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Irrationality of Growth Constants Associated with Polynomial Recursions

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Abstract

We consider integer sequences that satisfy a recursion of the form $x_{n+1} = P(x_n)$ for some polynomial P of degree $d > 1$. If such a sequence tends to infinity, then it satisfies an asymptotic formula of the form $x_n \sim A\alpha^{d^n}$, but little can be said about the

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constant α. In this paper, we show that α is always irrational or an integer. In fact, we prove a stronger statement: if a sequence $(G_n)_{n\geq 0}$ satisfies an asymptotic formula of the form $G_n = A\alpha^n + B + O(\alpha^{-\epsilon n})$, where A, B are algebraic and $\alpha > 1$, and the sequence contains infinitely many integers, then α is irrational or an integer.

1 Introduction

Integer sequences obtained by polynomial iteration, i.e., sequences that satisfy a recursion of the form

$$x_{n+1} = P(x_n),$$

occur in several areas of mathematics. One can find many interesting examples in Finch's book on mathematical constants [2, Chapter 6.10].

Let us give two concrete examples: the first is the sequence given by $x_0 = 0$ and $x_{n+1} = x_n^2 + 1$ for $n \geq 0$, which is entry A003095 in the On-Line Encyclopedia of Integer Sequences (OEIS) [6]. Among other things, x_n is the number of binary trees whose height (greatest distance from the root to a leaf) is less than n. This sequence grows very rapidly: there exists a constant $\beta \approx 1.2259024435$ (the digits are A076949 in the OEIS) such that $x_n = \lfloor \beta^{2^n} \rfloor$. However, this formula is not as explicit as it may seem, since the only known way to evaluate the constant β numerically involves all elements of the sequence: it can be expressed as

$$\beta = \prod_{n=1}^{\infty} \left(1 + x_n^{-2}\right)^{2^{-n-1}}.$$

Another well-known example is Sylvester's sequence (A000058 in the OEIS), which is given by $y_0 = 2$ and $y_{n+1} = y_n^2 - y_n + 1$. It arises in the context of Egyptian fractions, y_n being the smallest positive integer for each n such that

$$\frac{1}{y_0} + \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n} < 1.$$

There is also a pseudo-explicit formula for this sequence: for a constant $\gamma \approx 1.5979102180$, we have $y_n = \lfloor \gamma^{2^n} + \frac{1}{2} \rfloor$. However, again no formula for γ is known that does not involve the sequence elements. This is also the reason why little is known about the constants β and γ in these two examples and generally growth constants associated with similar sequences that satisfy a polynomial recursion.

In this short note, we will prove that the constants β and γ in these examples are—perhaps unsurprisingly—irrational, as are all non-integer growth constants associated with similar sequences that follow a polynomial recursion. The precise statement reads as follows:

Theorem 1. *Suppose that an integer sequence satisfies a recursion of the form $x_{n+1} = P(x_n)$ for some polynomial P of degree $d > 1$ with rational coefficients. Assume further that $x_n \to \infty$ as $n \to \infty$. Set*

$$\alpha = \lim_{n\to\infty} (x_n)^{d^{-n}}.$$

Then α is a real number greater than 1 that is either irrational or an integer.

It is natural to conjecture that the constants β and γ in our first two examples are not only irrational, but even transcendental. This is not always true for polynomial recursions in general, though: for example, consider the sequence given by $z_1 = 3$ and $z_{n+1} = z_n^2 - 2$. It is not difficult to prove that

$$z_n = L_{2^n} = \Big(\frac{1+\sqrt{5}}{2}\Big)^{2^n} + \Big(\frac{1-\sqrt{5}}{2}\Big)^{2^n}$$

for all $n \geq 1$, where L_n denotes the n-th Lucas number. Thus the constant α in Theorem 1 would be the golden ratio in this example.

In the following section, we briefly review the classical method to determine the asymptotic behavior of polynomially recurrent sequences. Theorem 1 will follow as a consequence of a somewhat stronger result, Theorem 2. This theorem and its proof, which makes use of the subspace theorem, will be given in Section 3.

