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On a Two-Parameter Family of Generalizations of Pascal's Triangle

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Abstract

We consider a two-parameter family of triangles whose (n, k)-th entry (counting the initial entry as the $(0, 0)$-th entry) is the number of tilings of N-boards (which are linear arrays of N unit square cells for any nonnegative integer N) with unit squares and $(1, m-1; t)$-combs for some fixed $m = 1, 2, \ldots$ and $t = 2, 3, \ldots$ that use n tiles in total of which k are combs. A $(1, m-1; t)$-comb is a tile composed of t unit square sub-tiles (referred to as teeth) placed so that each tooth is separated from the next by a gap of width $m - 1$. We show that the entries in the triangle are coefficients of the product of two consecutive generalized Fibonacci polynomials each raised to some nonnegative integer power. We also present a bijection between the tiling of an $(n + (t-1)m)$-board with k $(1, m-1; t)$-combs with the remaining cells filled with squares and the k-subsets of $\{1, \ldots, n\}$ such that no two elements of the subset differ by a multiple of m up to $(t-1)m$. We can therefore give a combinatorial proof of how the number of such k-subsets is related to the coefficient of a polynomial. We also derive a recursion relation for the number of closed walks from a particular node on a class of directed pseudographs and apply it obtain an identity concerning the $m = 2$, $t = 5$ instance of the family of triangles. Further identities of the triangles are also established mostly via combinatorial proof.

1 Introduction

In a recent paper [3], that we will henceforth refer to as AE22, we considered two one-parameter families of generalizations of Pascal's triangle. Regarding the triangles as lower triangular matrices, the members of both families have ones in the leftmost column and the repetition of 1 followed by $m-1$ zeros along the leading diagonal, where m is a positive integer. In the case of the first family, the rest of the entries are obtained using Pascal's recurrence, i.e., $\binom{n}{k}_m = \binom{n-1}{k}_m + \binom{n-1}{k-1}_m$, where $\binom{n}{k}_m$ is the (n,k)-th entry (counting the first entry as being in row $n=0$ and column $k=0$) of the m-th triangle of the family. We showed that this is equivalent to the triangles being row-reversed $(1/(1-x^m), x/(1-x))$ Riordan arrays. A $(p(x), q(x))$ *Riordan array*, where $p(x) = p_0 + p_1x + p_2x^2 + \cdots$ and $q(x) = q_1x + q_2x^2 + \cdots$, is an infinite lower triangular matrix whose (n,k)-th entry is the coefficient of x^n in the series expansion of $p(x)(q(x))^k$ [13, 4]. The row-reversed version of a Riordan array has the entries up to and including the leading diagonal in each row placed in reverse order [3].

The main focus of AE22 was on a second family of triangles whose (n,k)-th entry (denoted by $\left\langle {n \atop k} \right\rangle_m$) is the number of ways to tile N-boards (which are linear arrays of $N \geq 0$ unit square cells) using k $(1, m-1)$-fences and $n-k$ squares (and thus n tiles in total). A $(1, m-1)$-*fence* is a tile composed to two unit-square sub-tiles separated by a gap of width $m-1$ [5, 6]. The two families of triangles coincide for $m = 1, 2$ and the $m = 1$ case is Pascal's triangle, i.e., $\binom{n}{k}_1 = \left\langle {n \atop k} \right\rangle_1 = \binom{n}{k}$ and $\binom{n}{k}_2 = \left\langle {n \atop k} \right\rangle_2$ for all n and k. We showed that for $j \geq 0$, $k \geq 0$, $m \geq 1$, and $r = 0, \ldots, m-1$, the entry $\left\langle {mj+r-k \atop k} \right\rangle_m$ is the coefficient of x^k in $f_j^{m-r}(x) f_{j+1}^r(x)$, where in this instance the Fibonacci polynomial $f_n(x)$ is defined by $f_n(x) = f_{n-1}(x) + x f_{n-2}(x) + \delta_{n,0}$, $f_{n<0}(x) = 0$, where $\delta_{i,j}$ is 1 if $i = j$ and zero otherwise. By first identifying a bijection between the tilings of an $(n+m)$-board with k $(1, m-1)$-fences and $n+m-2k$ squares and the subsets of $\mathbb{N}_n = \{1, \ldots, n\}$ containing k elements none of which differ from another element in the subset by m, we showed that the number of such subsets, $S^{(m)}(n,k) = \left\langle {n+m-k \atop k} \right\rangle_m$. We thus arrived at a combinatorial proof of the relation between $S^{(m)}(n,k)$ and the coefficient of x^k in the product of nonnegative integer powers of two successive Fibonacci polynomials.

Here we generalize the second family of triangles by considering the analogous n-tile tilings of N-boards with $(1, m-1; t)$-combs and squares for positive integer m and $t = 2, 3, \ldots$. A $(w, g; t)$-*comb* contains t sub-tiles of dimensions $w \times 1$ (referred to as *teeth*) separated from one another by gaps of width g [2]. A $(1, m-1; 2)$-comb is evidently a $(1, m-1)$-fence and so the $t = 2$ instances of the triangles we introduce here coincide with the second family of triangles in AE22. Our choice of this particular way to generalize the second family stems from the result that the additional combinatorial interpretation of entries in the triangle as numbers of k-subsets of $\mathbb{N}_n$ satisfying rules on disallowed differences also applies in the case of the generalized triangles.

After introducing the two-parameter family of triangles (along with a less compact version of the triangles which is used in some proofs) in §2, we show how entries of the triangles are related to some generalized Fibonacci polynomials in §3. Then, in §4, we give a bijection between the tilings of an $(n+(t-1)m)$-board with k $(1, m-1; t)$-combs and $n+(t-1)m-kt$ squares and the k-subsets of $\mathbb{N}_n$ such that no two elements of the subset differ by any element of the set $\{m, 2m, \ldots, (t-1)m\}$. This enables us to relate the number of such subsets to coefficients of products of powers of two successive generalized Fibonacci polynomials. The remainder of the paper concerns finding identities satisfied by entries in the triangle. Most of the identities are obtained via the enumeration of metatiles with a certain length or number of tiles, which can be problematic if the metatiles contain an arbitrary number of tiles. A *metatile* is a gapless grouping of tiles that completely covers a whole number of cells and cannot be split into smaller metatiles. In most cases, there are infinitely many possible metatiles and there have been various approaches to the enumeration problem: obtaining the symbolic representation of all the families of metatiles [7], obtaining a recursion relation for the number of metatiles of a certain length and thus expressing the number in terms of a known sequence [8], identifying a bijection between the metatiles and a set of objects whose number is known [2], and constructing a directed pseudograph (that we refer to as a digraph) to represent the placing of tiles [6]. We will use the first and last of these approaches and these are described further in §5. Recursion relations for numbers of tilings corresponding to a particular class of digraph are derived in the appendix and these are used to obtain identities for the $m = 2$, $t = 5$ triangle in §6 where further identities concerning the triangles are also derived, mostly via combinatorial proof.

2 The two-parameter family of triangles

For $m = 1, 2, \ldots$ and $t = 2, 3, \ldots$, let $\left\langle {n \atop k} \right\rangle_{m,t}$ denote the number of n-tile tilings of N-boards that use k $(1, m-1; t)$-combs (and $n-k$ squares). We define $\left\langle {0 \atop 0} \right\rangle_{m,t} = 1$ and that $\left\langle {n \atop k<0} \right\rangle_{m,t} = \left\langle {n \atop k>n} \right\rangle_{m,t} = 0$. As a $(1, 0; t)$-comb is just a t-omino and the number of n-tile tilings using n t-ominoes and $n-k$ squares is simply $\binom{n}{k}$ for any t, we have $\left\langle {n \atop k} \right\rangle_{1,t\geq 2} = \binom{n}{k}$, which is Pascal's triangle (A007318). The triangles corresponding to $m = 2, 3, 4, 5$ with $t = 2$ are A059259, A350110, A350111, and A350112, respectively [3]. We show examples of the starts of triangles for combs with at least 3 teeth in Figs. 1–4.

We can also create a triangle of $\left[{n \atop k} \right]_{m,t}$ where this denotes the number of tilings of an n-board that use k $(1, m-1; t)$-combs (and therefore $n - kt$ squares) again with $\left[{0 \atop 0} \right]_{m,t} = 1$. The two triangles are related via the following identity.

Identity 1. *For $m \geq 1$, $t \geq 2$, and $n \geq k \geq 0$ we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{m,t} = \left\langle \begin{matrix} n-(t-1)k \\ k \end{matrix} \right\rangle_{m,t}.$$

Proof. If a tiling contains $n-(t-1)k$ tiles of which k are $(1, m-1; t)$-combs (and so $n-kt$ are squares), the total length is $n - kt + kt = n$. □

We will refer to the ray of entries given by $\left\langle {n-\mu k \atop k} \right\rangle_{m,t}$ for $k = 0, \ldots, \lfloor n/(\mu+1) \rfloor$ as the n-th $(1, \mu)$-*antidiagonal*. A $(1, 1)$-antidiagonal is therefore what is normally referred to simply as

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	**1**													
1	**1**	**0**												
2	**1**	**0**	**1**											
3	**1**	**1**	**2**	**0**										
4	**1**	**2**	**4**	**0**	**1**									
5	**1**	**3**	**6**	**3**	**3**	**0**								
6	**1**	**4**	**9**	8	**9**	**0**	**1**							
7	**1**	**5**	13	17	**18**	**6**	**4**	**0**						
8	**1**	**6**	18	30	**36**	20	**16**	**0**	**1**					
9	**1**	**7**	24	48	66	55	**40**	**10**	**5**	**0**				
10	**1**	**8**	31	72	114	120	**100**	40	**25**	**0**	**1**			
11	**1**	**9**	39	103	186	234	221	135	**75**	**15**	**6**	**0**		
12	**1**	**10**	48	142	289	420	456	350	**225**	70	**36**	**0**	**1**	
13	**1**	**11**	58	190	431	709	876	805	581	280	**126**	**21**	**7**	**0**

Figure 1: The start of a Pascal-like triangle (A354665 in the OEIS [14]) whose (n,k)-th entry, $\left\langle {n \atop k} \right\rangle_{2,3}$, is the number of n-tile tilings using k $(1,1;3)$-combs (and $n-k$ squares). Entries in bold font (and those in bold font in Figs. 2–4) are covered by identities in §6.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	**1**													
1	**1**	**0**												
2	**1**	**0**	**1**											
3	**1**	**0**	**2**	**0**										
4	**1**	**1**	**4**	**0**	**1**									
5	**1**	**2**	**6**	**0**	**3**	**0**								
6	**1**	**3**	**9**	**4**	**9**	**0**	**1**							
7	**1**	**4**	**12**	10	**18**	**0**	**4**	**0**						
8	**1**	**5**	**16**	21	**36**	**10**	**16**	**0**	**1**					
9	**1**	**6**	21	36	**60**	30	**40**	**0**	**5**	**0**				
10	**1**	**7**	27	57	**100**	81	**100**	**20**	**25**	**0**	**1**			
11	**1**	**8**	34	84	158	168	**200**	70	**75**	**0**	**6**	**0**		
12	**1**	**9**	42	118	243	322	**400**	231	**225**	**35**	**36**	**0**	**1**	
13	**1**	**10**	51	160	361	560	736	560	**525**	140	**126**	**0**	**7**	**0**

