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A Matrix Approach to Generalized Delannoy and Schröder Arrays

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Abstract

We construct a set of Pascal-like infinite matrices that contains the generalized Delannoy arrays associated with weighted lattice paths. From each of our Delannoy matrices we obtain several Schröder arrays. We construct the matrices combining two Pascal translation matrices and a diagonal matrix, and we find explicit formulas for the entries of the matrices, and recurrence relations and generating functions for the central generalized Delannoy and Schröder numbers. We also express the entries of all the generalized Delannoy matrices in terms of Jacobi polynomials.

1 Introduction

The Pascal triangle is the oldest and most studied two-dimensional array of integers. Its origin is in elementary algebra, but it has numerous properties of interest in combinatorics and number theory. The representation of the Pascal triangle as an infinite lower triangular matrix makes it a linear transformation on the vector space of formal power series. During the last 40 years numerous generalizations of Pascal matrices have been studied from diverse points of view. One of the connections of Pascal matrices with combinatorics comes from the problems of counting diverse types of paths on lattices. From some of these problems come some interesting arrays of integers, for example the Delannoy and the Schröder arrays, that have been extensively studied by numerous authors. The entries for the Delannoy array A008288 and the Schröder array A033877 in Sloane's On-line Encyclopedia of Integer Sequences [10] contain hundreds of references and useful links.

Several authors have used Riordan matrices to construct matrices that are useful to study diverse aspects of the Delannoy and Schröder arrays and their generalizations. Among them we can mention Barry [3], Cheon, Kim, and Shapiro [5], Ramírez and Sirvent [8], Schröder [9], L. Yang et al. [13], and S.-L. Yang et al. [14]. Other relevant references about Delannoy numbers are the papers by Banderier and Schwer [1] and Sulanke [11]. Dziemiańczuk [6] studied weighted lattice paths with weight functions that depend on the position on the lattice.

In the present paper, we construct Pascal-like matrices by generalizing the Pascal triangle represented as a symmetric infinite matrix $T = [t_{n,k}]$ with $t_{n,k} = \binom{n+k}{k}$ for $n \geq 0$ and $k \geq 0$. Let P be the lower triangular matrix with entries $p_{n,k} = \binom{n}{k}$. Then it is easy to see that $PP^{\mathsf{T}} = T$. This is the construction of the symmetric matrix representation of the Pascal triangle that we generalize as follows. We extend first the lower triangular P to $P(z) = [\binom{n}{k} z^{n-k}]$, where z is a complex number. The matrix $P(z)$ represents a translation operator. Then we take an infinite diagonal matrix E, with nonzero entries in the diagonal, and construct a matrix $A = P(z)EP^{\mathsf{T}}(x)$, where z and x are complex numbers. The matrix A is a generalization of T and depends on the parameters z and x and also on the entries of E. Choosing suitable diagonal matrices E we obtain matrices with interesting combinatorial identities. When the entries of E are the terms of a geometric sequence $1, u, u^2, u^3, \ldots$, the corresponding matrix A is a generalized Delannoy matrix, associated with weighted lattice paths with weights z, x and $u - zx$.

Using shift matrices we combine a generalized Delannoy matrix D with a scalar multiple of a shifted D and obtain a generalized Schröder matrix G. We also show that G can be obtained as $P(z)MP^{\mathsf{T}}(x)$, where M is a simple Riordan matrix, and also as DU, where U is an upper triangular invertible Toeplitz matrix associated with a rational function.

In our matrices the central generalized Delannoy or Schröder numbers appear in the main diagonal, in the same way as the central binomial coefficients appear in the symmetric Pascal matrix T. Something similar can be observed in the approach of Woan [12], who uses Hankel matrices.

Our approach seems to be an alternative to the Riordan matrix approach to the construction of generalized Pascal-like matrices. Replacing the Pascal matrices $P(z)$ with Stirling matrices or q-Pascal matrices we can develop theories analogous to the one presented here.

2 Pascal-like infinite matrices

In this section we use Pascal and diagonal matrices to construct a set of Pascal-like infinite matrices whose elements satisfy certain simple two dimensional recurrence relations, and that contains the generalized Delannoy matrices.

We consider infinite matrices $A = [a_{n,k}]$ with complex entries and non-negative indices

n and k. If A and B are infinite matrices the usual matrix multiplication AB is not well defined whenever $\sum_{j=0}^{\infty} a_{n,j} b_{j,k}$ is divergent for some pair (n,k). If A is a lower triangular or a lower m-Hessenberg matrix, then AB is well defined for every infinite matrix B. Similarly, if A is an upper triangular or an upper m-Hessenberg matrix, then BA is well defined for every infinite matrix B.

