Journal of Integer Sequences, Vol. 24 (2021),
Article 21.9.5

Sum of Reciprocals of Germain Primes

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Abstract

A prime p is a Germain prime if $2p+1$ is also prime. We compute the sum of the reciprocals of the Germain primes and related sequences. Since we do not know whether there are infinitely many Germain primes, all we can do is bound the sum in an interval.

1 Introduction

In 1919, Brun [4] proved that the sum

$$B = \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{29} + \frac{1}{31} + \cdots$$

of the reciprocals of all twin primes either converges or is a finite sum. Various authors [17] have estimated Brun's constant B. Sebah [16] computed the sum of the reciprocals of all twin primes $< 10^{16}$ and found that $B > 1.830$. Klyve [8] showed that $B < 2.347$ as reported in Crandall and Pomerance [5]. Platt and Trudgian [14] used the number of twin primes $< 4 \cdot 10^{18}$ computed by Oliveira e Silva [13] and found that $1.840503 < B < 2.288513$. Several authors [3, 2, 10, 11, 13, 14, 16, 17] agree that the most probable value of B is about 1.90216, but this estimate is not rigorous. These results show how little we know about the distribution of twin primes, even whether there are infinitely many of them. The twin primes are Sequence A001097 in the OEIS [12].

A prime p is a Germain prime if $2p + 1$ is also prime. They are named after Sophie Germain, who proved nearly 200 years ago that the first case of Fermat's Last Theorem is true when the exponent is a Germain prime. The Germain primes are Sequence A005384 in the OEIS. Computing the sum of the reciprocals of the Germain primes is similar to computing Brun's constant B. First, it is easy to prove using a sieve that this sum either converges or is a finite sum. Second, we do not know whether there are infinitely many Germain primes, so the best lower bound for the sum is the partial sum up to the limit to which we can compute it directly. One can compute a most probable value for the sum just as for B.

Little extra work is required to study the primes p for which $2^k p+1$ is also prime for any fixed $k \geq 1$. The same methods apply to the sum of the reciprocals of the primes p for which $p + 2^k$ is also prime. See Lee and Park [9] for the case of p, $p + 8$ both prime, for example.

2 Notation and easy results

Throughout this work p denotes a prime. Let

$$c_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \approx 0.660161815846869573927812110\,0$$

be the twin prime constant as computed by Harley (See Lee and Park [9]).

Let a and b be positive integers with $\gcd(a, b) = 1$ and $2 \mid ab$. Let $\mathcal{S}_{a,b} = \{p : ap + b \text{ is prime}\}$. For real $x > 0$, let $S_{a,b}(x)$ denote the sum of $\frac{1}{p}$ over all primes $p \leq x$ with $p \in \mathcal{S}_{a,b}$. Also let $S'_{a,b}(x)$ be the sum of $\frac{1}{p} + \frac{1}{ap+b}$ over all primes $p \leq x$ with $p \in \mathcal{S}_{a,b}$. Clearly one sum is finite if and only if the other is finite. Let $S_{a,b} = \lim_{x\to\infty} S_{a,b}(x)$ and $S'_{a,b} = \lim_{x\to\infty} S'_{a,b}(x)$.

Thus $S'_{1,2} = B$, Brun's constant, and

$$S_{2,1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{11} + \frac{1}{23} + \frac{1}{29} + \frac{1}{41} + \frac{1}{53} + \cdots$$

is the sum of the reciprocals of the Germain primes. Also,

$$S_{4,1} = \frac{1}{3} + \frac{1}{7} + \frac{1}{13} + \frac{1}{37} + \frac{1}{43} + \frac{1}{67} + \frac{1}{73} + \frac{1}{79} + \cdots$$

and

$$S_{8,1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{11} + \frac{1}{17} + \frac{1}{29} + \frac{1}{71} + \frac{1}{101} + \frac{1}{107} + \cdots .$$

See Sequences A023212 and A023228 in the OEIS. The case of $S_{1,8}$ was studied by Lee and Park [9].

We show first that the limits $S_{a,b}$ and $S'_{a,b}$ always exist. Let $\pi_{a,b}(x)$ be the number of primes $p \leq x$ for which $ap + b$ is also prime.

