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An Extension of the Euclid-Euler Theorem to Certain α-Perfect Numbers

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Abstract

In a posthumously published work, Euler proved that all even perfect numbers are of the form $2^{p-1}(2^p-1)$, where 2^p-1 is a prime number. In this article, we extend Euler's method for certain α-perfect numbers for which Euler's result can be generalized. In particular, we use Euler's method to prove that if N is a 3-perfect number divisible by 6; then either $2 \parallel N$ or $3 \parallel N$. As well, we prove that if N is a $\frac{5}{2}$-perfect number divisible by 5, then $2^4 \parallel N$, $5^2 \parallel N$, and $31^2 \mid N$. Finally, for $p \in \{17, 257, 65537\}$, we prove that there are no $\frac{2p}{p-1}$-perfect numbers divisible by p.

1 Introduction

A perfect number is equal to the sum of its proper divisors. These numbers have been studied since the ancient Greek times. An old known result is Proposition 36 of Book IX

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in the *Elements*, where Euclid showed that if $2^p - 1$ is a prime number then $2^{p-1}(2^p - 1)$ is perfect. Around two thousand years later, L. Euler [2, p. 630] proved that every even perfect number is of Euclid's form. In the same page, there is Euler's formula for an odd perfect number. If $\sigma(N) = 2N$ and N is odd, then

$$N = p^e k^2, \tag{1}$$

where p is a prime number, $(p, k) = 1$, and $p \equiv e \equiv 1 \pmod 4$.

Let σ to be the sum of divisors function and

$$I(N) = \frac{\sigma(N)}{N}$$

to be the *index function* of N. Clearly, $I(N)$ is multiplicative. For p a prime number and a a non-negative integer, the index function $I(p^a)$ is monotonically increasing in a but monotonically decreasing in p. Also, for any prime number p and any positive integer a, we have

$$\frac{p+1}{p} \leq I(p^a) < \frac{p}{p-1}. \tag{2}$$

Moreover, if $I(N) = \frac{c}{d}$ and $\gcd(c, d) = 1$, we have

$$\text{if } p \text{ is a prime number and } p \mid d, \text{ then } p \mid N; \tag{3}$$

and

$$\text{if } p \text{ is a prime number and } p \mid c, \text{ then } p \mid \sigma(N). \tag{4}$$

For any rational number α, a positive integer N is an α-*perfect* number if $I(N) = \alpha$. We say α is an *abundancy index* if $\alpha \in \mathrm{Im}(I)$. The case $\alpha = 2$ corresponds to a *perfect number* and, when α is an integer, N is a *multiperfect number*. More than 5700 multiperfect numbers have been found (see Achim Flammenkamp's The Multiply Perfect Numbers Page [3]). Many other rational values for α were studied, some of them with connections with odd perfect numbers [7, 11]. When a rational α is not in the image of $I(N)$, α is called an *abundancy outlaw*. Holdener and Stanton [5] gave a list of abundancy outlaws as well as rational numbers for which their abundancy status is unknown. Given a prime p, we say that a rational α is a p-*abundancy outlaw* if there are no α-perfect numbers divisible by p.

We will present a generalization of Euler's result on perfect numbers that can be applied to certain α-perfect numbers. In particular, it can be applied to the 3-perfect numbers and $\frac{5}{2}$-perfect numbers. For $p \in \{17, 257, 65537\}$, we also prove that $\frac{2p}{p-1}$ is a p-abundancy outlaw.

The only known 3-perfect numbers are the following

$$2^3 \cdot 3 \cdot 5,\ 2^5 \cdot 3 \cdot 7,\ 2^9 \cdot 3 \cdot 11 \cdot 31,\ 2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127,$$

$$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73, \text{ and } 2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151.$$

Dickson [1, p. 33] gives a historical account of the 3-perfect numbers. It is conjectured the numbers above are the only 3-perfect numbers. Clearly, N is a 3-perfect number with $2 \parallel N$

if and only if $\frac{N}{2}$ is an odd perfect number. Therefore, if the above list of 3-perfect numbers is complete, then there are no odd perfect numbers. Also, it is easy to see that $\frac{5}{2}$ is not an abundancy outlaw, since $I(24) = \frac{5}{2}$.

In order to apply our generalization of the Euclid-Euler theorem we will need to find all solutions of certain two and three-variable exponential diophantine equations. We will use the general method developed by Styer [10] which formalizes and extends a method used by Guy, Lacampagne, and Selfridge [4] (also see the paper [8] for similar results).