For z in the complex numbers we define the Pascal matrix $P(z)$ as the lower triangular infinite matrix whose (n,k) entry is $\binom{n}{k} z^{n-k}$, for $n \geq k \geq 0$, and it is zero if $k > n$. That is,

$$P(z) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ z & 1 & 0 & 0 & 0 & \dots \\ z^2 & 2z & 1 & 0 & 0 & \dots \\ z^3 & 3z^2 & 3z & 1 & 0 & \dots \\ z^4 & 4z^3 & 6z^2 & 4z & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is well-known that the set $\{P(z) : z \in \mathbb{C}\}$ is a group under matrix multiplication which is isomorphic to the additive group of complex numbers, and is also isomorphic to the group of translation operators on the vector space of polynomials in one complex variable.

Define the matrix $A = [a_{n,k}]$ by

$$A = P(z) P^{\mathsf{T}}(x) = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 & \dots \\ z & 2zx & 3zx^2 & 4zx^3 & 5zx^4 & \dots \\ z^2 & 3z^2x & 6z^2x^2 & 10z^2x^3 & 15z^2x^4 & \dots \\ z^3 & 4z^3x & 10z^3x^2 & 20z^3x^3 & 35z^3x^4 & \dots \\ z^4 & 5z^4x & 15z^4x^2 & 35z^4x^3 & 70z^4x^4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then we have $a_{n,k} = \binom{n+k}{k} z^n x^k$, which can be interpreted as the sum of all the products of the form $b_1 b_2 b_3 \cdots b_{n+k}$, where n of the b_i are equal to z and the rest are equal to x. Notice that the recurrence relation $a_{n,k} = x a_{n,k-1} + z a_{n-1,k}$ holds for all $n \geq 1$ and $k \geq 1$. For $m \geq 0$ the entries $a_{n,k}$ with indices that satisfy $n + k = m$ are the coefficients of $(z + x)^m$.

The product $P^{\mathsf{T}}(x) P(z)$ is not defined for all pairs (x, z). The convergence of the series involved in the multiplication requires some conditions on x and z.

If we take $z = x = 1$ then A becomes the classical Pascal triangle written as a symmetric matrix whose entries in the main diagonal are the central binomial coefficients. Notice that $P(1)$ is the Pascal triangle written as a lower triangular matrix.

We describe next a simple way to generalize the matrix A introducing a diagonal matrix as a factor. Let $c_0, c_1, c_2, \dots$ be a sequence of complex numbers and let $E = \mathrm{Diag}(c_0, c_1, c_2, \dots)$, that is,

$$E = \begin{bmatrix} c_0 & 0 & 0 & 0 & 0 & \dots \\ 0 & c_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & c_2 & 0 & 0 & \dots \\ 0 & 0 & 0 & c_3 & 0 & \dots \\ 0 & 0 & 0 & 0 & c_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $B = P(z)EP^{\mathsf{T}}(x)$. Then we have

$$B = \begin{bmatrix} c_0 & c_0 x & c_0 x^2 & c_0 x^3 & \dots \\ c_0 z & c_0 zx + c_1 & c_0 zx^2 + 2c_1 x & c_0 zx^3 + 3c_1 x^2 & \dots \\ c_0 z^2 & c_0 z^2 x + 2c_1 z & c_0 z^2 x^2 + 4c_1 zx + c_2 & c_0 z^2 x^3 + 6c_1 zx^2 + 3c_2 & \dots \\ c_0 z^3 & c_0 z^3 x + 3c_1 z^2 & c_0 z^3 x^2 + 6c_1 z^2 x + 3c_2 z & c_0 z^3 x^3 + 9c_1 z^2 x^2 + 9c_2 zx + c_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The definition of B gives us

$$b_{n,k} = \sum_{j=0}^{\min(n,k)} \binom{n}{j}\binom{k}{j} c_j z^{n-j} x^{k-j}. \qquad (1)$$

Let us note that $b_{n,n} = c_n$ plus a function that depends only on $c_0, c_1, \dots, c_{n-1}, z, x$. This fact clearly implies that for any pair (z, x) and any given sequence $r_0, r_1, r_2, \dots$ we can find a unique sequence $c_0, c_1, c_2, \dots$ such that $b_{n,n} = r_n$ for $n \geq 0$.