Figure 2: The start of a Pascal-like triangle (A354666) with entries $\left\langle {n \atop k} \right\rangle_{2,4}$.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	**1**													
1	**1**	**0**												
2	**1**	**0**	**1**											
3	**1**	**0**	**2**	**0**										
4	**1**	**0**	**4**	**0**	**1**									
5	**1**	**1**	**6**	**0**	**3**	**0**								
6	**1**	**2**	**9**	**0**	**9**	**0**	**1**							
7	**1**	**3**	**12**	**5**	**18**	**0**	**4**	**0**						
8	**1**	**4**	**16**	12	**36**	**0**	**16**	**0**	**1**					
9	**1**	**5**	**20**	25	**60**	**15**	**40**	**0**	**5**	**0**				
10	**1**	**6**	**25**	42	**100**	42	**100**	**0**	**25**	**0**	**1**			
11	**1**	**7**	31	66	**150**	112	**200**	**35**	**75**	**0**	**6**	**0**		
12	**1**	**8**	38	96	**225**	224	**400**	112	**225**	**0**	**36**	**0**	**1**	
13	**1**	**9**	46	134	325	424	**700**	364	**525**	**70**	**126**	**0**	**7**	**0**

Figure 3: The start of a Pascal-like triangle (A354667) with entries $\langle {n \atop k} \rangle_{2,5}$.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	**1**													
1	**1**	**0**												
2	**1**	**0**	**0**											
3	**1**	**0**	**0**	**1**										
4	**1**	**0**	**1**	**2**	**0**									
5	**1**	**1**	**3**	**4**	**0**	**0**								
6	**1**	**2**	5	**8**	**0**	**0**	**1**							
7	**1**	**3**	8	**12**	**0**	**3**	**3**	**0**						
8	**1**	**4**	12	**18**	**9**	**12**	**9**	**0**	**0**					
9	**1**	**5**	16	**27**	25	29	**27**	**0**	**0**	**1**				
10	**1**	**6**	21	42	51	66	**54**	**0**	**6**	**4**	**0**			
11	**1**	**7**	27	62	95	135	**108**	**36**	**30**	**16**	**0**	**0**		
12	**1**	**8**	34	88	160	234	**216**	126	95	**64**	**0**	**0**	**1**	
13	**1**	**9**	42	122	252	396	432	321	280	**160**	**0**	**10**	**5**	**0**

Figure 4: The start of a Pascal-like triangle (A354668) with entries $\langle {n \atop k} \rangle_{3,3}$.

an antidiagonal. As a consequence of Identity 1, the $(1, t-1)$-antidiagonals of the $\left\langle {n \atop k} \right\rangle_{m,t}$ triangle are the rows of the $\left[{n \atop k} \right]_{m,t}$ triangle. In the rest of the paper we therefore only give identities for the $\left\langle {n \atop k} \right\rangle_{m,t}$ triangle as it is more 'compact' in the sense that its rows contain fewer trailing zeros. However, as in AE22, some of the identities are more straightforward to prove by considering the tiling of an n-board, in which case we need to consider $\left[{n \atop k} \right]_{m,t}$. The following bijection (which is established in the proof of Theorem 2.1 in [2]) will be used in such proofs.

Lemma 2. *For $t \geq 2$, $j \geq 0$, and $r = 0, \ldots, m$, where $m \geq 1$, there is a bijection between the tilings of an $(mj + r)$-board using k $(1, m-1; t)$-combs and $mj + r - kt$ squares and the tilings of an ordered m-tuple of r $(j+1)$-boards followed by $m - r$ j-boards using k t-ominoes and $mj + r - kt$ squares.*

3 Relation of the triangles to polynomials

For $t \geq 2$ we define a $(1, t)$-bonacci polynomial as follows:

$$f_n^{(t)}(x) = f_{n-1}^{(t)}(x) + x f_{n-t}^{(t)}(x) + \delta_{n,0}, \quad f_{n<0}^{(t)}(x) = 0. \tag{1}$$

The $(1, 2)$-bonacci polynomials $f_n^{(2)}(x)$ are the Fibonacci polynomials used in AE22. We refer to the sequence defined by

$$f_n^{(t)} = f_{n-1}^{(t)} + f_{n-t}^{(t)} + \delta_{n,0}, \quad f_{n<0}^{(t)} = 0, \tag{2}$$

for $t \geq 2$ as the $(1, t)$-bonacci numbers. The $t = 2, \ldots, 8$ cases are, respectively, the Fibonacci numbers (A000045), the Narayana's cows sequence (A000930) and sequences A003269, A003520, A005708, A005709, and A005710 in the OEIS.

Lemma 3. *The sum of the coefficients of $f_n^{(t)}(x)$ is $f_n^{(t)}(1) = f_n^{(t)}$.*

Proof. The sum of the coefficients of $f_n^{(t)}(x)$ can be expressed as $f_n^{(t)}(1)$. Putting $x = 1$ into (1) gives (2) with $f_n^{(t)}$ replaced by $f_n^{(t)}(1)$. □

In the next lemma and theorem (which are generalizations of Lemma 13 and Theorem 14 in AE22) we employ the coefficient operator $[x^k]$, which denotes the coefficient of x^k in the term it precedes.

Lemma 4. *Let $f(t, n, k) = [x^k] f_n^{(t)}(x)$ and let $b(t, n, k)$ be the number of tilings of an n-board with squares and t-ominoes that use exactly k t-ominoes. Then $f(t, n, k) = b(t, n, k)$ for all n and k.*

Proof. This follows from Theorem 10 of AE22. The metatiles are the square and t-omino. □

Theorem 5. *For $j \geq 0$, $k \geq 0$, $m \geq 1$, $t \geq 2$, and $r = 0, \ldots, m-1$ we have*

$$\left\langle \begin{matrix} mj + r - (t-1)k \\ k \end{matrix} \right\rangle_{m,t} = [x^k]\big(f_j^{(t)}(x)\big)^{m-r}\big(f_{j+1}^{(t)}(x)\big)^r. \tag{3}$$

Proof. Identity 1 gives us $\left\langle {mj+r-(t-1)k \atop k} \right\rangle_{m,t} = \left[{mj+r \atop k} \right]_{m,t}$. From Lemma 2, we have that $\left[{mj+r \atop k} \right]_{m,t}$ equals the number of ways to tile an ordered m-tuple of r $(j+1)$-boards followed by $m-r$ j-boards using k t-ominoes (and $mj + r - kt$ squares). The number of such tilings of the m-tuple of boards is

$$\sum_{\substack{k_1 \geq 0,\, k_2 \geq 0,\, \ldots,\, k_m \geq 0, \\ k_1 + k_2 + \cdots + k_m = k}} \left(\prod_{i=1}^{r} b(t, j+1, k_i) \right) \left(\prod_{i=r+1}^{m} b(t, j, k_i) \right),$$

in which the first product is omitted when $r = 0$. The coefficient of x^k in $\big(f_{j+1}^{(t)}(x)\big)^r \big(f_j^{(t)}(x)\big)^{m-r}$ is

$$\begin{aligned}
&[x^k]\left(\prod_{i=1}^{r} \sum_{k_i=0}^{\lfloor (j+1)/t \rfloor} f(t, j+1, k_i) x^{k_i} \right) \left(\prod_{i=r+1}^{m} \sum_{k_i=0}^{\lfloor j/t \rfloor} f(t, j, k_i) x^{k_i} \right) \\
&\quad = [x^k] \sum_{k_1 \geq 0, k_2 \geq 0, \ldots, k_m \geq 0} \left(\prod_{i=1}^{r} f(t, j+1, k_i) \right) \left(\prod_{i=r+1}^{m} f(t, j, k_i) \right) x^{k_1 + k_2 + \cdots + k_m} \\
&\quad = \sum_{\substack{k_1 \geq 0,\, k_2 \geq 0,\, \ldots,\, k_m \geq 0, \\ k_1 + k_2 + \cdots + k_m = k}} \left(\prod_{i=1}^{r} f(t, j+1, k_i) \right) \left(\prod_{i=1}^{r} f(t, j, k_i) \right).
\end{aligned}$$

The result then follows from Lemma 4. □

The following identity gives the sums of the $(1, t-1)$-antidiagonals of the $\left\langle {n \atop k} \right\rangle_{m,t}$ triangle. It is a generalization of Identity 15 in AE22. This, in turn, is a generalization of the well-known result that the sum of the elements in the n-th antidiagonal of Pascal's triangle is the Fibonacci number $f_n^{(2)}$, counting the initial 1 in the triangle as the zeroth antidiagonal.

Identity 6. *For $t \geq 2$, $j \geq 0$, $m \geq 1$, and $r = 0, \ldots, m-1$ we have*

$$\sum_{k=0}^{\lfloor (mj+r)/t \rfloor} \left\langle \begin{matrix} mj + r - (t-1)k \\ k \end{matrix} \right\rangle_{m,t} = \big(f_j^{(t)}\big)^{m-r} \big(f_{j+1}^{(t)}\big)^r.$$

Proof. Summing (3) over all permitted k gives the sum of all coefficients of

$$F(x) = \big(f_j^{(t)}(x)\big)^{m-r} \big(f_{j+1}^{(t)}(x)\big)^r,$$

which is $F(1)$ and equals $\big(f_j^{(t)}\big)^{m-r} \big(f_{j+1}^{(t)}\big)^r$ by Lemma 3. □

4 Relation of the triangles to restricted combinations

We now look at $S^{(m,t)}(n,k)$, the number of subsets of $\mathbb{N}_n$ of size k such that the difference of any two elements of the subset does not equal any element in the set $\mathcal{Q} = \{m, 2m, \ldots, (t-1)m\}$. For example, $S^{(2,3)}(5,0) = 1$, $S^{(2,3)}(5,1) = 5$, $S^{(2,3)}(5,2) = 6$, and $S^{(2,3)}(5,k>2) = 0$ since the possible subsets of $\mathbb{N}_5$ such that no two elements in the subset differ by 2 or 4 are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{4,5\}$, $\{1,4\}$, and $\{2,5\}$. There is a formula for $S^{(m,t)}(n,k)$ in terms of sums of products of binomial coefficients [12]. Here we will show that $S^{(m,t)}(n,k) = \left\langle {n+(t-1)(m-k) \atop k} \right\rangle_{m,t}$ and hence obtain an expression for the number of subsets in terms of coefficients of products of $(1,t)$-bonacci polynomials, which is a generalization of earlier results [11, 3]. We first establish the following bijection.