2 Generalizing Euler's method

Let α be a rational number and $N > 1$ be an α-perfect number, then $I(N) = \alpha$. Using (2) and the multiplicativity of $I(N)$, there exist positive integers r and m, prime numbers p_i, and positive integers a_i, with $1 \leq i \leq r$; such that

$$N = m \prod_{i=1}^{r} p_i^{a_i}, \tag{5}$$

$$\alpha \prod_{i=1}^{r} \frac{p_i - 1}{p_i} \leq 1, \tag{6}$$

and

$$\gcd\left(m, \prod_{i=1}^{r} p_i^{a_i}\right) = 1.$$

Therefore,

$$\alpha m \prod_{i=1}^{r} p_i^{a_i} = \sigma(N) = \sigma(m) \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

Hence, for some integers k and d, we have

$$\sigma(m) = \frac{\beta k}{d} \prod_{i=1}^{r} p_i^{a_i+1} \tag{7}$$

and

$$m = \frac{k}{d} \prod_{i=1}^{r} \left(p_i^{a_i+1} - 1\right), \tag{8}$$

where

$$\beta = \alpha \prod_{i=1}^{r} \frac{p_i - 1}{p_i} \leq 1$$

and

$$d = \gcd\left(\beta \prod_{i=1}^{r} p_i^{a_i+1}, \prod_{i=1}^{r} \left(p_i^{a_i+1} - 1\right)\right).$$

We will try to find a lower bound for $\sigma(m)$, by summing divisors of m that are explicitly indicated in Eq. (8). A comparison of the lower bound with Eq. (7), will give us contradictions or conditions on the form of N. From now on, we will always consider $\beta = 1$.

3 The Euclid-Euler theorem: $r = 1$ and $\alpha = \frac{p}{p-1}$

Suppose $N = p^a m$ is an α-perfect number, where p is a prime number and $\alpha = \frac{p}{p-1}$ satisfies Condition (6) with equality. Then $I(N) = \alpha$.

By (3), if $p \neq 2$ then $2 \mid N$. Therefore,

$$I(N) \geq \frac{3}{2}\frac{p+1}{p} > \frac{p}{p-1} = \alpha.$$

Hence, $p = 2$ and we have the case studied by Euler [2]. The Euler theorem is a well known result, but we write its proof here to give context for our results.

Theorem 1 (Euler)**.** *If N is an even perfect number, then $N = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are prime numbers.*

Proof. Suppose $N = 2^a m$ and $\sigma(N) = 2N$. Then

$$2^{a+1}m = \sigma(N) = (2^{a+1} - 1)\sigma(m).$$

Since $\gcd(2^{a+1}-1, 2^{a+1}) = 1$, then there exists a positive integer k such that $m = k(2^{a+1}-1)$ and $\sigma(m) = k2^{a+1}$. Since k and m divide m, and $\sigma(m) = k + m$, then $2^{a+1} - 1$ is a prime number and $k = 1$. Now, $2^{a+1} - 1$ can only be a prime number if $a + 1$ is a prime number p. Therefore, $N = 2^{p-1}(2^p - 1)$. ∎

4 3-perfect numbers divisible by 6: $r = 2$ and $\alpha = 3$

In this section, we will consider $r = 2$, $p_1 = 2$, and $p_2 = 3$ in Eq. (5). We will prove that if a 3-perfect number N is divisible by 6, then either $2 \parallel N$ or $3 \parallel N$. Although Steuerwald [9] prove this result in 1954, we will use the method introduced in Section 2. Before we proceed, we need results about the difference between powers of 2 and 3. These results are special cases of Pillai's conjecture and Mihăilescu's theorem [6], which proves the Catalan conjecture. It is worth noting that the solutions of equation $|2^a - 3^b| = 1$ were first determined by Levi ben Gershon, also known as Gersonides (1288–1344), in his treatise *De Numeris Harmonicis*.

Lemma 2. *The only solutions of the diophantine equation*

$$2^a - 3^b = -1 \tag{9}$$

are $(1,1)$ and $(3,2)$. Also, the only solutions of the diophantine equation

$$2^a - 3^b = 2^c - 1, \tag{10}$$

are $(2,1,1)$, $(4,2,3)$, and $(a,0,a)$, for all $a \in \mathbb{Z}$.

Proof. We start by solving the equation $|2^a - 3^b| = 1$. Since 2 is a primitive root of 3^b, for any $b \geq 1$; then the solutions of $2^x \equiv \pm 1 \pmod{3^b}$ are $x = 3^{b-1}k$, for $k \geq 0$. But, for $b \geq 3$ we have
$$2^{3^{b-1}} > 2 \cdot 3^b.$$
Therefore, $b \in \{0, 1, 2\}$ and so we obtain the solutions $(a, 0, a)$ and $(2, 1, 1)$ of Eq. (10). Also, we obtain the solutions $(1, 1)$ and $(3, 2)$ of Eq. (9).

