using (5), (6) and (7) gives for $n \geq 30092$,

$$F(n) \geq \frac{2n}{\log(n)-1} - \frac{2n}{\log(2n)-1.1} - \frac{2c\log(n)}{\log(\log(n))} = G(n), \quad \text{say.} \tag{8}$$

We set $n = e^x$ with $x \geq \log(30092) \approx 10.31$ and $a = 1.1 - \log(2) \approx 0.40$. Then,

$$\frac{1}{2}G(n) = \frac{1}{2}G(e^x) = (1-a)\frac{e^x}{(x-1)(x-a)} - c\frac{x}{\log(x)}.$$

Using

$$\frac{x-y}{\log(x)-\log(y)} < \frac{x+y}{2} \quad (0 < y < x),$$

(see Mitrinović [5, p. 273]) with $y = 1$ gives

$$\begin{aligned}
\frac{1}{2(1-a)}(x-1)(x-a)G(e^x) &= e^x - \frac{c}{1-a}x(x-a)\frac{x-1}{\log(x)} \\
&> e^x - \frac{c}{1-a}x(x-a)\frac{x+1}{2} \\
&> 1 + \sum_{k=1}^{5}\frac{x^k}{k!} - \frac{6}{5}x(x+1)\Big(x-\frac{2}{5}\Big) \\
&= 1 + \frac{x}{600}P(x)
\end{aligned}$$

with

$$P(x) = 5x^4 + 25x^3 - 620x^2 - 132x + 888.$$

Since P is positive on $[10,\infty)$, we obtain $G(e^x) > 0$ for $x \geq \log(30092)$. From (8) we conclude that $F(n) > 0$ for $n \geq 30092$. □

Finally, we show that (3) improves Zhang's inequality (2).

Lemma 2. *For all integers $n \geq 2$, we have*

$$2\omega(n) \geq \omega(2n). \tag{9}$$

Equality holds in (9) if and only if $n = p^k$, where p is an odd prime number and k is a positive integer.

Proof. Let

$$n = \prod_{j=1}^{r} {p_j}^{k_j},$$

where $p_1, \cdots, p_r$ are prime numbers with $p_1 < \cdots < p_r$ and $k_1, \cdots, k_r$ are positive integers. If $p_1 = 2$, then

$$2\omega(n) - \omega(2n) = 2r - r = r > 0,$$

and if $p_1 > 2$, then

$$2\omega(n) - \omega(2n) = 2r - (r + 1) = r - 1 \geq 0,$$

with equality if and only if $n = {p_1}^{k_1}$. □

Let $n \geq 71$. If $n = p^k$, where $p \geq 3$ is a prime number and $k \geq 1$ is an integer, then $n \notin \{78, 100, 102, 126\}$. From Theorem 1 and Lemma 2 we conclude that

$$2\pi(n) - \pi(2n) > 2\omega(n) = \omega(2n).$$

And, if $n \neq p^k$, then

$$2\pi(n) - \pi(2n) \geq 2\omega(n) > \omega(2n).$$

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