Lemma 7. *For $m, n \geq 1$, $t \geq 2$, and $k \geq 0$, there is a bijection between the k-subsets of $\mathbb{N}_n$ such that all pairs of elements taken from a subset do not differ by an element from the set $\mathcal{Q} = \{m, 2m, \ldots, (t-1)m\}$, and the tilings of an $(n+(t-1)m)$-board with k $(1, m-1; t)$-combs and $n + (t-1)m - kt$ squares.*

Proof. We label the cells of the $(n + (t-1)m)$-board from 1 to $n + (t-1)m$. If a k-subset contains element i then we place a comb so that its left tooth occupies cell i. Notice that if $i = n$ then the rightmost tooth occupies the final cell on the board. After placing combs corresponding to each element of the subset, the rest of the board is filled with squares of which there must be $n + (t-1)m - kt$. Conversely, the tiling of any $(n + (t-1)m)$-board with k combs corresponds to a k-subset where no two elements differ by an element of $\mathcal{Q}$ since the remaining teeth of a comb whose leftmost tooth occupies cell i lie on cells $i + m, i + 2m, \ldots, i + (t-1)m$, which means none of these cells can be occupied by the leftmost tooth of another comb. □

Corollary 8. *For $m, n \geq 1$, $t \geq 2$, and $k \geq 0$ we have $S^{(m,t)}(n,k) = \left\langle {n+(t-1)(m-k) \atop k} \right\rangle_{m,t}$.*

Proof. Lemma 7 gives us $S^{(m,t)}(n,k) = \left[{n+(t-1)m \atop k} \right]_{m,t}$. The result then follows from Identity 1. □

Corollary 9. *For $m, n \geq 1$, and $t \geq 2$, the sum of the elements in the n-th $(1, t-1)$-antidiagonal of $\left\langle {n \atop k} \right\rangle_{m,t}$ is the number of subsets of $\mathbb{N}_{n-(t-1)m}$ chosen so that no two elements of the subsets differ by any member of the set $\{m, \ldots, (t-1)m\}$.*

Proof. The elements in the $(1, t-1)$-antidiagonal are, for $k \geq 0$,

$$\left\langle {n-(t-1)k \atop k} \right\rangle_{m,t} = S^{(m,t)}(n-(t-1)m, k)$$

by Corollary 8. Summing over all k then gives the result. □

The next two corollaries follow from Theorem 5 and Identity 6, respectively.

Corollary 10. *For $j, k \geq 0$, $m \geq 1$, $t \geq 2$, and $r = 0, \ldots, m-1$ we have*

$$S^{(m,t)}(mj + r, k) = [x^k]\big(f^{(t)}_{j+t-1}(x)\big)^{m-r}\big(f^{(t)}_{j+t}(x)\big)^r.$$

Corollary 11. *For $j \geq 0$, $m \geq 1$, $t \geq 2$, and $r = 0, \ldots, m-1$, the number of subsets of $\mathbb{N}_{mj+r}$ each of which lack pairs of elements that differ by a multiple of m up to $(t-1)m$ is $\big(f^{(t)}_{j+t-1}\big)^{m-r}\big(f^{(t)}_{j+t}\big)^r$.*

5 Metatiles and digraphs

The simplest metatiles when tiling with squares (S) and $(1, m-1; t)$-combs (C) are the *free square* (S), what we will refer to as an *m-comb* (C^m), which is m interlocking combs with no gaps, and the *filled comb* ($CS^{(m-1)(t-1)}$), which is a comb with all the gaps filled with squares. The $m = 2$, $t = 3$ instances of these are the first three metatiles depicted in Fig. 5(a).

When $m = 1$, the only metatiles are the two individual tiles themselves: a square and a comb, which, as the gaps are of zero width, is just a t-omino. When $m > 1$, the only case when there is a finite number of metatiles is when $t = 2$ [10]. There are two cases when there is a single infinite sequence of metatiles: the $(m, t) = (3, 2)$ case, which was dealt with in AE22, and when $m = 2$ and $t = 3$. In the latter case, the metatiles are S, C^2, and CSC^jS for $j \geq 0$, as illustrated in Fig. 5(a). This infinite sequence of metatiles is analogous to that found for the $(m, t) = (3, 2)$ case [3, §6]: CS has a single remaining unit-width slot which can be filled either with an S, thus completing the metatile, or with the left tooth of a C (to give CSC) which again results in a slot of unit width.

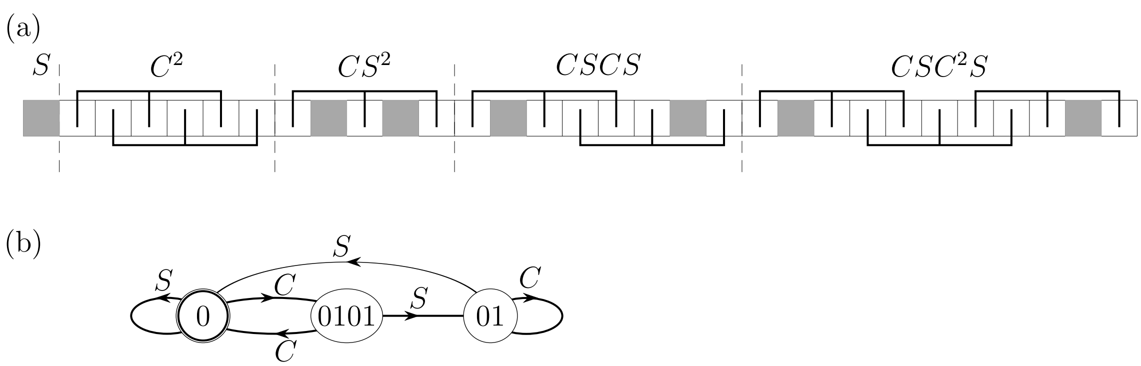

Figure 5: Metatiles when tiling with squares and $(1, 1; 3)$-combs ($m = 2$, $t = 3$). (a) A 31-board tiled with all the metatiles containing less than 6 tiles. Shaded (white) cells are occupied by squares (combs). Bold lines indicate which teeth belong to the same comb. Dashed lines show boundaries between metatiles. The symbolic representation is above each metatile. (b) The digraph for generating metatiles.

For a particular choice of types of tiles, a systematic way to generate all metatiles and, in the simpler cases, obtain finite-order recursion relations for the number of tilings is via a directed pseudograph (henceforth referred to as a *digraph*) in which each arc represents the addition of a tile and each node represents the current state of the yet-to-be-completed metatile [6, 9]. Any such digraph contains a *0 node* which represents the empty board or the completed metatile. The remaining nodes are named using binary strings: the i-th digit of the string is 0 (1) if the i-th cell, starting at the first unoccupied cell of the incomplete metatile and ending at its last occupied cell, is empty (filled). Thus all nodes (except the 0 node) start with 0 and end with 1. There is a bijection between each possible metatile and each path on the digraph which starts and finishes at the 0 node without visiting it in between. To obtain the symbolic representation of the metatile, one simply reads off the names of the arcs along the path and then simplifies the resulting expression by, for example, replacing CC by C^2. The digraph for generating metatiles when tiling with squares and $(1, 1; 3)$-combs is shown in Fig. 5(b).

In such metatile-generating digraphs, a *cycle* is a path starting and finishing at a given node but not repeating any other node or arc in between [6]. Where there is no possibility of ambiguity, we refer to cycles just by the arcs they contain. An *inner cycle* is a cycle that does not include the 0 node. For example, the digraph in Fig. 5(b) has just one inner cycle, i.e., C (which connects the 01 node to itself). Note that, as we have already seen in the tiling corresponding to Fig. 5(b), if a digraph has an inner cycle, there are infinitely many possible metatiles as, if the walk reaches a node in an inner cycle, the cycle can be traversed an arbitrary number of times before it is exited and the walk returns to the 0 node.

If all of the inner cycles of a digraph have at least one node in common, that node (or any one of those nodes) is said to be the *common node*. E.g., in the digraph in Fig. 5(b) the 01 node is the common node. A *common circuit* is a simple path from the 0 node to the common node (with no node or arc reached more than once) followed by a simple path from the common node back to the 0 node. The nodes and arcs in the return journey, again, cannot be repeated (although they can coincide with nodes or arcs traversed in the outward journey). E.g., CSS is the common circuit in Fig. 5(b).

An *outer cycle* is a cycle that starts at the 0 node but never reaches the common node. Thus there are two such cycles in Fig. 5(b): S and C^2. In a digraph with a common node, the outer cycles correspond to metatiles which are not members of an infinite family; the remaining metatiles correspond to the outbound part of the common circuit followed by an arbitrary number of trips around the inner cycle(s) (in any order, if there are more than one) followed by the inbound part of the common circuit (these are the metatiles CSC^jS for $j = 0, 1, 2, \ldots$ in Fig. 5).

6 Further identities

We start by deriving identities that apply to all the triangles and later on obtain recursion relations for some particular instances of the triangles after constructing the corresponding metatile-generating digraphs. The following three identities arise from considering the simplest types of n-tile tilings.

Identity 12. *For $n \geq 0$, $m \geq 1$, and $t \geq 2$ we have* $\left\langle {n \atop 0} \right\rangle_{m,t} = 1$.

Proof. There is only one way to create an n-tile tiling without using any combs, namely, the all-square tiling. □

Identity 13. *For $n \geq 0$, $m \geq 1$, and $t \geq 2$ we have* $\left\langle {n \atop n} \right\rangle_{m,t} = \delta_{n \bmod m,0}$.

Proof. The only way to tile without squares is the all m-comb tiling. This can only occur if the number of tiles is a multiple of m. □

Identity 14. *For $n, m \geq 1$ and $t \geq 2$ we have*

$$\left\langle {n \atop 1} \right\rangle_{m,t} = \begin{cases} 0, & \text{if } n < (m-1)(t-1)+1; \\ n-(m-1)(t-1), & \text{otherwise.} \end{cases}$$

Proof. Any n-tile tiling using exactly one $(1, m-1; t)$-comb must have a filled comb, which itself contains $(m-1)(t-1)+1$ tiles. Thus there can be no n-tile tilings using 1 comb that use less than this number of tiles. If $n \geq (m-1)(t-1)+1$, the tiling consists of a filled comb and $n-(m-1)(t-1)-1$ free squares, which gives a total of $n-(m-1)(t-1)$ metatile positions in which the filled comb can be placed. □

The pattern of zeros seen in the triangles is a result of the following identity.