Now, suppose $c \geq 2$. If c is even, then $3 \mid 2^c - 1$ and we obtain a contradiction. If c is odd, we have $2^a \equiv 1 \pmod 3$ so a is even. Therefore, there exists a positive integer a' such that $a = 2a'$. As $c = 2c' + 1$, for some positive integer c', we have $2^{2a'} - 3^b = 2^{2c'+1} - 1$. So $(-1)^{b+1} \equiv -1 \pmod 4$. Hence, b is even and we have $b = 2b'$, for some non-negative integer b'. Then we have
$$2^{2a'} - 3^{2b'} = 2^{2c'+1} - 1.$$
Therefore,
$$(-1)^{a'} - (-1)^{b'} \equiv 2 \cdot (-1)^{c'} - 1 \pmod 5.$$
If a' and b' have the same parity we have a contradiction. If a' is odd and b' is even we have a contradiction as well. Therefore, a' is even and b' is odd. So we have $4 \mid a$ and $b \equiv 2 \pmod 4$, which implies $1 - (-1) \equiv 2 \cdot (-1)^{c'} - 1 \pmod 5$. Hence, c' is odd and $c \equiv 3 \pmod 4$. Since $c > 1$, then $a > c$. Therefore, we have $2^c(2^{a-c} - 1) = 3^b - 1$. Hence, $2^c \parallel 3^b - 1$.

Since $b \equiv 2 \pmod 4$, then $2^3 \parallel 3^b - 1$. Hence, $c = 3$ and we can rewrite Eq. (10) as
$$\left(2^{\frac{a}{2}} - 3^{\frac{b}{2}}\right)\left(2^{\frac{a}{2}} + 3^{\frac{b}{2}}\right) = 7.$$

Therefore, $a = 4$ and $b = 2$. Hence, if $c \geq 2$, the only solution of Eq. (10) is $(4, 2, 3)$. Therefore, we obtain the stated result, $\square$

We can now state and give a new proof of Steuerwald's theorem on 3-perfect numbers.

Theorem 3. *Suppose N is a 3-perfect number of the form $N = 2^a\, 3^b m$, where $a, b \geq 1$ and* $\gcd(6, m) = 1$. *Then $a = 1$ and $b \neq 1$, or $a \neq 1$ and $b = 1$.*

Proof. Let $N = 2^a\, 3^b m$ such that $a, b \geq 1$, $\gcd(6, m) = 1$, and $\sigma(N) = 3N$. Then
$$3N = 2^a\, 3^{b+1} m = \sigma(N) = (2^{a+1} - 1)\, \frac{3^{b+1} - 1}{2}\, \sigma(m).$$

Therefore,
$$\frac{\sigma(m)}{m} = \frac{2^{a+1}\, 3^{b+1}}{(2^{a+1} - 1)(3^{b+1} - 1)}.$$
Let $d = \gcd\left(2^{a+1}\, 3^{b+1}, (2^{a+1} - 1)(3^{b+1} - 1)\right)$. It is easy to see that $d = 2^s\, 3^t$, where $1 \leq s \leq a + 1$ and $0 \leq t \leq b + 1$. Since
$$\gcd\left(\frac{2^{a+1} 3^{b+1}}{2^s\, 3^t}, \frac{(2^{a+1} - 1)(3^{b+1} - 1)}{2^s\, 3^t}\right) = 1,$$

then

$$\sigma(m) = \frac{2^{a+1}\,3^{b+1}}{2^s\,3^t}k \text{ and } m = \frac{2^{a+1}-1}{3^t}\,\frac{3^{b+1}-1}{2^s}k,$$

for some positive integer k.

Let us consider the following three cases, which will establish the claim.

Case A: Suppose that $t \neq 0$ and let

$$M = \max\left(\frac{2^{a+1}-1}{3^t}, \frac{3^{b+1}-1}{2^s}\right).$$

Then we have

$$\begin{aligned}
\frac{\sigma(m)}{k} &= \frac{2^{a+1}3^{b+1}}{2^s\,3^t} \\
&= \frac{((2^{a+1}-1)+1)\left(\left(3^{b+1}-1\right)+1\right)}{2^s\,3^t} \\
&= \frac{2^{a+1}-1}{3^t}\,\frac{3^{b+1}-1}{2^s} + \frac{2^{a+1}-1}{3^t}\,\frac{1}{2^s} + \frac{3^{b+1}-1}{2^s}\,\frac{1}{3^t} + \frac{1}{2^s\,3^t} \\
&< \frac{m}{k} + \frac{M}{2} + \frac{M}{3} + 1 \\
&< \frac{m}{k} + M + 1.
\end{aligned}$$