Theorem 1. *Suppose* $\gcd(a,b)=1$ *and* $2 \mid ab$*. Define* $S_{a,b}(x)$*,* $S'_{a,b}(x)$*,* $S_{a,b}$ *and* $S'_{a,b}$ *as above. Then the limits in the definitions of* $S_{a,b}$ *and* $S'_{a,b}$ *exist.*

Proof. Since $S_{a,b}(x)$ is the sum of positive terms we need only show that it is bounded. According to Theorem 3.12 of Halberstam and Richert [7], we have

$$\pi_{a,b}(x) \le \frac{cx}{(\log x)^2}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right).$$

for some constant c. Therefore,

$$\begin{aligned} S_{a,b}(x) &= \sum_{p\le x, p\in \mathcal{S}_{a,b}} \frac{1}{p} = \sum_{t=1}^{x} \frac{\pi_{a,b}(t)-\pi_{a,b}(t-1)}{t} \\ &\le \sum_{t=2}^{x}\left(\frac{1}{t}-\frac{1}{t+1}\right)\pi_{a,b}(t) \qquad (\text{since } \pi_{a,b}(1)=0) \\ &\le \sum_{t=2}^{x} \frac{1}{t(t+1)}\frac{ct}{(\log t)^2}\left(1+O\left(\frac{\log\log t}{\log t}\right)\right) = O(1) \end{aligned}$$

as $x\to\infty$. The proof for $S'_{a,b}$ is identical. □

Fix x_0. Since $1/p>0$ always, we have $S_{a,b}\ge S_{a,b}(x_0)$ for every (a,b). These are the best lower bounds for $S_{a,b}$ that we know how to compute. Table 1 shows these lower bounds for a few values of x_0. We computed lower bounds for Brun's constant for comparison, although a better lower bound is found in Platt and Trudgian [14]. The sums were computed in double precision, but only nine decimal places are shown in the table. Table 2 shows the corresponding values of $\pi_{a,b}$. The data for twin primes is exactly the same as Nicely [10] found.

The most probable value for $S_{a,b}$ is based on the Hardy-Littlewood [6] (or see Bateman and Horn [1]) approximation to $\pi_{a,b}(x)$. This heuristic estimate is

$$\pi_{a,b}(x) \approx 2c_2\int_2^x \frac{dt}{(\log t)^2}\prod_{p>2,\, p\mid ab}\frac{p-1}{p-2}$$

for fixed coprime integers a, b. When ab is a power of 2, which we always assume in this work, the product over primes $p>2$ is 1. Assuming this heuristic and using partial summation, one finds

$$S_{a,b}-S_{a,b}(x_0) \approx 2c_2\int_{x_0}^{\infty}\frac{dt}{t(\log t)^2} = \frac{2c_2}{\log x_0}.$$

Thus, after computing $S_{a,b}(x_0)$, the most probable value for $S_{a,b}$ is $S_{a,b}(x_0)+2c_2/\log x_0$. The constant $2c_2$ is replaced by $4c_2$ for $S'_{1,2}$ because we add the reciprocals of two (consecutive) primes in that case. These values are shown in Table 3.

x_0	$S'_{1,2}(x_0)$	$S_{2,1}(x_0)$	$S_{4,1}(x_0)$	$S_{8,1}(x_0)$
10^2	1.330990366	1.268745760	0.654987903	0.898299886
10^3	1.518032463	1.350207371	0.718127473	0.985924806
10^4	1.616893558	1.395040698	0.758673876	1.021448995
10^5	1.672799585	1.422243022	0.783466978	1.045080342
10^6	1.710776931	1.440222094	0.800867822	1.061448591
10^7	1.738357044	1.453223143	0.813394490	1.073447436
10^8	1.758815621	1.463079361	0.822866962	1.082581870
10^9	1.774735958	1.470771610	0.830302440	1.089777846
10^{10}	1.787478503	1.476946485	0.836290746	1.095594911
10^{11}	1.797904311	1.482013752	0.841220054	1.100393501
10^{12}	1.806592419	1.486246659	0.845347507	1.104420686
$2 \cdot 10^{12}$	1.808931050	1.487387756	0.846461664	1.105509109
$5 \cdot 10^{12}$	1.811852563	1.488814135	0.847855248	1.106871419
10^{13}	1.813943761	1.489835793	0.848854028	1.107848344

Table 1: Some values of $S'_{1,2}(x)$ and $S_{a,b}(x)$.

x_0	$\pi_{1,2}(x_0)$	$\pi_{2,1}(x_0)$	$\pi_{4,1}(x_0)$	$\pi_{8,1}(x_0)$
10^2	8	10	9	6
10^3	35	37	31	34
10^4	205	190	176	161
10^5	1224	1171	1057	1019
10^6	8169	7746	7422	7090
10^7	58980	56032	53709	51464
10^8	440312	423140	407198	392990
10^9	3424506	3308859	3198946	3095744
10^{10}	27412679	26569515	25773602	25030013
10^{11}	224376048	218116524	212205881	206597032
10^{12}	1870585220	1822848478	1777532673	1734464714
$2 \cdot 10^{12}$	3552770943	3464491157	3380477421	3300430590
$5 \cdot 10^{12}$	8312493003	8112446112	7921805792	7740062386
10^{13}	15834664872	15462601989	15107599980	14768799353