Identity 15. *For $j \geq 1$, $m, t \geq 2$, $p = 1, \ldots, m-1$, and $r = 1-(t-2)p, \ldots, p$ we have*

$$\left\langle {mj-r \atop mj-p} \right\rangle_{m,t} = 0.$$

Proof. We first derive an expression for K, the maximum number of combs that can be used in the tiling of an $(mJ+R)$-board, where $R = 0, \ldots, m-1$. From Lemma 2, we have that K is also the maximum number of t-ominoes that can be used in the tiling of R $(J+1)$-boards and $m-R$ J-boards. Then it is straightforward to show that

$$K = \begin{cases} \dfrac{m\big(J-(J \bmod t)\big)}{t}, & \text{if } J \bmod t < t-1; \\ \dfrac{m(J-t+1)}{t} + R, & \text{if } J \bmod t = t-1. \end{cases} \tag{4}$$

From Identity 1 we have

$$\left\langle {mj-r \atop mj-p} \right\rangle_{m,t} = \left[{tmj-r-(t-1)p \atop mj-p} \right]_{m,t}.$$

Writing $tmj - r - (t-1)p$ in the form $mJ + R$, if $J = tj - s$, where $s = 1, \ldots, t$, then $R = sm - r - (t-1)p$. The condition that $0 \le R < m$ gives $(s-1)m < r + (t-1)p \le sm$. This condition is compatible with the minimum and maximum values $r + (t-1)p$ can take which are, respectively, 2 and tm. From (4) we find for $s > 1$ that $K = m(j-1)$, which is always less than $mj - p$. When $s = 1$, we have $K = mj - p - r - (t-2)p$. Since $r + (t-2)p \ge 1$ we have $K < mj - p$ in this case as well. □

The following identity explains the entries that appear at the vertical boundaries of the nonzero parts of the triangles and start with rising powers of ascending positive integers. Identities 16 and 17 reduce to Identity 24 of AE22 when $t = 2$.

Identity 16. *For $j, m \ge 1$, $t \ge 2$, $s = 0, \ldots, t-2$, and $r = 0, \ldots, m$ we have*

$$\left\langle {m(j+s-1)+r \atop m(j-1)} \right\rangle_{m,t} = \binom{j+s-1}{s}^{m-r} \binom{j+s}{s+1}^{r}$$

$$= \begin{cases} j^r, & \text{if } s = 0; \\ \left(\dfrac{j(j+1)\cdots(j+s-1)}{s!} \right)^m \left(\dfrac{j+s}{s+1} \right)^r, & \text{if } s > 0. \end{cases}$$

Proof. From Identity 1 we have

$$\left\langle {m(j+s-1)+r \atop m(j-1)} \right\rangle_{m,t} = \left[{m(t(j-1)+s)+r \atop m(j-1)} \right]_{m,t}.$$

By Lemma 2, this is the number of ways to tile $m - r$ boards of length $t(j-1) + s$ and r boards of length $t(j-1)+s+1$ with $m(j-1)$ t-ominoes (and $sm + r$ squares). As $s + 1 < t$, each of the m boards always contains exactly $j - 1$ t-ominoes. A board of length $t(j-1)+s$ has s squares and so there are $j + s - 1$ metatile positions in which to put the squares (the rest being filled by t-ominoes) and thus $\binom{j+s-1}{s}$ ways to tile it. Likewise, a board of length $t(j-1)+s+1$ has $j + s$ metatile positions and so there are $\binom{j+s}{s+1}$ ways to tile it. The result follows from the numbers of each type of board. □

The next identity explains the rising powers of integers on non-vertical rays of entries at the boundaries of the nonzero parts of the triangles.

Identity 17. *For $m, j \ge 1$, $t \ge 2$, and $p = 0, \ldots, m$ we have*

$$\left\langle {mj + (t-2)p \atop mj - p} \right\rangle_{m,t} = \binom{j+t-2}{t-1}^{p}.$$

Proof. From Identity 1 we have

$$\left\langle {mj + (t-2)p \atop mj - p} \right\rangle_{m,t} = \left[{tmj - p \atop mj - p} \right]_{m,t} = \left[{m(jt-1) + m - p \atop (m-p)j + p(j-1)} \right]_{m,t},$$

which is also the number of ways to tile $m-p$ jt-boards and p boards of length $jt-1$ using $(m-p)j+p(j-1)$ t-ominoes and $(t-1)p$ squares. The jt-boards are completely filled by j t-ominoes and the p $(jt-1)$-boards each have $j-1$ t-ominoes and $t-1$ squares. As on these p shorter boards there are $j+t-2$ tiles in total, there are $\binom{j+t-2}{t-1}$ ways to tile each of them, which leads to a total of

$$\binom{j+t-2}{t-1}^p$$

tilings for the set of boards. □

The following two identities are generalizations of Identities 25 and 26 in AE22.

Identity 18. *For $j, m \geq 1$ and $t \geq 2$ we have*

$$\left\langle {mj+t-1 \atop mj-1} \right\rangle_{m,t} = m\binom{j+t-1}{t}.$$

Proof. From Identity 1 we obtain

$$\left\langle {mj+t-1 \atop mj-1} \right\rangle_{m,t} = \left[{tmj \atop mj-1} \right]_{m,t},$$

which, from Lemma 2, is the number of ways to tile an m-tuple of jt-boards with $mj-1$ t-ominoes and t squares. As the length of each board is a multiple of t, all the squares must lie on the same board. On such a board there are $j-1$ t-ominoes and t squares making $j+t-1$ tiles in total. Hence there are $\binom{j+t-1}{t}$ possible ways to tile it. As there are m possible boards on which to place all the squares, the result follows. □

Identity 19. *For $t \geq 2$ and $m, j \geq 1$ provided $mj \geq 2$ we have*

$$\left\langle {mj+2(t-1) \atop mj-2} \right\rangle_{m,t} = \begin{cases} \binom{m}{2}, & \textit{if } j=1, m>1; \\ m\binom{j+2(t-1)}{2t} + \binom{m}{2}\binom{j+t-1}{t}^2, & \textit{if } m, j>1; \\ \binom{j+2(t-1)}{2t}, & \textit{if } m=1, j>1. \end{cases}$$

Proof. From Identity 1 we obtain

$$\left\langle {mj+2(t-1) \atop mj-2} \right\rangle_{m,t} = \left[{tmj \atop mj-2} \right]_{m,t},$$

which, from Lemma 2, is the number of ways to tile an m-tuple of jt-boards with $mj-2$ t-ominoes and $2t$ squares. If $j>1$, all $2t$ squares can be on the same jt-board which, with the $j-2$ t-ominoes on that board, makes $j-2+2t$ tiles in total and hence $\binom{j+2(t-1)}{2t}$ tilings of it. With m boards to choose from, this gives the first term on the right-hand sides of the identity when $j>1$. Otherwise, if $m>1$, two of the boards have t squares each. There are $\binom{j+t-1}{t}$ ways to tile each of those boards and $\binom{m}{2}$ ways to choose them. □

The following identity is a generalization of the previous two.

Identity 20. *For $s \geq 1$, $t \geq 2$, and $m, j \geq 1$ provided $mj \geq s$ we have*

$$\left\langle \begin{matrix} mj + s(t-1) \\ mj - s \end{matrix} \right\rangle_{m,t} = \sum_{\substack{r_i \geq 1; \\ r_1 + \cdots + r_p = s}} \binom{m}{p} \prod_{i=1}^{p} \binom{j + r_i(t-1)}{r_i t},$$

where the sum is over compositions of s, p is the number of parts of the composition, and $\binom{a}{b}$ is understood to equal zero if $a < b$.

Proof. From Identity 1 we have

$$\left\langle \begin{matrix} mj + s(t-1) \\ mj - s \end{matrix} \right\rangle_{m,t} = \left[\begin{matrix} tmj \\ mj - s \end{matrix} \right]_{m,t},$$

which, from Lemma 2, is the number of ways to tile an m-tuple of jt-boards with $mj - s$ t-ominoes and st squares. We partition the squares into p parts of sizes $r_i t$ where $r_i \in \mathbb{Z}^+$ such that $r_1 + \cdots + r_p = s$. A jt-board containing $r_i t$ squares has $j - r_i$ t-ominoes and thus $j - r_i + r_i t$ tiles in total and so

$$\binom{j + r_i(t-1)}{r_i t}$$

possible ways to tile it. There are $\binom{m}{p}$ ways to choose which of the m boards have any squares. □

In order to truly deserve to be called a Pascal-like triangle, a triangle ought to have a portion where Pascal's recurrence is obeyed. We now show that this is the case for our triangles by using a result from a study on restricted combinations [12] to extend and prove Conjecture 30 of AE22.

Theorem 21. *For integers $k \geq 0$, $m \geq 1$, $t \geq 2$, and $n > (m-1)(t-1)k$,*

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{m,t} = \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle_{m,t} + \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle_{m,t}. \tag{5}$$

Proof. The result holds for $k = 0$ since $\left\langle \begin{smallmatrix} n \geq 0 \\ 1 \end{smallmatrix} \right\rangle_{m,t} = 1$ by Identity 12 and $\left\langle \begin{smallmatrix} n<0 \\ k \end{smallmatrix} \right\rangle_{m,t} = \left\langle \begin{smallmatrix} n \\ k<0 \end{smallmatrix} \right\rangle_{m,t} = 0$ by definition. Mansour and Sun give the result in Theorem 3.5 of their paper [12], when rewritten in our own notation, that for any integers $m, k \geq 1$, and $t \geq 2$,

$$S^{(m,t)}(N,k) = S^{(m,t)}(N-1,k) + S^{(m,t)}(N-t,k-1), \tag{6}$$

provided that $N \geq m(t-1)(k-1)$. However, the condition for this relation between numbers of subsets to hold should read $N > m(t-1)(k-1)$ (personal communication with Mansour). Corollary 8 gives us $n = N + (t-1)(m-k)$ and we can rewrite (6) as

$$\left\langle \begin{matrix} N + (t-1)(m-k) \\ k \end{matrix} \right\rangle_{m,t} = \left\langle \begin{matrix} N + (t-1)(m-k) - 1 \\ k \end{matrix} \right\rangle_{m,t} + \left\langle \begin{matrix} N + (t-1)(m-k+1) - t \\ k-1 \end{matrix} \right\rangle_{m,t},$$

which reduces to (5). The corrected condition becomes $n-(t-1)(m-k) > m(t-1)(k-1)$, which gives the condition in our theorem. □

We now turn to obtaining recursion relations for particular instances of the triangles. For all but the last triangle we consider, we require the following theorem which extends a result proved elsewhere for tilings of an n-board when the digraph has a common node [6, Theorem 5.4 and Identity 5.5] to also include n-tile tilings of boards.

Theorem 22. *For a digraph possessing a common node, let $l_{\mathrm{o}i}$ be the length of the i-th outer cycle ($i=1,\dots,N_{\mathrm{o}}$), let L_r be the length of the r-th inner cycle ($r=1,\dots,N$) and let K_r be the number of combs it contains, and let $l_{\mathrm{c}i}$ be the length of the i-th common circuit ($i=1,\dots,N_{\mathrm{c}}$) and let $k_{\mathrm{c}i}$ be the number of combs it contains. Then for all integers n and k,*

$$B_n = \delta_{n,0} + \sum_{r=1}^{N}(B_{n-L_r} - \delta_{n,L_r}) + \sum_{i=1}^{N_{\mathrm{o}}}\left(B_{n-l_{\mathrm{o}i}} - \sum_{r=1}^{N} B_{n-l_{\mathrm{o}i}-L_r}\right) + \sum_{i=1}^{N_{\mathrm{c}}} B_{n-l_{\mathrm{c}i}}, \tag{7}$$

$$\begin{aligned} B_{n,k} = \delta_{n,0}\delta_{k,0} &+ \sum_{r=1}^{N}(B_{n-L_r,k-K_r} - \delta_{n,L_r}\delta_{k,K_r}) + \sum_{i=1}^{N_{\mathrm{o}}}\left(B_{n-l_{\mathrm{o}i},k-k_{\mathrm{o}i}} - \sum_{r=1}^{N} B_{n-l_{\mathrm{o}i}-L_r,k-k_{\mathrm{o}i}-K_r}\right) \\ &+ \sum_{i=1}^{N_{\mathrm{c}}} B_{n-l_{\mathrm{c}i},k-k_{\mathrm{c}i}}, \end{aligned} \tag{8}$$

where $B_{n<0} = B_{n,k<0} = B_{n<k,k} = 0$. If the lengths of the cycles and circuits are calculated as the number of tiles (the total contribution made to the number of cells occupied) then B_n is the number of n-tile tilings (the number of tilings of an n-board) and $B_{n,k}$ is the number of such tilings that use k combs.