Therefore,

$$\sigma(m) < m + Mk + k. \tag{11}$$

By definition of M, we have $m = uMk$, with $u \in \left\{\frac{2^{a+1}-1}{3^t}, \frac{3^{b+1}-1}{2^s}\right\}$. By definition of s and t, we have $u \in \mathbb{Z}$. Then $Mk \mid m$. If $Mk \neq m$ and $M \neq 1$, then m, Mk, and k are different divisors of m. Thus,

$$\sigma(m) \geq m + Mk + k. \tag{12}$$

The combination of inequalities (11) and (12), give us a contradiction.

If $Mk = m$ or $M = 1$, then

$$\frac{2^{a+1}-1}{3^t} = 1 \text{ or } \frac{3^{b+1}-1}{2^s} = 1.$$

Therefore, $2^{a+1} - 3^t = 1$ or $3^{b+1} - 2^s = 1$. By Lemma 2, we have $a = 1$ or $b = 1$.

Case B: Suppose that $t = 0$ and

$$2^{a+1} - 1 \neq \frac{3^{b+1}-1}{2^s}.$$

Let

$$M' = \min\left(2^{a+1} - 1, \frac{3^{b+1} - 1}{2^s}\right).$$

If $M' = 1$ then

$$2^{a+1} - 1 = 1 \text{ or } \frac{3^{b+1} - 1}{2^s} = 1.$$

Therefore, $2^{a+1} = 2$ or $3^{b+1} - 2^s = 1$. As $a, b \geq 1$, by Lemma 2, we conclude that $b = 1$.

If $M' \neq 1$ then

$$m, \left(2^{a+1} - 1\right) k, \frac{3^{b+1} - 1}{2^s} k, \text{ and } k,$$

are different divisors m. Therefore,

$$\begin{aligned}
\sigma(m) &\geq m + (2^{a+1} - 1)k + \frac{3^{b+1} - 1}{2^s} k + k \\
&> (2^{a+1} - 1) \frac{3^{b+1} - 1}{2^s} k + \frac{2^{a+1} - 1}{2^s} k + \frac{3^{b+1} - 1}{2^s} k + \frac{k}{2^s} \\
&= \sigma(m),
\end{aligned}$$

where the strict inequality results from $s \geq 1$. So, we obtain a contradiction.

Case C: Suppose that $t = 0$ and

$$2^{a+1} - 1 = \frac{3^{b+1} - 1}{2^s}.$$

Then $2^{a+1+s} - 3^{b+1} = 2^s - 1$. By Lemma 2, we have $(a, b) \in \{(0, 0), (0, 1)\}$. Since $a, b \geq 1$ we obtain a contradiction.

Hence, we must have $a = 1$ or $b = 1$.

Next, we prove that we cannot have $a = 1$ and $b = 1$, simultaneously.

If $a = 1$, let $N = 2N'$, where N' is odd. Then

$$6N' = 3N = \sigma(N) = 3\sigma(N')$$

Therefore, $\sigma(N') = 2N'$. So N' is an odd perfect number. By Euler's formula (1), if $3 \mid N'$ then $3^2 \mid N'$. Hence, we cannot have $b = 1$. Thus, we have $a = 1$ and $b \neq 1$, or $a \neq 1$ and $b = 1$. $\square$

5 Fermat prime numbers: $r = 2$ and $\alpha = \frac{2p}{p-1}$

In Eq. (5), if $r = 2$, $p_1 = 2$, and $p_2 \neq 3$, then α is not an integer. Moreover, if p_2 is not a Fermat number, then we must have a prime $p \notin \{p_1, p_2\}$ such that $p \mid \frac{p_2-1}{2}$. Using (3), we obtain $p \mid N$. For this reason, and in order to keep $r = 2$ without any other known prime divisors of N, we will only consider the cases where $p_2 \neq 3$ and p_2 is a Fermat prime number. Let $F_n = 2^{2^n} + 1$ denote the n-th Fermat number, for any non-negative integer n. The next result states some properties of Fermat numbers that will be used later.