Let D_z and D_x denote the operators of differentiation with respect to z and x, respectively. Then equation (1) can be written as

$$b_{n,k} = \sum_{j \geq 0} c_j \frac{D_z^j}{j!} \frac{D_x^j}{j!} z^n x^k.$$

Using the Leibniz rule we obtain the equations

$$\frac{D_z^{j-1}}{(j-1)!} z^{n-1} = \frac{D_z^j}{j!} z^n - z \frac{D_z^j}{j!} z^{n-1}, \qquad \frac{D_x^{j-1}}{(j-1)!} x^{k-1} = \frac{D_x^j}{j!} x^k - x \frac{D_x^j}{j!} x^{k-1}.$$

Multiplying these equations we obtain

$$\frac{D_z^{j-1} D_x^{j-1}}{((j-1)!)^2} z^{n-1} x^{k-1} = \frac{D_z^j}{j!} \frac{D_x^j}{j!} z^n x^k + zx \frac{D_z^j}{j!} \frac{D_x^j}{j!} z^{n-1} x^{k-1} - z \frac{D_z^j}{j!} \frac{D_x^j}{j!} z^{n-1} x^k - x \frac{D_z^j}{j!} \frac{D_x^j}{j!} z^n x^{k-1}.$$

This equation gives us the recurrence relation

$$b_{n,k} = \hat{b}_{n-1,k-1} + z b_{n-1,k} + x b_{n,k-1} - zx b_{n-1,k-1}, \qquad n \geq 1,\ k \geq 1, \qquad (2)$$

where

$$\hat{b}_{n,k} = \sum_{j\geq 0} c_{j+1} \binom{n}{j} \binom{k}{j} z^{n-j} x^{k-j}.$$

Compare this equation with (1) and note the shift in the indices of the c_i. This means that the numbers $\hat{b}_{n,k}$ are the entries of $\hat{B} = P(z)\mathrm{Diag}(c_1, c_2, c_3, \ldots)P^{\mathsf{T}}(x)$.

The definition of B also gives us $P^{-1}(z)B = EP^{\mathsf{T}}(x)$ and $B(P^{\mathsf{T}}(x))^{-1} = P(x)E$. Notice that all the matrix products in both equations are well defined. Since $P^{-1}(z) = P(-z)$, the first equation yields

$$\sum_{j=0}^{n} \binom{n}{j} (-z)^{n-j} b_{j,k} = c_n \binom{k}{n} x^{k-n}.$$

In particular, if $k = n$ we get the relation

$$b_{n,n} = c_n - \sum_{j=0}^{n-1} \binom{n}{j} (-z)^{n-j} b_{j,n}, \tag{3}$$

which expresses $b_{n,n}$ in terms of the preceding entries in the n-th column of B. From the equation $B(P^{\mathsf{T}}(x))^{-1} = P(x)E$ we obtain an analogous result.

For $m \geq 1$ let B_m denote the initial $m \times m$ section of B, that is, $B_m = [b_{n,k}]$, where $0 \leq n \leq m-1$ and $0 \leq k \leq m-1$. Since B_m is the product of the initial $m \times m$ sections of $P(z)$, E, and $P^{\mathsf{T}}(x)$, it is clear that $\det(B_m) = \det(E_m) = c_0 c_1 \cdots c_{m-1}$. Note that $\det(B_m)$ is independent of z and x.

For certain sequences $c_0, c_1, c_2, \ldots$ and pairs (z, x) the corresponding matrices B have interesting combinatorial properties. For example, if $z = x = 1$ and $c_k = k!$ then the matrix B is

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ 1 & 3 & 7 & 13 & 21 & 31 & \ldots \\ 1 & 4 & 13 & 34 & 73 & 136 & \ldots \\ 1 & 5 & 21 & 73 & 209 & 501 & \ldots \\ 1 & 6 & 31 & 136 & 501 & 1546 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The sequences on the diagonals of B, starting with the main diagonal, correspond to the sequences A002720, A000262, A052852, A062147, and A062266 of Sloane's On-line Encyclopedia of Integer Sequences [10]. The recurrence relation (2) yields connections among those sequences.