In the proofs of Identities 23, 24, 26, 27, 29, and 30, which use Theorem 22 (and Identities 32 and 33, which use Theorem 36), the lengths of the cycles and circuits are the number of tiles they contain. In the proofs of Identities 25, 28, 31, and 34, the lengths of the cycles and circuits are the total number of cells that the tiles along the arcs occupy. An S occupies 1 cell whereas a C occupies t cells. Thus if L is the length of a cycle or circuit containing K combs when finding the recursion relations for n-tile tilings then $L' = L + (t-1)K$ is the length of that cycle or circuit when the recursion relations are for the tilings of an n-board.

Identity 23. *For all $n, k \in \mathbb{Z}$ we have*

$$\begin{aligned} \left\langle {n \atop k} \right\rangle_{2,3} = \delta_{n,0}\delta_{k,0} - \delta_{n,1}\delta_{k,1} + \left\langle {n-1 \atop k} \right\rangle_{2,3} + \left\langle {n-1 \atop k-1} \right\rangle_{2,3} - \left\langle {n-2 \atop k-1} \right\rangle_{2,3} + \left\langle {n-2 \atop k-2} \right\rangle_{2,3} \\ + \left\langle {n-3 \atop k-1} \right\rangle_{2,3} - \left\langle {n-3 \atop k-3} \right\rangle_{2,3}. \end{aligned} \tag{9}$$

Proof. The digraph for tiling with squares and $(1,1;3)$-combs has a single inner cycle connecting the 01 node to itself by a C (Fig. 5(b)). Hence 01 is the common node and $L_1 = K_1 = 1$. There are 2 outer cycles (S and C^2) and so $l_{o1} = 1$, $k_{o1} = 0$, and $l_{o2} = k_{o2} = 2$. There is a single common circuit (CS^2) which gives $l_{c1} = 3$ and $k_{c1} = 1$. □

Identity 24. *If B_n is the sum of the n-th row of $\langle {n \atop k} \rangle_{2,3}$ then, for all n, we have*

$$B_n = \delta_{n,0} - \delta_{n,1} - \delta_{n,2} + 2B_{n-1},$$

where $B_{n<0} = 0$.

Proof. Sum each term in (9) over all k or use (13). □

As defined above, $(B_n)_{n\geq 0} = 1, 1, 2, 4, 8, 16, 32, 64, 128, 256, \dots$ is A011782.

Identity 25. *If A_n is the sum of the n-th $(1,2)$-antidiagonal of $\langle {n \atop k} \rangle_{2,3}$ then, for all n, we have*

$$A_n = \delta_{n,0} - \delta_{n,3} + A_{n-1} + A_{n-3} - A_{n-4} + A_{n-5} + A_{n-6} - A_{n-9},$$

where $A_{n<0} = 0$.

Proof. By Identity 1, the n-th $(1,2)$-antidiagonal of $\langle {n \atop k} \rangle_{2,3}$ is the n-th row of $\left[{n \atop k} \right]_{2,3}$. From the definition of the latter triangle, A_n is the number of tilings of an n-board using squares and $(1,1;3)$-combs and is given by (13) (with B_n replaced by A_n) applied to the same digraph as in the proof of Identity 23 but with the following changes made to the lengths: $L_1 = 3$, $l_{o2} = 6$, $l_{c1} = 5$. □

As defined above, $(A_n)_{n\geq 0} = 1, 1, 1, 1, 1, 2, 4, 6, 9, 12, 16, 24, 36, 54, 81, 117, \dots$ is A224809. From Corollary 9, A_n is the number of subsets of $\mathbb{N}_{n-4}$ chosen so that no two elements differ by 2 or 4.

Identity 26. *For all $n, k \in \mathbb{Z}$ we have*

$$\begin{aligned} \left\langle {n \atop k} \right\rangle_{2,4} &= \delta_{n,0}\delta_{k,0} - \delta_{n,2}(\delta_{k,1} + \delta_{k,2}) + \left\langle {n-1 \atop k} \right\rangle_{2,4} + \left\langle {n-2 \atop k-1} \right\rangle_{2,4} + 2\left\langle {n-2 \atop k-2} \right\rangle_{2,4} \\ &\quad - \left\langle {n-3 \atop k-1} \right\rangle_{2,4} - \left\langle {n-3 \atop k-2} \right\rangle_{2,4} + \left\langle {n-4 \atop k-1} \right\rangle_{2,4} + \left\langle {n-4 \atop k-2} \right\rangle_{2,4} - \left\langle {n-4 \atop k-3} \right\rangle_{2,4} - \left\langle {n-4 \atop k-4} \right\rangle_{2,4}. \end{aligned} \quad (10)$$

Proof. The digraph for tiling with squares and $(1,1;4)$-combs has 2 inner cycles (SC and C^2) both of which pass though the 0101 and 01 nodes (Fig. 6). We choose 0101 as the common node. We see that $L_1 = L_2 = 2$, $K_1 = 1$, and $K_2 = 2$. There are 2 outer cycles (S and C^2) and so $l_{o1} = 1$, $k_{o1} = 0$, and $l_{o2} = k_{o2} = 2$. There are 2 common circuits: $CS\{S, C\}S$ where $X\{Y, Z\}$ means XY and XZ. Hence $l_{c1} = l_{c2} = 4$, $k_{c1} = 1$, and $k_{c2} = 2$. □

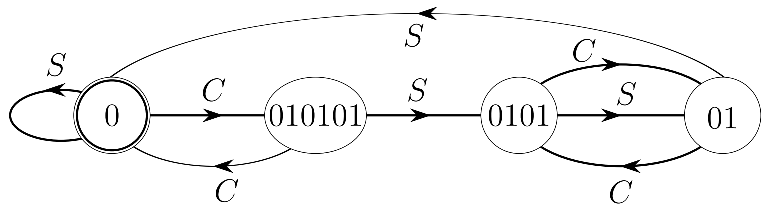

Figure 6: Digraph for generating metatiles when tiling with squares and $(1,1;4)$-combs ($m=2$, $t=4$).

Identity 27. *If B_n is the sum of the n-th row of $\left\langle {n \atop k} \right\rangle_{2,4}$ then, for all n, we have*

$$B_n = \delta_{n,0} - 2\delta_{n,2} + B_{n-1} + 3B_{n-2} - 2B_{n-3},$$

where $B_{n<0} = 0$.

Proof. Sum each term in (10) over all k or use (13). □

As defined above, $(B_n)_{n\geq 0} = 1, 1, 2, 3, 7, 12, 27, 49, 106, 199, 419, \ldots$ is A099163.

The proofs of the following identity and Identities 31 and 34 are analogous to that of Identity 25. We just need to find the modified lengths of the cycles and circuits in the digraph before using the theorem giving the recursion relation.

Identity 28. *If A_n is the sum of the n-th $(1,3)$-antidiagonal of $\left\langle {n \atop k} \right\rangle_{2,4}$ then, for all n, we have*

$$A_n = \delta_{n,0} - \delta_{n,5} - \delta_{n,8} + A_{n-1} + A_{n-5} - A_{n-6} + A_{n-7} + 2A_{n-8} - A_{n-9} + A_{n-10} - A_{n-13} - A_{n-16},$$

where $A_{n<0} = 0$.

Proof. We use the same digraph and associated parameters as in the proof of Identity 26 except that $L_1 = 5$, $L_2 = 8$, $l_{\mathrm{o}2} = 8$, $l_{\mathrm{c}1} = 7$, and $l_{\mathrm{c}2} = 10$. □

As defined above, $(A_n)_{n\geq 0} = 1, 1, 1, 1, 1, 1, 1, 2, 4, 6, 9, 12, 16, 20, 25, 35, \ldots$ is A224808. From Corollary 9 we have that A_n is the number of subsets of $\mathbb{N}_{n-6}$ chosen so that no two elements differ by 2, 4, or 6.

Identity 29. *For all $n, k \in \mathbb{Z}$ we have*

$$\begin{aligned}\left\langle {n \atop k} \right\rangle_{4,2} = {}& \delta_{n,0}\delta_{k,0} - \delta_{n,2}\delta_{k,1} - \delta_{n,3}\delta_{k,2} - \delta_{n,4}\delta_{k,4} + \left\langle {n-1 \atop k} \right\rangle_{4,2} + \left\langle {n-2 \atop k-1} \right\rangle_{4,2} - \left\langle {n-3 \atop k-1} \right\rangle_{4,2} + \left\langle {n-3 \atop k-2} \right\rangle_{4,2} \\ & + \left\langle {n-4 \atop k-1} \right\rangle_{4,2} + \left\langle {n-4 \atop k-3} \right\rangle_{4,2} + 2\left\langle {n-4 \atop k-4} \right\rangle_{4,2} + \left\langle {n-5 \atop k-2} \right\rangle_{4,2} + 2\left\langle {n-5 \atop k-3} \right\rangle_{4,2} - \left\langle {n-5 \atop k-4} \right\rangle_{4,2} \\ & - \left\langle {n-6 \atop k-3} \right\rangle_{4,2} - \left\langle {n-6 \atop k-5} \right\rangle_{4,2} - \left\langle {n-7 \atop k-4} \right\rangle_{4,2} - \left\langle {n-7 \atop k-5} \right\rangle_{4,2} - \left\langle {n-7 \atop k-6} \right\rangle_{4,2} - \left\langle {n-8 \atop k-7} \right\rangle_{4,2} - \left\langle {n-8 \atop k-8} \right\rangle_{4,2}. \end{aligned} \quad (11)$$

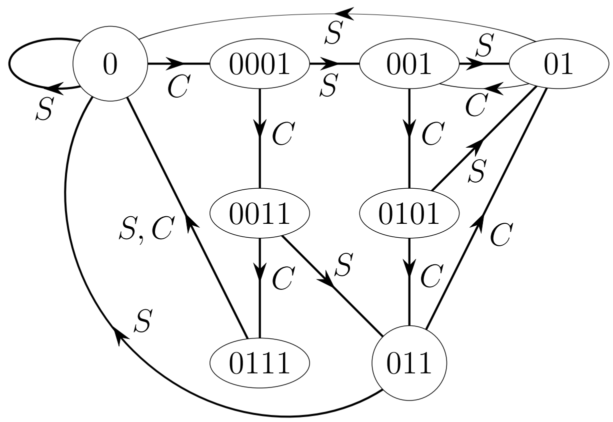

Figure 7: Digraph for generating metatiles when tiling with squares and $(1,3;2)$-combs ($m=4$, $t=2$).