Lemma 4. *Let n be a non-negative integer and F_n be the n-th Fermat number. Then*

1. $F_{n-1} \mid F_n^{2^n} - 1$, *for* $n \geq 1$*;*

2. $\mathrm{ord}_{2^{2^n+s}}(F_n) = 2^s$, *for* $n \geq 1$ *and* s *a non-negative integer.*

Proof. Let $n \geq 1$. Clearly, $F_n = (F_{n-1} - 1)^2 + 1$. So there exists an integer t, such that

$$\begin{aligned}
F_n^{2^n} - 1 &= \left((F_{n-1} - 1)^2 + 1\right)^{2^n} - 1 \\
&= -1 + \left(\left(F_{n-1}^2 - 2F_{n-1}\right) + 2\right)^{2^n} \\
&= -1 + \sum_{j=0}^{2^n} \binom{2^n}{j} \left(F_{n-1}^2 - 2F_{n-1}\right)^j 2^{2^n - j} \\
&= -1 + tF_{n-1} + 2^{2^n} \\
&= tF_{n-1} + F_0 F_1 \cdots F_{n-1}.
\end{aligned}$$

Hence, $F_{n-1} \mid F_n^{2^n} - 1$.

The statement *2.* is clearly true for $s = 0$. We prove by induction that

$$F_n^{2^{s-1}} \equiv 1 + 2^{2^n+s-1} \pmod{2^{2^n+s}}, \tag{13}$$

for any $s \geq 1$.

Since $F_n \equiv F_n \pmod{2^{2^n+1}}$, the congruence (13) is valid for $s = 1$. Now, suppose (13) is valid for s. Then there exists an integer t such that

$$\begin{aligned}
F_n^{2^s} &= \left(F_n^{2^{s-1}}\right)^2 \\
&= \left(1 + \left(2^{2^n+s-1} + 2^{2^n+s} t\right)\right)^2 \\
&= 1 + 2^{2^n+s} + 2^{2^n+s+1} t + 2^{2^{n+1}+2s-2} + 2^{2^{n+1}+2s} t^2 + 2^{2^{n+1}+2s} t \\
&\equiv 1 + 2^{2^n+s} \pmod{2^{2^n+s+1}}.
\end{aligned}$$

Therefore, Eq. (13) is true for any $s \geq 1$. So we have

$$\mathrm{ord}_{2^{2^n+s}}(F_n) \neq 2^{s-1}.$$

Since $\mathrm{ord}_{2^{2^n+s}}(F_n)$ is a power of 2 and

$$F_n^{2^s} \equiv 1 \pmod{2^{2^n+s}},$$

then statement *2.* is obtained. □

Before applying Euler's method we still need two technical results about some exponential diophantine equations. Our proof of these results is inspired by Styer's work [10].

Lemma 5. *Let $n \geq 1$ and F_n be the n-th Fermat number. Suppose there exists a non-negative integer s and a prime number q_n, such that*

$$q_n \mid F_n^{2^s} - 1 \tag{14}$$

and

$$\mathrm{ord}_{F_n}(2) \mid \mathrm{ord}_{q_n}(2). \tag{15}$$

Then the exponential diophantine equation

$$2^a - F_n^b = 2^c - 1, \tag{16}$$

has no solutions (a, b, c) with $b \geq 1$ and $c \geq 2^n + s$.

Proof. Suppose there exists a non-negative integer s and a prime number q_n satisfying Cond. (14) and (15). Also, suppose that exists a solution (a, b, c) of Eq. (16) with $b \geq 1$ and $c \geq 2^n + s$. Since $b \geq 1$, we have $a > c$. Therefore, we have

$$F_n^b \equiv 1 \pmod{2^{2^n+s}}.$$

By Lemma 4, we have $\mathrm{ord}_{2^{2^n+s}}(F_n) = 2^s$. Hence, $2^s \mid b$. Now, Cond. (14) and Eq. (16) imply $2^a - 1 \equiv 2^c - 1 \pmod{q_n}$. Let $t = \mathrm{ord}_{q_n}(2)$. Then $a \equiv c \pmod{t}$. Hence, exists a' such that $a = c + a't$. But, by Eq. (15), $\mathrm{ord}_{F_n}(2) \mid t$, which implies

$$2^c - 1 \equiv 2^a \equiv 2^c \left(2^t\right)^{a'} \equiv 2^c \pmod{F_n}.$$

Since the previous equation has no solutions, we obtain a contradiction. Thus, we obtain the stated result. $\square$

Lemma 6. *Let F_n be the n-th Fermat number and consider the following exponential diophantine equations*

$$2^a - F_n^b = -1 \tag{17}$$

and

$$2^a - F_n^b = 2^c - 1. \tag{18}$$

Then

(a) when $n \in \{1, 2, 3, 4\}$, Equation (17) only holds for $(a, b) = (2^n, 1)$;