2 Asymptotic formulas for polynomially recurrent sequences

There is a classical technique for the analysis of polynomial recursions. The 1973 paper of Aho and Sloane [1] already contains a treatment of the two examples given in the introduction (along with many other examples). See also the book of Greene and Knuth [3, Chapter 2.2.3] for a discussion of the method.

Let the polynomial P in the recursion $x_{n+1} = P(x_n)$ be given by

$$P(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_0.$$

Note that

$$P(x) = c_d \Big(x + \frac{c_{d-1}}{dc_d}\Big)^d + O(x^{d-2}).$$

So if we perform the substitution $y_n = c_d^{1/(d-1)}(x_n + \frac{c_{d-1}}{dc_d})$, the recursion becomes

$$\begin{aligned} y_{n+1} &= c_d^{1/(d-1)}\Big(P(x_n) + \frac{c_{d-1}}{dc_d}\Big) \\ &= c_d^{d/(d-1)}\Big(x_n + \frac{c_{d-1}}{dc_d}\Big)^d + O(x_n^{d-2}) \\ &= y_n^d + O(y_n^{d-2}). \end{aligned}$$

Let us assume that $x_n \to \infty$, thus also $y_n \to \infty$. It is easy to see that the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are increasing from some point onwards in this case. We can also assume, without

loss of generality, that none of the y_n is zero: if not, we simply choose a later starting point. Taking the logarithm, we obtain

$$\log y_{n+1} = d \log y_n + O(y_n^{-2})$$

or equivalently

$$\log\left(\frac{y_{n+1}}{y_n^d}\right) = O(y_n^{-2}). \tag{1}$$

Next express $\log y_n$ as follows:

$$\begin{aligned}\log y_n &= d \log y_{n-1} + \log\left(\frac{y_n}{y_{n-1}^d}\right) = d^2 \log y_{n-2} + d \log\left(\frac{y_{n-1}}{y_{n-2}^d}\right) + \log\left(\frac{y_n}{y_{n-1}^d}\right) \\ &= \cdots = d^n \log y_0 + \sum_{k=0}^{n-1} d^{n-k-1} \log\left(\frac{y_{k+1}}{y_k^d}\right).\end{aligned}$$

Extending to an infinite sum (which converges since $\log(y_{k+1}/y_k^d)$ is bounded) yields

$$\log y_n = d^n\Big(\log y_0 + \sum_{k=0}^{\infty} d^{-k-1} \log\left(\frac{y_{k+1}}{y_k^d}\right)\Big) - \sum_{k=n}^{\infty} d^{n-k-1} \log\left(\frac{y_{k+1}}{y_k^d}\right).$$

Set

$$\log \alpha = \log y_0 + \sum_{k=0}^{\infty} d^{-k-1} \log\left(\frac{y_{k+1}}{y_k^d}\right),$$

so that

$$\log y_n = d^n \log \alpha - \sum_{k=n}^{\infty} d^{n-k-1} \log\left(\frac{y_{k+1}}{y_k^d}\right).$$

In view of (1) and the fact that $y_n \le y_{n+1} \le \cdots$ for sufficiently large n, this gives

$$\log y_n = d^n \log \alpha + O(y_n^{-2}),$$

and thus finally

$$y_n = \alpha^{d^n} + O\big(\alpha^{-d^n}\big).$$

This means that

$$x_n = c_d^{-1/(d-1)} \alpha^{d^n} - \frac{c_{d-1}}{d c_d} + O\big(\alpha^{-d^n}\big).$$

3 Application of the subspace theorem

We now combine the asymptotic formula from the previous section with an application of the subspace theorem to prove our main result on polynomial recursions. In fact, we first state and prove a somewhat stronger result that implies Theorem 1.