If we take $z = x = 1$ and $c_k = k + 1$ for $k \geq 0$ the matrix B is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 5 & 7 & 9 & \dots \\ 1 & 5 & 12 & 22 & 35 & \dots \\ 1 & 7 & 22 & 50 & 95 & \dots \\ 1 & 9 & 35 & 95 & 210 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The sequence $(b_{2,k})$ is A000326, called pentagonal numbers. The sequence $(b_{3,k})$ is A002412, called hexagonal pyramidal numbers. The sequence $(b_{4,k})$ is A002418, called 4dim figurate numbers, and the sequence $(b_{5,k})$ is A051843.

We obtain another interesting example when we take $z = x = 1$ and c_k equal to the k-th Fibonacci number. In Section 3, we take $c_k = u^k$ for $k \geq 0$, where u is a nonzero complex number, and we obtain the generalized Delannoy arrays.

3 Generalized Delannoy matrices

In this section we study the family of generalized Delannoy matrices that we obtain using the construction presented in Section 2. The generalized Delannoy matrices have been studied by several authors using the Riordan matrix approach, or considering combinatorial problems related with weighted lattice paths.

We consider now sequences $(c_k)_{k\geq 0}$ that are geometric progressions of the form $c_k = u^k$, for $k \geq 0$, where u is a nonzero number. For $u \neq 0$ let $E(u) = \text{Diag}(1, u, u^2, u^3, \dots)$. Then $uE(u) = \text{Diag}(u, u^2, u^3, \dots)$.

Let us define the matrix

$$D(z, u, x) = P(z)E(u)P^{\mathsf{T}}(x), \qquad z, x, u \in \mathbb{C},\ u \neq 0.$$

The matrix $D(z, u, x)$ is called *generalized Delannoy matrix*. It is a particular case of the matrices studied in Section 2. Note that $D(z, u, x) = D^{\mathsf{T}}(x, u, z)$ and also that $D(1, 1, 0)$ is the classic Pascal lower triangular matrix, $D(z, 1, 0) = P(z)$, and $D(1, 1, 1)$ is the symmetric Pascal matrix.

The three factors $P(z)$, $E(u)$, and $P^{\mathsf{T}}(x)$ are invertible infinite matrices, but in general, $D(z, u, x)$ is not invertible, because the product $P^{\mathsf{T}}(-x)E(1/u)P(-z)$ may not be defined. However, if we compute the product $P^{\mathsf{T}}(-x)(E(1/u)(P(-z)D(z, u, x)))$, in the order indicated by the parenthesis, then we obtain the infinite identity matrix.

In order to simplify the notation we will denote the entries of $D(z, u, x)$ just by $d_{n,k}$. The matrix $\hat{D}(z, u, x)$ that corresponds to the shifted sequence $u, u^2, u^3, \dots$ is equal to $uD(z, u, x)$, and therefore the recurrence relation (2) becomes in this case

$$d_{n,k} = zd_{n-1,k} + xd_{n,k-1} + (u - zx)d_{n-1,k-1}, \qquad n, k \geq 1. \tag{4}$$

This is the recurrence satisfied by the weighted Delannoy paths from $(0,0)$ to (n,k) on the lattice $\mathbb{N}\times\mathbb{N}$, that are constructed using the steps $(1,0)$, with weight z, $(0,1)$, with weight x, and $(1,1)$, with weight $u-xz$. Several authors have studied such weighted paths. See, for example, the papers by Cheon, Kim, and Shapiro [5], and Ramírez and Sirvent [8].

The explicit formula (1) becomes

$$d_{n,k}=\sum_{j=0}^{k}\binom{n}{j}\binom{k}{j}u^j z^{n-j}x^{k-j}. \tag{5}$$

Compare this formula with the formula for the entries of the lower triangular Pascal-like matrices constructed by Barry [3, Sec. 2] using Riordan matrices with one parameter.

Using (3) and (5) we obtain

$$d_{n,n}=u^n-\sum_{j=0}^{n-1}\binom{n}{j}(-z)^{n-j}d_{j,n}=\sum_{j=0}^{n}\binom{n}{j}^2 u^j(zx)^{n-j}, \qquad n\geq 1.$$

The first sum gives the n-th central generalized Delannoy number in terms of the preceding terms in the n-th column. The second sum is an explicit formula for $d_{n,n}$ in terms of the parameters z,u,x.

It is easy to verify that, if $y\neq 0$, then the main diagonals of the matrices $D(z,u,x)$, $D(y,zx,u/y)$, and $D(y,u,zx/y)$ coincide.