Table 2: Some values of $\pi_{a,b}(x_0)$.

x_0	$S'_{1,2}$	$S_{2,1}$	$S_{4,1}$	$S_{8,1}$
10^2	1.904399633	1.555450394	0.941692537	1.185004520
10^3	1.900305309	1.541343794	0.909263896	1.177061229
10^4	1.903598191	1.538393015	0.902026193	1.164801312
10^5	1.902163292	1.536924875	0.898148831	1.159762196
10^6	1.901913353	1.535790305	0.896436033	1.157016803
10^7	1.902188263	1.535138753	0.895310100	1.155363045
10^8	1.902167938	1.534755520	0.894543121	1.154258029
10^9	1.902160239	1.534483751	0.894014581	1.153489986
10^{10}	1.902160356	1.534287412	0.893631672	1.152935837
10^{11}	1.902160541	1.534141868	0.893348170	1.152521616
10^{12}	1.902160630	1.534030764	0.893131612	1.152204792
$2 \cdot 10^{12}$	1.902160522	1.534002492	0.893076401	1.152123846
$5 \cdot 10^{12}$	1.902160560	1.533968133	0.893009246	1.152025417
10^{13}	1.902160571	1.533944198	0.892962433	1.151956749

Table 3: Most probable values of $S'_{1,2}$ and $S_{a,b}$.

The consistency of the estimates in the columns instills confidence, although there is no proof that they are anywhere near the true values. One might consider "higher order" terms in this approximation, but Shanks and Wrench [17] tell why this is unlikely to provide a closer estimate; see Sections 3 and 5 of their paper.

3 Upper bound on the limit

The upper bound is based on Lemma 5 of Riesel and Vaughan [15], which we quote here using Inequality (3.20) from their proof, as was done by Klyve [8].

Theorem 2. *Let a and b be integers with $a > 0$, $b \neq 0$ and $\gcd(a,b) = 1$. Let*

$$R(x,a,b) = \sup_{I} \sum_{p \in I, ap+b \text{ prime}} 1,$$

where the supremum is taken over all intervals of length x. Suppose that L and $C = C(L)$ are related by Table 4. Then, whenever $x \geq e^L$ we have

$$R(x,a,b) < \left(\frac{16 c_2 x}{(\log x)(C + \log x)} + 2\sqrt{x} \right) \prod_{p>2,\, p|ab} \frac{p-1}{p-2}\,.$$

L	C	L	C
24	0.97	48	8.2054
25	2.31	60	8.302
26	3.40	82	8.3503
27	4.28	100	8.3708
28	5.00	127	8.3905
29	5.58	147	8.404
31	6.45	174	8.4102
34	7.24	214	8.4201
36	7.56	278	8.4301
42	8.04	396	8.44004
44	8.11	690	8.45001

Table 4: L and C of Theorem 2.

The product in Theorem 2 is 1 when ab is a power of 2, as we always assume. Klyve [8] found it more convenient not to have the $2\sqrt{x}$ term in the theorem. He gave an alternate version, which we quote here.

Theorem 3. *Let a and b be coprime positive integers with ab a power of 2. Suppose that L and $D = D(L)$ are related by Table 5. Then, whenever $x \geq e^L$ we have*

$$\pi_{a,b}(x) < \frac{16c_2 x}{(\log x)(D + \log x)}.$$

The same bound applies when $\pi_{a,b}(x)$ is replaced by the count over any other interval of length $x \geq e^L$.

In Table 5 we correct a tiny error in Klyve [8]. He had $D = 2.30$ when $L = 25$; it should be $D = 2.296$.

L	D	L	D
24	0.95	48	8.20
25	2.296	60	8.30
26	3.39	82	8.35
27	4.27	100	8.37
28	4.99	127	8.39
29	5.57	147	8.40
31	6.44	174	8.41
34	7.23	214	8.42
36	7.55	278	8.43
42	8.03	396	8.44
44	8.10	690	8.45

Table 5: L and D of Theorem 3.

Note that the upper bound in Theorem 3 is independent of a and b, so long as ab is a power of 2.

Theorem 4. *We have* $1.4898 < S_{2,1} < 1.8027$, $0.8488 < S_{4,1} < 1.1617$ *and* $1.1078 < S_{8,1} < 1.4208$.

Proof. The lower bounds come from Table 1 with $x_0 = 10^{13}$.