Proof. The digraph for tiling with squares and $(1,3;2)$-combs (which are also called $(1,3)$-fences) has 3 inner cycles all of which contain the nodes 001 and 01 (Fig. 7). We choose 001 as the common node. The cycles, given as lists of arcs starting from 001, are $\{S, C\{S, C^2\}\}C$. Hence $L_i = 2,3,4$ and $K_i = 1,2,4$, respectively, for $i = 1,2,3$. There are 5 outer cycles: $S, C^2\{S\{S, CS\}, C\{S, C\}\}$. Thus $l_{\mathrm{o}i} = 1,4,5,4,4$ and $k_{\mathrm{o}i} = 0,2,3,3,4$, respectively, for $i = 1,\ldots,5$. There are 8 common circuits: $C\{S, CSC^2\}\{S^2, C\{S^2, C\{S, CS\}\}\}$. Hence $l_{\mathrm{c}i} = 4,5,5,6,7,8,8,9$ and $k_{\mathrm{c}i} = 1,2,3,4,4,5,6,7$, respectively, for $i = 1,\ldots,8$. □

Identity 30. *If B_n is the sum of the n-th row of $\left\langle {n \atop k} \right\rangle_{4,2}$ then for all n we have*

$$B_n = \delta_{n,0} - \delta_{n,2} - \delta_{n,3} - \delta_{n,4} + B_{n-1} + B_{n-2} + 4B_{n-4} + 2B_{n-5} - 2B_{n-6} - 3B_{n-7} - 2B_{n-8},$$

where $B_{n<0} = 0$.

Proof. Sum each term in (11) over all k or use (13). □

As defined above, $(B_n)_{n\geq 0} = 1,1,1,1,5,12,21,34,70,155,318,610,\ldots$ has the generating function $(1-x-x^3)/((1-2x)(1-x^2)(1+2x^2+x^3+x^4))$.

Identity 31. *If A_n is the sum of the n-th antidiagonal of $\left\langle {n \atop k} \right\rangle_{4,2}$ then for all n we have*

$$A_n = \delta_{n,0} - \delta_{n,3} - \delta_{n,5} - \delta_{n,8} + A_{n-1} + A_{n-3} - A_{n-4} + 2A_{n-5} + 2A_{n-7} + 4A_{n-8} - 2A_{n-9} - 2A_{n-11} - A_{n-12} - A_{n-13} - A_{n-15} - A_{n-16},$$

where $A_{n<0} = 0$.

Proof. We use the same digraph and associated parameters as in the proof of Identity 29 except that $L_1 = 3$, $L_2 = 5$, $L_3 = 8$, $l_{\mathrm{o}i} = 6,8,7,8$ for $i = 2,\ldots,5$, and for $i = 1,\ldots,8$, $l_{\mathrm{c}i} = 5,7,8,10,11,13,14,16$. □

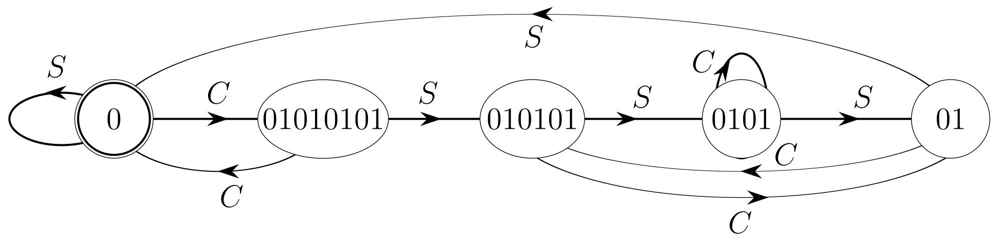

Figure 8: Digraph for generating metatiles when tiling with squares and $(1, 1; 5)$-combs ($m = 2$, $t = 5$).

As defined above, $(A_n)_{n\geq 0} = 1, 1, 1, 1, 1, 1, 2, 4, 8, 16, 24, 36, 54, 81, 135, 225, \ldots$ (after removing the first four 1s) is A031923. From Corollary 9 we have that A_n is the number of subsets of $\mathbb{N}_{n-4}$ chosen so that no two elements differ by 4.

Identity 32. *For all $n, k \in \mathbb{Z}$ we have*

$$\left\langle {n \atop k} \right\rangle_{2,5} = \delta_{n,0}\delta_{k,0} - \delta_{n,1}\delta_{k,1} - \delta_{n,2}\delta_{k,2} + \delta_{n,3}(\delta_{k,3} - \delta_{k,1}) + \left\langle {n-1 \atop k} \right\rangle_{2,5} + \left\langle {n-1 \atop k-1} \right\rangle_{2,5} - \left\langle {n-2 \atop k-1} \right\rangle_{2,5}$$
$$+ 2\left\langle {n-2 \atop k-2} \right\rangle_{2,5} + \left\langle {n-3 \atop k-1} \right\rangle_{2,5} - \left\langle {n-3 \atop k-2} \right\rangle_{2,5} - 2\left\langle {n-3 \atop k-3} \right\rangle_{2,5} - \left\langle {n-4 \atop k-1} \right\rangle_{2,5}$$
$$+ \left\langle {n-4 \atop k-2} \right\rangle_{2,5} + \left\langle {n-4 \atop k-3} \right\rangle_{2,5} - \left\langle {n-4 \atop k-4} \right\rangle_{2,5} + \left\langle {n-5 \atop k-1} \right\rangle_{2,5} - 2\left\langle {n-5 \atop k-3} \right\rangle_{2,5} + \left\langle {n-5 \atop k-5} \right\rangle_{2,5}. \quad (12)$$

Proof. The digraph for tiling with squares and $(1, 1; 5)$-combs has 3 inner cycles but no common node (Fig. 8). If the loop at the 0101 node were not present, the digraph would have a common node. Using terminology and notation we introduce in the appendix, the loop at 0101 is an errant loop and has length $L_0 = 1$ and number of combs $K_0 = 1$. We take the 010101 node as the pseudo-common node (we could have also chosen the 01 node instead). There are two common circuits, $CSCS$ and CS^4, the first of which is plain. Thus $l_{\text{c1}} = l_{\text{pc1}} = 4$, $k_{\text{c1}} = k_{\text{pc1}} = 2$, $l_{\text{c2}} = 5$, $k_{\text{c2}} = 1$, $N_{\text{c}} = 2$, and $N_{\text{pc}} = 1$. Of the other two inner cycles, C^2 is plain, S^2C is not. Thus $L_1 = 2$, $K_1 = 2$, $L_2 = 3$, and $K_2 = 1$. The outer cycles are S and C^2 and are both plain. Hence $l_{\text{o1}} = 1$, $k_{\text{o1}} = 0$, $l_{\text{o2}} = k_{\text{o2}} = 2$, and $N_{\text{o}} = 2$. The identity then follows from applying (14). □

Identity 33. *If B_n is the sum of the n-th row of $\langle {n \atop k} \rangle_{2,5}$ then for all n we have*

$$B_n = \delta_{n,0} - \delta_{n,1} - \delta_{n,2} + 2B_{n-1} + B_{n-2} - 2B_{n-3},$$

where $B_{n<0} = 0$.

Proof. Sum each term in (12) over all k or use (13). □

As defined above, $(B_n)_{n\geq 0} = 1, 1, 2, 3, 6, 11, 22, 43, 86, 171, 342, 683, \ldots$ is A005578.

Identity 34. *If A_n is the sum of the n-th $(1,4)$-antidiagonal of $\left\langle {n \atop k} \right\rangle_{2,5}$ then for all n we have*

$$A_n = \delta_{n,0} - \delta_{n,5} - \delta_{n,7} - \delta_{n,10} + \delta_{n,15} + A_{n-1} + A_{n-5} - A_{n-6} + A_{n-7} - A_{n-8} + A_{n-9} + 2A_{n-10} - A_{n-11} + A_{n-12} - 2A_{n-15} + A_{n-16} - 2A_{n-17} - A_{n-20} + A_{n-25}$$

where $A_{n<0} = 0$.

Proof. We use the same digraph and associated parameters as in the proof of Identity 32 except that $L_0 = 5$, $L_1 = 10$, $L_2 = 7$, $l_{\mathrm{o}2} = 10$, $l_{\mathrm{c}1} = l_{\mathrm{pc}1} = 12$, and $l_{\mathrm{pc}2} = 9$. □

As defined above, $(A_n)_{n\geq 0} = 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 48, 64, \ldots$ is A224811. From Corollary 9 we have that A_n is the number of subsets of $\mathbb{N}_{n-8}$ chosen so that no two elements differ by 2, 4, 6, or 8.

7 Discussion

In this paper and AE22 we considered tiling-derived triangles whose entries were shown to be numbers of k-subsets of $\mathbb{N}_n$ such that no two elements of the subset differ by an element in a set $\mathcal{Q}$ of disallowed differences. In AE22, $\mathcal{Q} = \{m\}$ for fixed $m \in \mathbb{Z}^+$, whereas in the present paper, $\mathcal{Q} = \{m, 2m, \ldots, (t-1)m\}$, where $t = 2, 3, \ldots$. One is then led to ask whether there is a correspondence between restricted combinations specified by other types of $\mathcal{Q}$ and tilings. When $\mathcal{Q} = \mathbb{N}_q$ for some $q \in \mathbb{Z}^+$, using the same ideas as in the proof of Lemma 7, it is straightforward to show that there is a bijection between the tilings of an $(n+q)$-board using k $(q+1)$-ominoes and squares and the number of k-subsets. However, the corresponding n-tile tilings triangles are just Pascal's triangle for any q. In order to obtain a tiling interpretation of restricted combinations with other classes of $\mathcal{Q}$ one needs a form of a tiling where some parts of the tiles are allowed to overlap with parts of other tiles. This is explored in depth in other work [1]. Whether or not such tiling schemes can be used to generate further aesthetically pleasing families of number triangles remains to be seen.

From some of the entries in the OEIS that give the sums of the $(1, t-1)$-antidiagonals of the triangle (see A224809, A224808, and A224811) it appears that the number of subsets of $\mathbb{N}_{n-(t-1)m}$ whose elements do not differ by an element of the set $\{m, 2m, \ldots, (t-1)m\}$ is also the number of permutations π of $\mathbb{N}_n$ such that $\pi(i) - i \in \{-m, 0, (t-1)m\}$ for all $i \in \mathbb{N}_n$. This is indeed true in general as we demonstrate combinatorially using combs and fences elsewhere [1].