Theorem 2. *Assume that the sequence $(G_n)_{n\geq 0}$ attains an integral value infinitely often, and that it satisfies an asymptotic formula of the form*

$$G_n = A\alpha^n + B + O(\alpha^{-\epsilon n}),$$

where $\alpha > 1$, A and B are algebraic numbers with $A \neq 0$, $\epsilon > 0$, and the constant implied by the O-term does not depend on n. Then the number α is either irrational or an integer.

In order to prove the irrationality of α we make use of the following version of the subspace theorem, which is most suitable for our purposes, cf. [5, Chapter V, Theorem 1D].

Theorem 3 (Subspace theorem)**.** *Let K be an algebraic number field, let $\mathfrak{a}_K$ be its maximal order, and let $M(K)$ be the set of canonical absolute values of K. Moreover, let $S \subset M(K)$ be a finite set of absolute values that contains all of the Archimedian ones. For each $\nu \in S$, $|\cdot|_\nu$ denotes the valuation corresponding to ν, and n_ν denotes the local degree. For $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathfrak{a}_K^N$, we define*

$$\overline{|\mathbf{x}|} = \max_{\substack{1\leq i\leq N \\ 1\leq j\leq \deg K}} |x_i^{(j)}|,$$

the maximum being taken over all conjugates $x_i^{(j)}$ of all entries x_i of $\mathbf{x}$. Finally, for each $\nu \in S$, let $L_{\nu,1}, \ldots, L_{\nu,N}$ be N linearly independent linear forms in n variables with coefficients in K. Then for given $\delta > 0$, the solutions of the inequality

$$\prod_{\nu\in S}\prod_{i=1}^{N} |L_{\nu,i}(\mathbf{x})|_\nu^{n_\nu} < \overline{|\mathbf{x}|}^{-\delta},$$

with $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathfrak{a}_K^N$ and $\mathbf{x} \neq \mathbf{0}$, lie in finitely many proper subspaces of K^N.

Proof of Theorem 2. Let us assume contrary to the statement of Theorem 2 that $\alpha = p/q$ is rational, where p and q are coprime positive integers, $p > q$, and $q \neq 1$. Moreover, assume that their prime factorizations are

$$p = p_1^{n_1} \cdots p_k^{n_k} \quad \text{and} \quad q = q_1^{m_1} \cdots q_\ell^{m_\ell}.$$

We choose K in the subspace theorem to be the normal closure of $\mathbb{Q}(A, B)$, and we write $D = [K : \mathbb{Q}]$. We fix one embedding of K into $\mathbb{C}$, so that we can assume that $K \subseteq \mathbb{C}$. Moreover, let us write A and B as $A = \beta_1/Q$ and $B = \beta_2/Q$, where β_1 and β_2 lie in the maximal order $\mathfrak{a}_K$ of K, and Q is a positive integer such that the ideals (β_1, β_2) and (Q) are coprime.

If n is an index such that G_n is an integer, we deduce that there exists an algebraic integer a which may depend on n such that

$$G_n = \frac{\beta_1 p^n + \beta_2 q^n + a}{Qq^n} = A\alpha^n + B + \frac{a}{Qq^n} = A\alpha^n + B + O(\alpha^{-\epsilon n}).$$

Since we are assuming that G_n is a rational integer, we can write the algebraic integer a in the form $a = X - \beta_1 p^n - \beta_2 q^n$, with $X \in \mathbb{Z}$. Moreover, we know that

$$|a| < CQ \left(\frac{q^{1+\epsilon}}{p^\epsilon} \right)^n, \tag{2}$$

where C is the constant implied by the O-term.