(b) when $n \in \{2, 3, 4\}$, Equation (18) only holds for

$$(a, b, c) \in \{(a, 0, a) \mid a \in \mathbb{Z}\} \cup \{(2^n + 1, 1, 2^n)\};$$

(c) when $n = 1$, Equation (18) only holds for

$$(a, b, c) \in \{(a, 0, a) \mid a \in \mathbb{Z}\} \cup \{(3, 1, 2), (7, 3, 2), (5, 2, 3)\}.$$

Proof. Suppose $n \in \{1,2,3,4\}$, $2^a - F_n^b = -1$, and $a \geq 2^n+1$. Therefore, $2^{2^n+1} \mid F_n^b - 1$. So b is even and we have $b = 2b'$, for some positive integer b'. But, since $3 \mid 5^2-1$, $9 \mid 17^2-1$, $43 \mid 257^2-1$, and $11 \mid 65537^2-1$, we always obtain a contradiction. Therefore, $(2^n, 1)$ is the only solution of Eq. (17), and we obtain statement *(a)*.

Now, we consider Eq. (18) with $n \in \{1,2,3,4\}$. Notice that we always have $a \geq c$. If $b = 0$ then $a = c$ and so we have $(a,b,c) = (a,0,a)$. Since $2^c(2^{a-c}-1) = F_n^b - 1$ and $2^{2^n} \mid F_n^b - 1$, for any positive integer b, we have $c \geq 2^n$. Therefore, if $b = 1$ then $c = 2^n$ and $a = 2^n + 1$.

From now on, we assume $b \geq 2$. Since F_n is a Fermat number, we have $a \geq 2^{n+1}+1$. As $2^n + n < 2^{n+1}$, then Eq. (18) implies

$$F_n^b \equiv 1 - 2^c \pmod{2^{2^n+n}}. \tag{19}$$

By Lemma 4, we have $\mathrm{ord}_{2^{2^n+n}}(F_n) = 2^n$. Then, for each c, exist integers b_c and b_c' such that $b = b_c + 2^n b_c'$. Notice that b_c and b_c' also depend on which n we are considering. By Lemma 4, we have $F_{n-1} \mid F_n^{2^n} - 1$. Hence,

$$2^a - F_n^{b_c} \equiv 2^c - 1 \pmod{F_{n-1}}. \tag{20}$$

As $\mathrm{ord}_{F_{n-1}}(2) = 2^n$, then exist integers a_c and a_c' such that $a = a_c + 2^n a_c'$. But then

$$2^{a_c}(-1)^{a_c'} \equiv 2^c - 1 \pmod{F_n}. \tag{21}$$

We now analyze Eq. (18) for each $n \in \{1,2,3,4\}$.

Suppose $n = 1$.

Since $s = 2$ and $q_1 = 13$ satisfy the conditions of Lemma 5, then Eq. (18) has no solutions for $c \geq 4$. Therefore, $c = 2$ or $c = 3$.

Clearly, $(7,3,2)$ is the only solution of Eq. (18), with $c = 2$ and $a \leq 7$. If $a \geq 8$ and $c = 2$, then $5^b \equiv -3 \pmod{256}$. Therefore, $b \equiv 35 \pmod{64}$, i.e., there exists a non-negative integer b' such that $b = 35 + 64b'$. Then

$$2^a - 5^{35} \cdot 5^{64b'} = 3.$$

As $641 \mid 5^{64} - 1$ and $5^{35} \equiv 516 \pmod{641}$, then

$$2^a \equiv 519 \pmod{641}.$$

But this congruence has no solutions.

If $c = 3$ and $a \leq 5$, then $(5,2,3)$ is the only solution of Eq. (18). Suppose $c = 3$ and $a \geq 6$, then $5^b \equiv -7 \pmod{64}$. Therefore, we have $b \equiv 10 \pmod{16}$; i.e., there exists a non-negative integer b' such that $b = 10 + 16b'$. Then

$$2^a - 5^{10} \cdot 5^{16b'} = 7.$$

As $13 \mid 5^{16} - 1$ and $5^{10} \equiv -1 \pmod{13}$, then

$$2^a \equiv 6 \pmod{13}.$$

Therefore, $a \equiv 5 \pmod{12}$. On the other hand, since $5^{10} \equiv 9 \pmod{17}$ and $17 \mid 5^{16} - 1$, then

$$2^a \equiv -1 \pmod{17}.$$

Hence, we have $a \equiv 4 \pmod 8$, which gives us a contradiction. Thus, we obtain statement *(c)*.

Let us now consider $n = 2$.