Let a and b be positive integers with $\gcd(a, b) = 1$ and $2 \mid ab$. Let $0 < M < N$. Stieltjes integration by parts yields

$$\sum_{t=M}^{N} \frac{\pi_{a,b}(t) - \pi_{a,b}(t-1)}{t} = \frac{\pi_{a,b}(N)}{N} - \frac{\pi_{a,b}(M)}{M} + \int_M^N \frac{\pi_{a,b}(t)}{t^2} dt\,. \tag{1}$$

Now

$$S_{a,b} = S_{a,b}(x_0) + \sum_{t=x_0}^{\infty} \frac{\pi_{a,b}(t) - \pi_{a,b}(t-1)}{t}\,.$$

We will divide the interval $[x_0, \infty)$ into segments with boundaries at e^L for L in Table 5 to take advantage of the constants $D(L)$ in that table. We will use Stieltjes integration to bound $S_{a,b}(N) - S_{a,b}(M)$ on each interval $[M, N)$.

Use Theorem 3 to bound the integral above. If L, L' are consecutive entries in Table 5, then the integral over $[M, N) = [e^L, e^{L'})$ becomes

$$\begin{aligned}
\int_{e^L}^{e^{L'}} \frac{\pi_{a,b}(t)}{t^2} dt &\le \int_{e^L}^{e^{L'}} \frac{16c_2 t}{t^2(\log t)(D(L) + \log t)}\, dt \\
&= 16c_2 \int_L^{L'} \frac{ds}{s(s + D(L))} \qquad (s = \log t) \\
&= \frac{16c_2}{D(L)} (\log s - \log(s + D(L)))\Big|_L^{L'} \\
&= \frac{16c_2}{D(L)} \log\left(\frac{L'(L + D(L))}{L(L' + D(L))}\right)\,.
\end{aligned}$$

Note that $\log x_0 \approx 29.993606$. If $L \le \log x_0 < L'$ in Table 5, then $L = 29$, $L' = 31$ and the first integral is bounded by

$$\frac{16c_2}{D(L)} \log\left(\frac{L'(\log(x_0) + D(L))}{\log(x_0)(L' + D(L))}\right) \approx 0.010262\,.$$

The last integral is bounded by

$$\frac{16c_2}{8.45} \log\left(\frac{8.45 + 690}{690}\right) \approx 0.015216\,.$$

The values of these upper bounds are shown in Table 6.

M	N	Upper Bound	M	N	Upper Bound
x_0	e^{31}	0.0102619142161059	e^{100}	e^{127}	0.0208935182022260
e^{31}	e^{34}	0.0250838160282704	e^{127}	e^{147}	0.0106594237333466
e^{34}	e^{36}	0.0143023911370928	e^{147}	e^{174}	0.0105916018816358
e^{36}	e^{42}	0.0350847550483840	e^{174}	e^{214}	0.0108704126795364
e^{42}	e^{44}	0.0096317734983264	e^{214}	e^{278}	0.0109807270912696
e^{44}	e^{48}	0.0170051336041258	e^{278}	e^{396}	0.0110369750969163
e^{48}	e^{60}	0.0381486495943858	e^{396}	e^{690}	0.0111777020604581
e^{60}	e^{82}	0.0421828971523529	e^{690}	∞	0.0152151240070501
e^{82}	e^{100}	0.0212203397984129			

Table 6: M, N and the upper bound for $\int_M^N \pi_{a,b}(t)t^{-2}dt$.

The sum of all upper bounds gives

$$\int_{x_0}^{\infty} \frac{\pi_{a,b}(t)}{t^2} dt < 0.3143472\,,$$

which is the same for every (a, b) with ab a power of 2. The sum of the terms $\pi_{a,b}(N)/N - \pi_{a,b}(M)/M$ in Equation (1) telescopes to give $-\pi_{a,b}(x_0)/x_0$. Adding these values to the lower bounds from Table 1 gives

$$\begin{aligned} S_{2,1} &< 1.4898358 - 0.0015463 + 0.3143472 = 1.8026367, \\ S_{4,1} &< 0.8488541 - 0.0015108 + 0.3143472 = 1.1616905, \qquad \text{and} \\ S_{8,1} &< 1.1078484 - 0.0014769 + 0.3143472 = 1.4207187. \end{aligned}$$

□

Better upper and lower bounds would result from using a larger x_0. One probably can derive slightly smaller upper bounds (with the same x_0) by assuming the ERH as Klyve [8] did for $S'_{1,2}$.

4 Acknowledgments

I thank Harold Diamond and Carl Pomerance for valuable correspondence concerning this research. I am grateful to Nick Egbert for noticing the typo in Klyve's thesis [8].

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2010 *Mathematics Subject Classification*: Primary 11N05. Secondary 11L20, 11N37, 11N55.

Keywords: Brun's constant, Germain prime, twin prime, sum of reciprocals.

(Concerned with sequences A001097, A005384, A023212, and A023228.)

Received August 16 2021; revised version received October 2 2021. Published in *Journal of Integer Sequences*, October 2 2021.

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