In AE22 it was noted that the $\left\langle {n \atop k} \right\rangle_{1,2}$ and $\left\langle {n \atop k} \right\rangle_{2,2}$ triangles are row-reversed Riordan arrays and it was shown (in Corollary 37 of AE22) that the $\left\langle {n \atop k} \right\rangle_{m>2,2}$ triangles are not. From Theorem 35 of AE22, the $\left\langle {n \atop k} \right\rangle_{m\geq 2,t\geq 3}$ triangles are not row-reversed Riordan arrays since when tiling with $(1, m-1; t)$-combs and squares, the filled-comb metatile contains more than one square if $(m-1)(t-1) > 1$. The same theorem tells us that, except for the $m = 1$

cases, the triangles are also not Riordan arrays since there are metatiles containing more than one comb.

There are a number of types of tiling that lead to common-node-free digraphs that have only a few inner cycles. As far as we are aware, Theorem 36 is the first result giving recursion relations for a class of such cases. The theorem can be modified or generalized to cope with a wider variety of classes and we will present these results in future studies involving applications of tilings where instances of such digraphs arise.

8 Appendix: Recursion relations for 3-inner-cycle digraphs with a pseudo-common node

For a digraph lacking a common node, we refer to an inner cycle that can be represented as a single arc linking a node $\mathcal{E}$ to itself as an *errant loop* if the digraph would have a common node $\mathcal{P}$ if the errant loop arc were removed. The node $\mathcal{P}$ is then referred to as a *pseudo-common node*. Evidently, $\mathcal{E}$ and $\mathcal{P}$ cannot be the same node; if they were the same node, the original digraph would have a true common node. For a digraph with an errant loop, a *common circuit* is defined as two concatenated simple paths from the 0 node to $\mathcal{P}$ and from $\mathcal{P}$ to the 0 node. We also need to modify the definition of an outer cycle: it is now a cycle starting at the 0 node which does not include $\mathcal{P}$. An outer cycle, inner cycle, or common circuit is said to be *plain* if it does not include the errant loop node $\mathcal{E}$. See the proof of Identity 32 for examples.

We use the $N = 2$ case of the following lemma in the proof of Theorem 36.

Lemma 35. *For positive integers $j_0, j_1, \ldots, j_N$, where $N \geq 2$, we have*

$$\binom{j_1+\cdots+j_N}{j_1,\ldots,j_N}\binom{j_0+j_N-1}{j_0} = \sum_{r=1}^{N-1}\binom{j_1+\cdots+j_N-1}{j_1,\ldots,j_r-1,\ldots}\left(\binom{j_0+j_N-1}{j_0} - \binom{j_0+j_N-2}{j_0-1}\right) + \binom{j_1+\cdots+j_N}{j_1,\ldots,j_N}\binom{j_0+j_N-2}{j_0-1} + \binom{j_1+\cdots+j_N-1}{j_1,\ldots,j_N-1}\binom{j_0+j_N-2}{j_0}.$$

Proof. Using the result for multinomial coefficients that

$$\binom{j_1+\cdots+j_N}{j_1,\ldots,j_N} = \sum_{r=1}^{N}\binom{j_1+\cdots+j_N-1}{j_1,\ldots,j_r-1,\ldots,j_N},$$

we have

$$\binom{j_1+\cdots+j_N}{j_1,\ldots,j_N}\binom{j_0+j_N-1}{j_0} = \left(\sum_{r=1}^{N-1}\binom{j_1+\cdots+j_N-1}{j_1,\ldots,j_r-1,\ldots} + \binom{j_1+\cdots+j_N-1}{j_1,\ldots,j_N-1}\right)\binom{j_0+j_N-1}{j_0}$$

$$= \sum_{r=1}^{N-1} \binom{j_1 + \cdots + j_N - 1}{j_1, \ldots, j_r - 1, \ldots} \binom{j_0 + j_N - 1}{j_0}$$
$$+ \binom{j_1 + \cdots + j_N - 1}{j_1, \ldots, j_N - 1} \left(\binom{j_0 + j_N - 2}{j_0} + \binom{j_0 + j_N - 1}{j_0 - 1} \right)$$
$$= \sum_{r=1}^{N-1} \binom{j_1 + \cdots + j_N - 1}{j_1, \ldots, j_r - 1, \ldots} \binom{j_0 + j_N - 1}{j_0} + \binom{j_1 + \cdots + j_N - 1}{j_1, \ldots, j_N - 1} \binom{j_0 + j_N - 2}{j_0}$$
$$+ \left(\binom{j_1 + \cdots + j_N}{j_1, \ldots, j_N} - \sum_{r=1}^{N-1} \binom{j_1 + \cdots + j_N - 1}{j_1, \ldots, j_r - 1, \ldots, j_N} \right) \binom{j_0 + j_N - 2}{j_0 - 1},$$

which gives the required result on rearranging. □

Theorem 36. *For a digraph with an errant loop of length L_0 containing K_0 combs, a plain inner cycle of length L_1 containing K_1 combs, a non-plain inner cycle of length L_2 containing K_2 combs, and outer cycles that are all plain and have length $l_{\mathrm{o}i}$ and contain $k_{\mathrm{o}i}$ combs for $i = 1, \ldots, N_{\mathrm{o}}$, let $l_{\mathrm{c}i}$ be the length of the i-th common circuit and let $k_{\mathrm{c}i}$ be the number of combs it contains ($i = 1, \ldots, N_{\mathrm{c}}$), and let $l_{\mathrm{pc}i}$ be the length of the i-th plain common circuit and let $k_{\mathrm{pc}i}$ be the number of combs it contains ($i = 1, \ldots, N_{\mathrm{pc}}$). Then for all integers n and k we have*

$$B_n = \delta_{n,0} + \sum_{r=0}^{2} (B_{n-L_r} - \delta_{n,L_r}) + \delta_{n,L_0+L_1} - B_{n-L_0-L_1}$$
$$+ \sum_{i=1}^{N_{\mathrm{o}}} \left(B_{n-l_{\mathrm{o}i}} + B_{n-l_{\mathrm{o}i}-L_0-L_1} - \sum_{r=0}^{N} B_{n-l_{\mathrm{o}i}-L_r} \right) + \sum_{i=1}^{N_{\mathrm{c}}} B_{n-l_{\mathrm{c}i}} - \sum_{i=1}^{N_{\mathrm{pc}}} B_{n-l_{\mathrm{pc}i}-L_0}, \tag{13}$$

$$B_{n,k} = \delta_{n,0}\delta_{k,0} + \sum_{r=0}^{2} (B_{n-L_r,k-K_r} - \delta_{n,L_r}\delta_{k,K_r}) + \delta_{n,L_0+L_1}\delta_{k,K_0+K_1} - B_{n-L_0-L_1,k-K_0-K_1}$$
$$+ \sum_{i=1}^{N_{\mathrm{o}}} \left(B_{n-l_{\mathrm{o}i},k-k_{\mathrm{o}i}} + B_{n-l_{\mathrm{o}i}-L_0-L_1,k-k_{\mathrm{o}i}-K_0-K_1} - \sum_{r=0}^{N} B_{n-l_{\mathrm{o}i}-L_r,k-k_{\mathrm{o}i}-K_r} \right)$$
$$+ \sum_{i=1}^{N_{\mathrm{c}}} B_{n-l_{\mathrm{c}i},k-k_{\mathrm{c}i}} - \sum_{i=1}^{N_{\mathrm{pc}}} B_{n-l_{\mathrm{pc}i}-L_0,k-k_{\mathrm{pc}i}-K_0}, \tag{14}$$

where $B_{n<0} = B_{n,k<0} = B_{n<k,k} = 0$. If the lengths of the cycles and circuits are calculated as the number of tiles (the total contribution made to the number of cells occupied) then B_n is the number of n-tile tilings (the number of tilings of an n-board) and $B_{n,k}$ is the number of such tilings that use k combs.

Proof. To keep the algebra looking as simple as possible while retaining the essentials at the heart of the proof, we just prove the formula for B_n when there is a single outer cycle, one plain common circuit, and one non-plain common circuit. Their respective lengths are l_o, l_pc, and l_npc. It is straightforward to modify the proof we give here to include the sums over outer cycles and common circuits.

Conditioning on the final metatile gives

$$B_n = \delta_{n,0} + B_{n-l_\mathrm{o}} + \sum_{j_1 \geq 0} B_{n-l_\mathrm{pc}-j_1 L_1} + \sum_{\substack{j_0,j_1\geq 0,\\ j_2\geq 1}} \binom{j_1+j_2}{j_1}\binom{j_0+j_2-1}{j_0} B_{n-l_\mathrm{pc}-j_0L_0-j_1L_1-j_2L_2}$$
$$+ \sum_{e,j_1\geq 0} B_{n-l_\mathrm{npc}-eL_0-j_1L_1} + \sum_{\substack{e,j_0,j_1\geq 0,\\ j_2\geq 1}} \binom{j_1+j_2}{j_1}\binom{j_0+j_2-1}{j_0} B_{n-l_\mathrm{npc}-(j_0+e)L_0-j_1L_1-j_2L_2} \quad (15)$$

with $B_{n<0} = 0$. We now explain the origin of the four sums in (15) while referring to the digraph in Fig. 8 (taking the 010101 node as $\mathcal{P}$) for examples of metatiles. The first sum is from metatiles obtained by taking the first part of the plain common circuit to $\mathcal{P}$, then following the plain inner cycle j_1 times, and then returning to the 0 node via the second half of the plain common circuit (e.g., $CSC^{2j_1}CS$ is the symbolic representation of the metatiles corresponding to the terms in the sum).

The second sum corresponds to metatiles with the same start and end as with the first sum but on reaching $\mathcal{P}$ the plain and non-plain inner cycles are executed j_1 and j_2 times, respectively, in any order but the non-plain inner cycle is executed at least once. The number of ways of choosing the order is $\binom{j_1+j_2}{j_1}$. The errant loop is also traversed a total of j_0 times. Each time the path reaches $\mathcal{E}$ (during an execution of the non-plain inner cycle), it can detour and traverse the errant loop any number of times. This is the origin of the $\binom{j_0+j_2-1}{j_0}$ factor which has j_2-1 rather than j_2 as the non-plain inner cycle must be started before the errant loop can be traversed. E.g., the metatiles corresponding to the terms in the sum when $j_0 = j_1 = j_2 = 1$ are $CS\{C^2SCS, SCSC^2\}CS$.

The third and fourth sums are analogous to the first and second but $\mathcal{P}$ is reached via the first half of the non-plain common circuit, and after the inner cycles have been traversed j_r times (with $r = 1$ in the third sum and $r = 0, 1, 2$ in the fourth), the 0 node is returned to via the second half of the non-plain common circuit but the errant loop is executed an extra e times when the path reaches $\mathcal{E}$. E.g., the metatiles corresponding to the terms in the third sum are $CSC^{2j_1}SC^eS^2$, and in the fourth sum when $j_0 = j_1 = j_2 = 1$ they are $CS\{C^2SCS, SCSC^2\}SC^eS^2$.