Since $s = 4$ and $q_1 = 18913$ satisfy the conditions of Lemma 5, then Eq. (18) has no solutions for $c \geq 8$.

For each $c \in \{4, 5, 6, 7\}$, consider $b = b_c + 2^n b'_c$ to be the solutions of Eq. (19) and $a = a_c + 2^n a'_c$ to be the solutions of Eq. (20) (when these equations have solution). The values of b_c and a_c are given in Table 1.

c	b_c	a_c
4	3	3
5	2	No solutions
6	0	2
7	0	2

Table 1: Solutions of Eq. (19) and Eq. (20) for $n = 2$.

Hence, when $c = 5$, Eq. (20) has no solutions. Also, when $c \in \{4, 6, 7\}$, Eq. (21) has no solutions.

Next, we consider $n = 3$.

Since $s = 4$ and $q_3 = 193$ satisfy the conditions of Lemma 5, then Eq. (18) has no solutions for $c \geq 12$.

For each $c \in \{8, 9, 10, 11\}$, consider $b = b_c + 2^n b'_c$ to be the solutions of Eq. (19) and $a = a_c + 2^n a'_c$ to be the solutions of Eq. (20) (when these equations have solution). The values b_c and a_c are given in table 2.

c	b_c	a_c
8	7	7
9	6	No solutions
10	4	1
11	0	3

Table 2: Solutions of Eq. (19) and Eq. (20) for $n = 3$.

Hence, when $c = 9$, Eq. (20) has no solutions. Also, when $c \in \{8, 10, 11\}$, Eq. (21) has no solutions.

Finally, consider $n = 4$.

Since $s = 5$ and $q_4 = 38899171806337$ satisfy the conditions of Lemma 5, then Eq. (18) has no solutions for $c \geq 21$.

For each $c \in \{16, 17, 18, 19, 20\}$, consider $b = b_c + 2^n b'_c$ to be the solutions of Eq. (19) and $a = a_c + 2^n a'_c$ to be the solutions of Eq. (20) (when these equations have solution). The values b_c and a_c are given in table 3.

c	b_c	a_c
16	15	15
17	14	No solutions
18	12	No solutions
19	8	No solutions
20	0	4

Table 3: Solutions of Eq. (19) and Eq. (20) for $n = 4$.

Hence, when $c \in \{17, 18, 19\}$, Eq. (20) has no solutions. When $c \in \{16, 20\}$, Eq. (21) has no solutions.

Thus, statement *(b)* is obtained. ☐

We are now in conditions to prove a generalization of the Euclid-Euler theorem for $\frac{2F_n}{F_n-1}$-perfect numbers, when $n \in \{1, 2, 3, 4\}$. The proof of the next theorem follows the same steps as the proof of Theorem 3.

Theorem 7. *Let F_n be the n-th Fermat number. Then*

1. *if exists N such that $\frac{\sigma(N)}{N} = \frac{F_1}{2}$ and $F_1 \mid N$, then $2^4 \parallel N, F_1^2 \parallel N$, and $31^2 \mid N$.*
2. *if $n \in \{2, 3, 4\}$ then $\frac{2F_n}{F_n-1}$ is a F_n-abundancy outlaw.*

Proof. Let $n \in \{1, 2, 3, 4\}$ and $F_n = 2^{2^n} + 1$. We can write $N = 2^a F_n^b m$ such that $a, b \geq 1$, $\gcd(2F_n, m) = 1$, and

$$\sigma(N) = \frac{2F_n}{F_n - 1} N.$$

Then

$$\frac{2F_n}{F_n - 1} = \frac{\sigma(N)}{N} = \frac{2^{a+1} - 1}{2^a} \frac{F_n^{b+1} - 1}{F_n^b (F_n - 1)} \frac{\sigma(m)}{m}.$$

Therefore,

$$\frac{\sigma(m)}{m} = \frac{2^{a+1} F_n^{b+1}}{(2^{a+1} - 1)(F_n^{b+1} - 1)}.$$

Let

$$d = \gcd\left(2^{a+1} F_n^{b+1}, (2^{a+1} - 1)(F_n^{b+1} - 1)\right).$$

As $F_n - 1 \mid F_n^{b+1} - 1$, then $d = 2^s\, F_n^t$, where $2^n \le s \le a+1$ and $0 \le t \le b+1$. Since

$$\gcd\left(\frac{2^{a+1}F_n^{b+1}}{2^s\,F_n^t}, \frac{(2^{a+1}-1)(F_n^{b+1}-1)}{2^s\,F_n^t}\right) = 1,$$

then

$$\sigma(m) = \frac{2^{a+1}F_n^{b+1}}{2^s\,F_n^t}k \text{ and } m = \frac{2^{a+1}-1}{F_n^t}\,\frac{F_n^{b+1}-1}{2^s}k,$$

for some positive integer k.