The classical Delannoy matrix A008288 is

$$D(1,2,1)=\begin{bmatrix}1&1&1&1&1&\dots\\ 1&3&5&7&9&\dots\\ 1&5&13&25&41&\dots\\ 1&7&25&63&129&\dots\\ 1&9&41&129&321&\dots\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}. \tag{6}$$

For any nonzero y the matrices $D(y,1,2/y)$ and $D(y,2,1/y)$ have the same main diagonal as $D(1,2,1)$.

It is well-known that the entries in the main diagonal of $D(1,2,1)$, usually called central Delannoy numbers, are values of the Legendre polynomials at 3. Hetyei [7] found expressions in terms of Jacobi polynomials for the entries of a matrix that he calls asymmetric Delannoy matrix. Such matrix is just $D(1,1,2)$, has the same central entries as $D(1,2,1)$, and is essentially the same as the array A049600. We show next that the entries of all the generalized Delannoy matrices $D(z,u,x)$ can be expressed in terms of Jacobi polynomials.

The Jacobi polynomials with parameters α and β have the explicit formula

$$p_n^{(\alpha,\beta)}(t)=\sum_{j=0}^{n}\binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{n-j}\left(\frac{t-1}{2}\right)^j, \qquad n\geq 0. \tag{7}$$

Theorem 1. *If $u - zx$ is not zero let $a = (u + zx)/(u - zx)$. Then*

$$d_{n,k} = z^{n-k}(u - zx)^k p_k^{(n-k,0)}(a), \qquad n \geq k \geq 0. \tag{8}$$

If $u = zx$ let $r(n, k)$ be the leading coefficient of $p_k^{(n-k,0)}(t)$. Then

$$d_{n,k} = z^n (2x)^k r(n, k), \qquad n \geq k \geq 0. \tag{9}$$

Proof. From (7) we obtain

$$p_k^{(n-k,0)}(a) = \sum_{j=0}^{k} \binom{k}{j} \binom{n+j}{k} \left(\frac{zx}{u - zx} \right)^j,$$

and therefore

$$z^{n-k}(u - zx)^k p_k^{(n-k,0)}(a) = z^n x^k \sum_{j=0}^{k} \binom{k}{j} \binom{n+j}{k} \left(\frac{u}{zx} - 1 \right)^{k-j}. \tag{10}$$

On the other hand

$$d_{n,k} = z^n x^k \sum_{j=0}^{k} \binom{k}{j} \binom{n}{j} \left(\frac{u}{zx} \right)^j. \tag{11}$$

Let $v = \frac{u}{zx} - 1$. Then

$$\left(\frac{u}{zx} \right)^j = (v + 1)^j = \sum_{i=0}^{j} \binom{j}{i} v^i.$$

Substitution of this expression in (11) gives

$$d_{n,k} = z^n x^k \sum_{j=0}^{k} \sum_{i=0}^{j} \binom{k}{j} \binom{n}{j} \binom{j}{i} v^i. \tag{12}$$

Changing the summation order and using the identity

$$\binom{k}{j} \binom{j}{i} = \binom{k}{i} \binom{k-i}{k-j},$$

we obtain from (12)

$$d_{n,k} = z^n x^k \sum_{i=0}^{k} \binom{k}{i} \left(\sum_{j=i}^{k} \binom{n}{j} \binom{k-i}{k-j} \right) v^i.$$

By the Vandermonde identity the second sum is equal to $\binom{n+k-i}{k}$. Therefore $d_{n,k}$ is equal to the expression in the right-hand side of (10), with j replaced by $k - i$. This completes the proof of (8).

In order to prove (9) we write (8) in the form

$$d_{n,k} = z^{n-k}(u+zx)^k a^{-k} p_k^{(n-k,0)}(a) = z^{n-k}(u+zx)^k \hat{p}_k^{(n-k,0)}(1/a),$$

where $\hat{p}_k^{(n-k,0)}$ is the reversed polynomial of $p_k^{(n-k,0)}$. If u goes to zx then a goes to infinity and $1/a$ goes to zero. The constant term of $\hat{p}_k^{(n-k,0)}$ is equal to the leading coefficient of $p_k^{(n-k,0)}$. □

Let $d_{n,k}$ denote the entries of $D(z,u,x)$, and let $a = (u+zx)/(u-zx)$. We define the series

$$g(t) = \sum_{n=0}^{\infty} d_{n,n} t^n = \sum_{n=0}^{\infty} (u-zx)^n p_n^{(0,0)}(a) t^n.$$

The polynomials $p_n^{(0,0)}$ are the Legendre polynomials and have the generating function

$$\frac{1}{\sqrt{1-2wv+v^2}} = \sum_{n=0}^{\infty} p_n^{(0,0)}(w) v^n.$$

Taking $v = (u-zx)t$ and $w = a$ we obtain

$$g(t) = \frac{1}{\sqrt{1-2(u+zx)t+(u-zx)^2t^2}},$$

which is the generating function of the central generalized Delannoy numbers $d_{n,n}$ with parameters z, u, x.