Assume that K has signature (r, s). We choose

$$S = \{\infty_1, \ldots, \infty_{r+s}, \mathfrak{p}_{1,1}, \ldots, \mathfrak{p}_{k,t_k}, \mathfrak{q}_{1,1}, \ldots, \mathfrak{q}_{\ell,u_\ell}\},$$

where the valuations $\mathfrak{p}_{i,1}, \ldots, \mathfrak{p}_{i,t_i}$ are all valuations lying above p_i for $1 \le i \le k$, and the valuations $\mathfrak{q}_{j,1}, \ldots, \mathfrak{q}_{j,u_j}$ are all valuations lying above q_j for $1 \le j \le \ell$. Moreover, let

$$\mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \bar{\sigma}_{r+1}, \ldots, \sigma_{r+s}, \bar{\sigma}_{r+s}\},$$

so that the valuation ∞_i is given by $|x|_{\infty_i} = |\sigma_i^{-1} x|$, where $|\cdot|$ is the usual absolute value of $\mathbb{C}$. Finally, the conjugates of β_1 and β_2 are denoted by $\beta_j^{(i)} = \sigma_i(\beta_j)$. We have the formula

$$|x_1 \beta_1^{(i)} + x_2 \beta_2^{(i)} + x_3|_{\infty_i} = |x_1 \beta_1 + x_2 \beta_2 + x_3|$$

for arbitrary rational numbers x_1, x_2, x_3.

Next, we construct suitable linear forms to apply the subspace theorem. Let us write $x_1 = p^n$, $x_2 = q^n$, and $x_3 = a$, thus $N = 3$. We choose our linear forms as $L_{\nu,1}(\mathbf{x}) = x_1, L_{\nu,2}(\mathbf{x}) = x_2$ for all $\nu \in S$, and $L_{\nu,3}(\mathbf{x}) = x_3$ if ν lies above one of the valuations $p_1, \ldots, p_k$. We choose $L_{\nu,3}(\mathbf{x}) = \beta_1 x_1 + \beta_2 x_2 + x_3$ if ν lies above one of the valuations $q_1, \ldots, q_\ell$. Finally, if $\nu = \infty_i$, then we put $L_{\infty_i,3}(\mathbf{x}) = (\beta_1 - \beta_1^{(i)}) x_1 + (\beta_2 - \beta_2^{(i)}) x_2 + x_3$.

Using the product formula, cf. [4, pp. 99–100], and trivial estimates we obtain

$$\prod_{\nu \in S} |L_{\nu,1}(\mathbf{x})|_\nu^{n_\nu} = 1, \qquad \prod_{\nu \in S} |L_{\nu,2}(\mathbf{x})|_\nu^{n_\nu} = 1,$$

$$\prod_{\substack{\nu | q_j \\ 1 \le j \le \ell}} |L_{\nu,3}(\mathbf{x})|_\nu^{n_\nu} \le q^{-Dn}, \qquad \prod_{\substack{\nu | p_i \\ 1 \le i \le k}} |L_{\nu,3}(\mathbf{x})|_\nu^{n_\nu} \le 1.$$

Thus we are left to compute the quantities $|L_{\infty_i,3}(\mathbf{x})|_{\infty_i}$. We obtain

$$\begin{aligned} |L_{\infty_i,3}(\mathbf{x})|_{\infty_i} &= |(\beta_1 - \beta_1^{(i)}) x_1 + (\beta_2 - \beta_2^{(i)}) x_2 + x_3|_{\infty_i} \\ &= |\beta_1 p^n + \beta_2 q^n + a - \beta_1^{(i)} p^n - \beta_2^{(i)} q^n|_{\infty_i} \\ &= |X - \beta_1^{(i)} p^n - \beta_2^{(i)} q^n|_{\infty_i} \\ &= |X - \beta_1 p^n - \beta_2 q^n| = |a|. \end{aligned}$$

Combining all inequalities, we have

$$\prod_{\nu \in S} \prod_{i=1}^{3} |L_{\nu,i}(\mathbf{x})|_\nu^{n_\nu} \le q^{-Dn} |a|^D < (CQ)^D \left(\frac{q}{p} \right)^{\epsilon D n}. \tag{3}$$