Let us consider the following three cases, which will establish the claim.

Case A: Suppose that $t \neq 0$ and let

$$M = \max\left(\frac{2^{a+1}-1}{F_n^t}, \frac{F_n^{b+1}-1}{2^s}\right).$$

Then we have that

$$\begin{aligned}
\frac{\sigma(m)}{k} &= \frac{2^{a+1}F_n^{b+1}}{2^s\,F_n^t} \\
&= \frac{2^{a+1}-1}{F_n^t}\,\frac{F_n^{b+1}-1}{2^s} + \frac{2^{a+1}-1}{F_n^t}\,\frac{1}{2^s} + \frac{F_n^{b+1}-1}{2^s}\,\frac{1}{F_n^t} + \frac{1}{2^s\,F_n^t} \\
&< \frac{m}{k} + \frac{M}{F_n-1} + \frac{M}{F_n} + 1 \\
&< \frac{m}{k} + M + 1.
\end{aligned}$$

Therefore,

$$\sigma(m) < m + Mk + k. \tag{22}$$

If $Mk \neq m$ and $M \neq 1$, then m has at least the divisors m, Mk, and k. Thus,

$$\sigma(m) \ge m + Mk + k. \tag{23}$$

The combination of inequalities (22) and (23), give us a contradiction.

If $Mk = m$ or $M = 1$, then

$$\frac{2^{a+1}-1}{F_n^t} = 1 \text{ or } \frac{F_n^{b+1}-1}{2^s} = 1.$$

Thus $2^{a+1} - F_n^t = 1$ or $F_n^{b+1} - 2^s = 1$. By Lemma 6, the only solutions of these equations are $(a,t) = (0,0)$ and $(b,s) = (0,2^n)$. Hence, we obtain a contradiction because $a, b \ge 1$.

Case B: Suppose $t = 0$ and

$$2^{a+1} - 1 \neq \frac{F_n^{b+1}-1}{2^s}.$$

Let

$$M' = \min\left(2^{a+1} - 1, \frac{F_n^{b+1} - 1}{2^s}\right).$$

If $M' = 1$ then

$$2^{a+1} - 1 = 1 \text{ or } \frac{F_n^{b+1} - 1}{2^s} = 1.$$

Thus $2^{a+1} = 2$ or $F_n^{b+1} - 2^s = 1$. As $a, b \geq 1$, by Lemma 6, we have a contradiction.

If $M' \neq 1$ then m has at least the divisors

$$m, \left(2^{a+1} - 1\right) k, \frac{F_n^{b+1} - 1}{2^s} k \text{ and } k.$$

Therefore,

$$\begin{aligned}\sigma(m) &\geq m + (2^{a+1} - 1)k + \frac{F_n^{b+1} - 1}{2^s} k + k \\ &> (2^{a+1} - 1)\frac{F_n^{b+1} - 1}{2^s} k + \frac{2^{a+1} - 1}{2^s} k + \frac{F_n^{b+1} - 1}{2^s} k + \frac{k}{2^s} \\ &= \sigma(m),\end{aligned}$$

where the strict inequality results from $s \geq 2^n$. Hence, we obtain a contradiction.

Case C: Suppose $t = 0$ and

$$2^{a+1} - 1 = \frac{F_n^{b+1} - 1}{2^s}.$$

Then $2^{a+1+s} - F_n^{b+1} = 2^s - 1$. By Lemma 6, we have

$$\begin{aligned}(F_n, a, b, s) \in \{&(5, -1, -1, 1),\ (5, 0, 0, 2),\ (5, 1,1, 3),\ (5, 4, 2, 2),\ (17, 0, 0, 4), \\ &(257, 0, 0, 8),\ (65537, 0, 0, 16)\}.\end{aligned}$$

Since $a, b \geq 1$, we only have solutions for $F_n = 5$ and then $(F_n, a, b) = (5, 1, 1)$ or $(F_n, a, b) = (5, 4, 2)$.

If $(F_n, a, b) = (5, 1, 1)$ then

$$\frac{\sigma(N)}{N} = \frac{5}{2} = \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{\sigma(m)}{m}.$$

Therefore, $9 \mid m$. But then

$$\frac{\sigma(N)}{N} = \frac{5}{2} \geq \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{13}{9} > \frac{5}{2}.$$

Hence, $(F_n, a, b) = (5, 4, 2)$ and so $31^2 \mid N$.

$\square$

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