The sums of the entries on the anti-diagonals of $D(z,u,x)$ are

$$s_n = \sum_{k=0}^{n} d_{n-k,k}, \qquad n \geq 0.$$

Let $h(t)$ be the generating function of the sequence $(s_n)_{n\geq 0}$, that is,

$$h(t) = \sum_{n=0}^{\infty} s_n t^n.$$

Using the explicit formula for the entries $d_{n-k,k}$ it is easy to see that

$$h(t)(1-(z+x)t+(zx-u)t^2) = 1,$$

and therefore $h(t)$ is

$$h(t) = \frac{1}{1-(z+x)t+(zx-u)t^2} = \sum_{n=0}^{\infty} s_n t^n.$$

Notice that $h(t)$ is a rational function of t with quadratic denominator and coefficients that depend on the parameters z, u, x. There are many sequences of numbers and polynomials that have generating functions that are rational functions with quadratic denominator, for example, the Chebyshev polynomials. Therefore, replacing the parameters z, u, x by suitable polynomials we can construct matrices $D(z, u, x)$ whose sums s_n are some known polynomial sequences. Cheon, Kim, and Shapiro [5] present several examples of such polynomial sequences.

4 Generalized Schröder matrices

Define the shift matrix S by

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and let $R = S^{\mathsf{T}}$. Note that $RS = I$, the infinite identity matrix, but SR is not the identity since its $(0, 0)$ entry is zero. All the other entries on the main diagonal of SR are equal to 1. If A is an infinite matrix then multiplication of A by S or R, either on the left or on the right, produces a shift of A of one step in one of the four directions. That is, SA is A moved one step downwards and with the 0-th row equal to zero; AS moves A one step to the left and the 0-th column disappears; RA moves A one step upwards and the 0-th row disappears; and AR is A moved one step to the right and with the 0-th column equal to zero.

The matrices $D(z, u, x)$ satisfy the recurrence relation (4). Using the shift matrices S and R we can write the recurrence relation (4) as the matrix equation

$$RD(z, u, x)S = xRD(z, u, x) + zD(z, u, x)S + (u - xz)D(z, u, x).$$

The generalized Delannoy matrix $D(z, u, x)$ satisfies the recurrence relation (4) and the entries of $D(z, u, x)$ in the 0-th row and the 0-th column are the boundary values. It is clear that any matrix obtained by applying some of the shifts described above to $D(z, u, x)$ must also satisfy the recurrence relation (4), and so does any linear combination of such shifted matrices. Let us consider next a simple example.

For $z \neq 0$ we define $G(z, u, x) = D(z, u, x) - (x/z)RD(z, u, x)R$. Let $g_{n,k}$ denote the entries of $G(z, u, x)$. Then we have $g_{n,k} = d_{n,k} - (x/z)d_{n+1,k-1}$ and

$$G(z, u, x) = \begin{bmatrix} 1 & 0 & -ux/z & -2ux^2/z & \dots \\ z & u & 0 & -xu(zx_u)/z & \dots \\ z^2 & 2uz & u(zx + u) & 0 & \dots \\ z^3 & 3uz^2 & zu(2xz + 3u) & u(x^2z^2 + 3uxz + u^2) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The lower triangular part of $G(z,u,x)$ satisfies (4) with boundary values equal to zero on the diagonal entries with indices $(n,n+1)$, for $n \geq 0$, and the values $1, z, z^2, \ldots$ in the 0-th column. The upper triangular part also satisfies (4) with boundary values equal to zero in the positions $(n,n+1)$ and the values $-ux/z, -2ux^2/z, \ldots$ in the 0-th row.

The particular case

$$G(1,2,1) = \begin{bmatrix} 1 & 0 & -2 & -4 & -6 & \ldots \\ 1 & 2 & 0 & -6 & -16 & \ldots \\ 1 & 4 & 6 & 0 & -22 & \ldots \\ 1 & 6 & 16 & 22 & 0 & \ldots \\ 1 & 8 & 30 & 68 & 90 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{13}$$

obtained from the classical Delannoy matrix $D(1,2,1)$, displayed in (6), has the classical Schröder matrix A033877 as its lower triangular part. Because of the symmetry of the Delannoy matrix $D(1,2,1)$ the upper triangular part of $G(1,2,1)$ is the negative of the transpose of $G(1,2,1)$, shifted one step upwards.