Now choose $\delta > 0$ small enough so that

$$\left(\frac{q}{p}\right)^{\epsilon D} < p^{-\delta}.$$

In view of (2), the inequality $|a|_\nu \leq p^n$ holds for all valuations ν lying above ∞ for sufficiently large n, so that $\overline{|\mathbf{x}|} = |x_1| = p^n$. Hence we obtain

$$(CQ)^D \left(\frac{q}{p}\right)^{\epsilon D n} < (p^n)^{-\delta} = \overline{|\mathbf{x}|}^{-\delta}$$

for sufficiently large n. In view of (3), we have shown that

$$\prod_{\nu \in S} \prod_{i=1}^{n} |L_{\nu,i}(\mathbf{x})|_\nu^{n_\nu} < \overline{|\mathbf{x}|}^{-\delta}. \tag{4}$$

By the subspace theorem, all solutions to (4) lie in finitely many subspaces of K^3. Since by assumption there are infinitely many solutions, there exists one subspace $T \subseteq K^3$ which contains infinitely many solutions. Let T be defined by $t_1x_1 + t_2x_2 + t_3x_3 = 0$, with fixed algebraic integers $t_1, t_2, t_3 \in \mathfrak{a}_K$. Then there must be infinitely many integers n such that $t_1p^n + t_2q^n + t_3a = 0$, which is in contradiction to (2) and the assumption that $p > q > 1$. Thus we can conclude that α cannot be rational, unless $q = 1$ so that α is an integer. □

Now the proof of Theorem 1 is straightforward.

Proof of Theorem 1. As derived in Section 2, if an integer sequence satisfies a recursion of the form $x_{n+1} = P(x_n)$ for some polynomial P of degree $d > 1$ with rational coefficients, and $x_n \to \infty$ as $n \to \infty$, then an asymptotic formula of the form

$$x_n = A\alpha^{d^n} + B + O(\alpha^{-d^n})$$

holds. If α is rational, but not an integer, then we have an immediate contradiction to Theorem 2. □

4 Further generalizations

Let us remark that Theorem 2 can be extended to number fields:

Theorem 4. *Let L be a number field, and let $\mathfrak{a}_L$ be its maximal order. Assume that the sequence $(G_n)_{n\geq 0}$ attains values in $\mathcal{O}_L$ infinitely often, and that it satisfies an asymptotic formula of the form*

$$G_n = A\alpha^n + B + O(|\alpha|^{-\epsilon n}),$$

where $|\alpha| > 1$, A and B are algebraic numbers with $A \neq 0$, $\epsilon > 0$, and the constant implied by the O-term does not depend on n. Then the number α is either an algebraic integer in $\mathfrak{a}_L$, or $\alpha \notin L$.

The proof of this theorem is similar to the proof of Theorem 2. In particular, let K be the normal closure of $L(A, B)$, and assume that $\alpha = p/q$ with $p \in \mathfrak{a}_L$ and $q \in \mathbb{Z}$ with $q > 1$. Then we consider the prime ideal factorizations

$$(p) = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k} \quad \text{and} \quad (q) = \mathfrak{q}_1^{m_1} \cdots \mathfrak{q}_\ell^{m_\ell}$$

in K. We can construct the same linear forms as in the proof of Theorem 2 and use the subspace theorem to get a contradiction.

It is also possible to consider a higher-dimensional variant of Theorem 1. Let $f_1, \ldots, f_N \in \mathbb{Z}[X_1, \ldots, X_N]$ be polynomials of degree $d > 1$. Then we can consider a sequence $(\mathbf{x}_n)_{n \geq 0}$ with $\mathbf{x}_n = \left(x_n^{(1)}, \ldots, x_n^{(N)}\right) \in \mathbb{Z}^N$ for all $n \geq 0$ satisfying the polynomial recursion

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n) = (f_1(\mathbf{x}_n), \ldots, f_N(\mathbf{x}_n)).$$

With this notation at hand we pose the following problem:

Problem 5. Assume that $\max\left\{x_n^{(1)}, \ldots, x_n^{(N)}\right\} \to \infty$ as $n \to \infty$, and let

$$\alpha = \lim_{n \to \infty} \left(\max\left\{x_n^{(1)}, \ldots, x_n^{(N)}\right\} \right)^{d^{-n}}.$$

Is α necessarily irrational or an integer?

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