Representing (15) by $E(n)$, we write down

$$E(n) - E(n-L_0) - E(n-L_1) + E(n-L_0-L_1) - E(n-L_2)$$

and re-index the sums so that, where possible, the $B_{n-\alpha}$ inside the sums for any α appear the same as for $E(n)$ (e.g., $\sum_{j_1\geq 0} B_{n-L_1-l_\mathrm{pc}-j_1L_1} = \sum_{j_1\geq 1} B_{n-l_\mathrm{pc}-j_1L_1}$). This leaves

$$B_n - \sum_{r=0}^{2} B_{n-L_r} + B_{n-L_0-L_1} = \delta_{n,0} + B_{n-l_\mathrm{o}} - \sum_{r=0}^{2}(\delta_{n,L_r} + B_{n-l_\mathrm{o}-L_r}) + \delta_{n,L_0+L_1} + B_{n-l_\mathrm{o}-L_0-L_1}$$

$$+\sum_{j_1\ge0}\beta_{j_1L_1}-\sum_{j_1\ge1}\beta_{j_1L_1}-\sum_{j_1\ge0}\beta_{L_0+j_1L_1}+\sum_{j_1\ge1}\beta_{L_0+j_1L_1}-\sum_{j_1\ge0}\beta_{j_1L_1+L_2}$$

$$+\sum_{e,j_1\ge0}\hat\beta_{j_1L_1}-\sum_{\substack{e\ge0,\\ j_1\ge1}}\hat\beta_{j_1L_1}-\sum_{\substack{e\ge1,\\ j_1\ge0}}\hat\beta_{j_1L_1}+\sum_{e,j_1\ge1}\hat\beta_{j_1L_1}-\sum_{e,j_1\ge0}\hat\beta_{j_1L_1+L_2}$$

$$+\sum_{\substack{j_0,j_1\ge0,\\ j_2\ge1}}\binom{j_1+j_2}{j_1}\binom{j_0+j_2-1}{j_0}\beta_\lambda-\sum_{\substack{j_0\ge0,\\ j_1,j_2\ge1}}\binom{j_1+j_2-1}{j_1-1}\binom{j_0+j_2-1}{j_0}\beta_\lambda$$

$$-\sum_{\substack{j_0,j_2\ge1,\\ j_1\ge0}}\binom{j_1+j_2}{j_1}\binom{j_0+j_2-2}{j_0-1}\beta_\lambda+\sum_{j_0,j_1,j_2\ge1}\binom{j_1+j_2-1}{j_1-1}\binom{j_0+j_2-2}{j_0-1}\beta_\lambda$$

$$-\sum_{\substack{j_0,j_1\ge0,\\ j_2\ge2}}\binom{j_1+j_2-1}{j_1}\binom{j_0+j_2-2}{j_0}\beta_\lambda$$

$$+\text{ the above 3 lines with }\beta_\lambda\text{ replaced by }\hat\beta_\lambda\text{ and also summed over all }e\ge0,\quad(16)$$

where $\beta_a = B_{n-l_{\mathrm{pc}}-a}$, $\hat\beta_a = B_{n-l_{\mathrm{npc}}-eL_0-a}$, and $\lambda = j_0L_0 + j_1L_1 + j_2L_2$. On rearranging (16) it is immediately apparent where all but the last two sums in (13) come from. The first two sums in the second line of (16) reduce to $\beta_0 = B_{n-l_{\mathrm{pc}}}$. The next two sums reduce to $-\beta_{L_0} = -B_{n-l_{\mathrm{pc}}-L_0}$, which accounts for the final sum in (13). The first four sums in the third line of (16) reduce to $B_{n-l_{\mathrm{npc}}}$, which, when added to the $B_{n-l_{\mathrm{pc}}}$, accounts for the penultimate sum in (13).

We now complete the proof by showing that the remaining terms in (16) cancel out. We regroup terms in each of the sums in the fourth, fifth, and sixth lines in (16) to give

$$\sum_{\substack{j_0,j_1\ge0,\\ j_2\ge1}}\binom{j_1+j_2}{j_1}\binom{j_0+j_2-1}{j_0}\beta_\lambda=\sum_{\substack{j_0,j_1\ge1,\\ j_2\ge2}}\binom{j_1+j_2}{j_1}\binom{j_0+j_2-1}{j_0}\beta_\lambda$$

$$+\sum_{j_0,j_1\ge1}\binom{j_1+1}{j_1}\beta_{j_0L_0+j_1L_1+L_2}+\sum_{\substack{j_0\ge1,\\ j_2\ge2}}\binom{j_0+j_2-1}{j_0}\beta_{j_0L_0+j_2L_2}+\sum_{\substack{j_1\ge1,\\ j_2\ge2}}\binom{j_1+j_2}{j_1}\beta_{j_1L_1+j_2L_2}$$

$$+\sum_{j_0\ge1}\beta_{j_0L_0+L_2}+\sum_{j_1\ge1}\binom{j_1+1}{j_1}\beta_{j_1L_1+L_2}+\sum_{j_2\ge2}\beta_{j_2L_2}+\beta_{L_2},\quad(17a)$$

$$\sum_{\substack{j_0\ge0,\\ j_1,j_2\ge1}}\binom{j_1+j_2-1}{j_1-1}\binom{j_0+j_2-1}{j_0}\beta_\lambda=\sum_{\substack{j_0,j_1\ge1,\\ j_2\ge2}}\binom{j_1+j_2-1}{j_1-1}\binom{j_0+j_2-1}{j_0}\beta_\lambda$$

$$+\sum_{j_0,j_1\ge1}\binom{j_1}{j_1-1}\beta_{j_0L_0+j_1L_1+L_2}+\sum_{\substack{j_1\ge1,\\ j_2\ge2}}\binom{j_1+j_2-1}{j_1-1}\beta_{j_1L_1+j_2L_2}+\sum_{j_1\ge1}\binom{j_1}{j_1-1}\beta_{j_1L_1+L_2},\quad(17b)$$

$$\sum_{\substack{j_0,j_2\geq 1,\\ j_1\geq 0}} \binom{j_1+j_2}{j_1}\binom{j_0+j_2-2}{j_0-1}\beta_\lambda = \sum_{\substack{j_0,j_1\geq 1,\\ j_2\geq 2}} \binom{j_1+j_2}{j_1}\binom{j_0+j_2-2}{j_0-1}\beta_\lambda$$

$$+ \sum_{j_0,j_1\geq 1} \binom{j_1+1}{j_1}\beta_{j_0L_0+j_1L_1+L_2} + \sum_{\substack{j_0\geq 1,\\ j_2\geq 2}} \binom{j_0+j_2-1}{j_0-1}\beta_{j_0L_0+j_2L_2} + \sum_{j_0\geq 1}\beta_{j_0L_0+L_2}, \tag{17c}$$

$$\sum_{j_0,j_1,j_2\geq 1} \binom{j_1+j_2-1}{j_1-1}\binom{j_0+j_2-2}{j_0-1}\beta_\lambda = \sum_{\substack{j_0,j_1\geq 1,\\ j_2\geq 2}} \binom{j_1+j_2-1}{j_1-1}\binom{j_0+j_2-2}{j_0-1}\beta_\lambda$$

$$+ \sum_{j_0,j_1\geq 1} \binom{j_1}{j_1-1}\beta_{j_0L_0+j_1L_1+L_2}, \tag{17d}$$

$$\sum_{\substack{j_0,j_1\geq 0,\\ j_2\geq 2}} \binom{j_1+j_2-1}{j_1}\binom{j_0+j_2-2}{j_0}\beta_\lambda = \sum_{\substack{j_0,j_1\geq 1,\\ j_2\geq 2}} \binom{j_1+j_2-1}{j_1}\binom{j_0+j_2-2}{j_0}\beta_\lambda$$

$$+ \sum_{\substack{j_0\geq 1,\\ j_2\geq 2}} \binom{j_0+j_2-1}{j_0}\beta_{j_0L_0+j_2L_2} + \sum_{\substack{j_1\geq 1,\\ j_2\geq 2}} \binom{j_1+j_2-1}{j_1}\beta_{j_1L_1+j_2L_2} + \sum_{j_2\geq 2}\beta_{j_2L_2}. \tag{17e}$$

We denote the p-th sum (or term) on the right-hand side of (17x) by x_p where x is a–e. Then $\mathrm{a}_1 - \mathrm{b}_1 - \mathrm{c}_1 + \mathrm{d}_1 - \mathrm{e}_1 = 0$ by virtue of Lemma 35, a_2 cancels c_2, a_3 cancels $\mathrm{c}_3 + \mathrm{e}_2$, a_4 cancels $\mathrm{b}_3 + \mathrm{e}_3$, a_5 cancels c_4, $\mathrm{a}_6 - \mathrm{b}_4 + \mathrm{a}_8 = \sum_{j_1\geq 0}\beta_{j_1L_1+L_2}$ and therefore cancels the last sum in the second line of (16), a_7 cancels e_4, and b_2 cancels d_2. The simplification works in the same way for the terms represented by the last line of (16). Denoting sums or terms in the corresponding set of equations by $\hat{x}_p$, $\hat{\mathrm{a}}_6 - \hat{\mathrm{b}}_4 + \hat{\mathrm{a}}_8 = \sum_{e,j_1\geq 0}\hat{\beta}_{j_1L_1+L_2}$ and therefore cancels the last sum in the third line of (16).

The proof of (14) proceeds in an analogous way. Again considering the case where there is a single outer cycle (with k_{o} combs), a plain common circuit (with k_{pc} combs), and a non-plain common circuit (with k_{npc} combs), conditioning on the final metatile gives

$$B_{n,k} = \delta_{n,0}\delta_{k,0} + B_{n-l_{\mathrm{o}},k-k_{\mathrm{o}}} + \sum_{j_1\geq 0} B_{n-l_{\mathrm{pc}}-j_1L_1,n-k_{\mathrm{pc}}-j_1K_1}$$

$$+ \sum_{\substack{j_0,j_1\geq 0,\\ j_2\geq 1}} \binom{j_1+j_2}{j_1}\binom{j_0+j_2-1}{j_0} B_{n-l_{\mathrm{pc}}-\lambda,k-k_{\mathrm{pc}}-\kappa} + \sum_{e,j_1\geq 0} B_{n-l_{\mathrm{npc}}-eL_0-j_1L_1,k-k_{\mathrm{npc}}-eK_0-j_1K_1}$$

$$+ \sum_{\substack{e,j_0,j_1\geq 0,\\ j_2\geq 1}} \binom{j_1+j_2}{j_1}\binom{j_0+j_2-1}{j_0} B_{n-l_{\mathrm{npc}}-eL_0-\lambda,k-k_{\mathrm{npc}}-eK_0-\kappa} \tag{18}$$

with $B_{n,k>n} = B_{n,k<0} = 0$ and where $\kappa = j_0K_0 + j_1K_1 + j_2K_2$. Denoting (18) by $E(n,k)$, writing down

$$E(n,k) - E(n-L_0,k-K_0) - E(n-L_1,k-K_1) + E(n-L_0-L_1,k-K_0-K_1) - E(n-L_2,k-K_2),$$

and then proceeding in the same way as for the proof of (13) gives the required result. □

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