The matrices $G(z,u,x)$ satisfy the recurrence (4) and include the classical Schröder matrix. We call the matrices $G(z,u,x)$ *generalized Schröder matrices*.

Let us consider another example.

$$G(2,3,1/2) = \begin{bmatrix} 1 & 0 & -3/4 & -3/4 & -9/16 & \ldots \\ 2 & 3 & 0 & -3 & -33/8 & \ldots \\ 4 & 12 & 12 & 0 & -57/4 & \ldots \\ 8 & 36 & 66 & 57 & 0 & \ldots \\ 16 & 96 & 252 & 372 & 300 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The sequence in the main diagonal of $G(2,3,1/2)$ is A047891. Note that all the entries in the lower triangular part are integers.

Two different generalized Schröder matrices can coincide in the main diagonal. For example, $G(1,2,2)$ and $G(2,2,1)$ have the sequence A151374 as the main diagonal, and $G(1,3,2)$ and $G(2,3,1)$ have A103210 as the main diagonal.

We can see from the definition of $G(z,u,x)$ that its entries can be expressed in terms of Jacobi polynomials.

We will use next certain commutation properties of the Pascal matrices $P(z)$ to obtain other matrix expressions for the generalized Schröder matrices.

The Pascal matrices satisfy

$$RP(z) = P(z)(zI + R)$$

and

$$P^{\mathsf{T}}(x)R = R(I + xR)^{-1}P^{\mathsf{T}}(x).$$

Therefore
$$\frac{x}{z}RD(z,u,x)R = \frac{x}{z}P(z)(zI+R)E(u)R(I+xR)^{-1}P^{\mathsf{T}}(x),$$
and after some simplifications we obtain

$$G(z,u,x) = P(z)E(u)\left(I - \frac{ux}{z}R^2\right)(I+xR)^{-1}P^{\mathsf{T}}(x). \tag{14}$$

The matrix $(I - \frac{ux}{z}R^2)(I+xR)^{-1}$ is an upper triangular invertible Toeplitz matrix and represents a multiplication operator on the space of formal power series. The matrix $E(u) = \mathrm{Diag}(1,u,u^2,\ldots)$ corresponds to the substitution operator $t \to ut$. Therefore the product $E(u)(I - \frac{ux}{z}R^2)(I+xR)^{-1}$ is a Riordan matrix and (14) describes its connection with the generalized Schröder matrix $G(z,u,x)$. The construction of $G(z,u,x)$ in (14) can be generalized as follows. Let $M(f,g)$ be a Riordan matrix and let z and x be complex numbers. Then $P(z)M(f,g)P^{\mathsf{T}}(x)$ can be considered as a more general Schröder matrix. We do not study such generalization in the present paper.

Using commutation properties of the Pascal matrices with functions of R we obtain another construction for $G(z,u,x)$ as follows. Let

$$U(z,u,x) = I - xR - \frac{ux}{z}R^2(I-xR)^{-1}. \tag{15}$$

$U(z,u,x)$ is an upper triangular invertible Toeplitz matrix and satisfies

$$\left(I - \frac{ux}{z}R^2\right)(I+xR)^{-1}P^{\mathsf{T}}(x) = P^{\mathsf{T}}(x)U(z,u,x).$$

Therefore, from (14) and the definition of $D(z,u,x)$ we get

$$G(z,u,x) = D(z,u,x)U(z,u,x). \tag{16}$$

We can also determine $U(z,u,x)$ by computing $P^{\mathsf{T}}(-x)(E(1/u)(P(-z)G(z,u,x)))$ in the order determined by the parenthesis. This is equivalent to the multiplication of $G(z,u,x)$ on the left by the "inverse" of $D(z,u,x)$.

If $u \neq 0$ then every finite initial section of $D(z,u,x)$ is invertible and thus so is every finite initial section of $G(z,u,x)$. Since the determinants of all the finite sections of $U(z,u,x)$ are equal to 1, it is clear that the determinants of corresponding initial sections of $D(z,u,x)$ and $G(z,u,x)$ are equal.

We present next some simple examples. From (15) we obtain

$$U(1,2,1) = \begin{bmatrix} 1 & -1 & -2 & -2 & -2 & \ldots \\ 0 & 1 & -1 & -2 & -2 & \ldots \\ 0 & 0 & 1 & -1 & -2 & \ldots \\ 0 & 0 & 0 & 1 & -1 & \ldots \\ 0 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and multiplying the classical Delannoy matrix $D(1,2,1)$ of (6) on the right by $U(1,2,1)$ we obtain the Schröder matrix $G(1,2,1)$ that appears in (13).

Another simple example of a generalized Schröder matrix is

$$D(1,1,1)U(1,1,1) = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & \dots \\ 1 & 1 & 0 & -2 & -5 & \dots \\ 1 & 2 & 2 & 0 & -5 & \dots \\ 1 & 3 & 5 & 5 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The lower triangular part of this matrix is the classical Catalan triangle A009766. The Delannoy matrix $D(1,1,1)$ is the symmetric classical Pascal triangle and

$$U(1,1,1) = \begin{bmatrix} 1 & -1 & -1 & -1 & \dots \\ 0 & 1 & -1 & -1 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Using the explicit formula (5) for the entries of $D(z,u,x)$ and the definition of $G(z,u,x)$, or alternatively, using (16) and the easily obtained explicit expression for the entries of $U(z,u,x)$, we obtain the explicit formula for the entries $g_{n,k}$ of $G(z,u,x)$. It is

$$g_{n,k} = \frac{n-k+1}{n+1} \sum_{j=1}^{k-1} \binom{n+1}{j} \binom{k-1}{j-1} z^{n-j} x^{k-j} u^j + \binom{n}{k} z^{n-k} u^k. \tag{17}$$

We can see from (17) that the central generalized Schröder numbers are given by

$$g_{n,n} = \frac{1}{n} \sum_{j=1}^{n} \binom{n}{j} \binom{n}{j-1} z^{n-j} x^{n-j} u^j.$$

It is easy to verify that $g_{n,n}$ is equal to the coefficient of t^{n-1} in the expression as a linear combination of powers of t of the polynomial

$$\frac{1}{nzx}(t+u)^n(t+zx)^n.$$

Using the well-known expression for the central Catalan numbers C_n we see that, if $u = zx$ and $u \neq 0$ then $g_{n,n} = C_n u^n$. The generating function of the central Catalan numbers is

$$h_c(t) = \sum_{n=0}^{\infty} C_n t^n = \frac{1-\sqrt{1-4t}}{2t},$$

and satisfies

$$\frac{1}{h_c(t)} + t\,h_c(t) = 1. \tag{18}$$

Let $h(t)$ be the generating function of the central generalized Schröder numbers, that is,

$$h(t) = \sum_{n=0}^{\infty} g_{n,n} t^n.$$

We will show that $h(t)$ satisfies an equation analogous to (18). Let $\hat{h}(t) = \sum_{n=0}^{\infty} e_n t^n$ be the reciprocal of $h(t)$. Then, since $g_{0,0} = 1$, we have $e_0 = 1$, and

$$\sum_{k=0}^{n} g_{k,k}\, e_{n-k} = 0, \qquad n \geq 1,$$

and thus

$$e_n = -\sum_{k=1}^{n} g_{k,k}\, e_{n-k}, \qquad n \geq 1.$$

This recurrence relation gives

$$e_1 = -g_{1,1},\ e_2 = -zxg_{1,1},\ e_3 = -zxg_{2,2}, \ldots,\ e_n = -zxg_{n-1,n-1}, \ldots$$

and therefore

$$zxt\, h(t) + \hat{h}(t) = 1 - (u - zx)t.$$

Multiplying by $h(t)$ both sides of this equation we obtain

$$zxt\, h^2(t) - (1 - (u - zx)t)h(t) + 1 = 0. \tag{19}$$

Solving this quadratic equation we obtain the generating function of the central generalized Schröder numbers

$$h(t) = \frac{1 - (u - zx)t - \sqrt{1 - 2(u + zx)t + (u - zx)^2 t^2}}{2zxt}. \tag{20}$$

The minus sign just before the square root in (20) is the correct choice to obtain $h(t)$.

If we take $u = zx$ in the previous equation then $h(t)$ becomes

$$h(t) = \frac{1 - \sqrt{1 - 4zxt}}{2zxt},$$

which is the generating function of the central generalized Catalan numbers $C_n z^n x^n$.

Since the right-hand side of (19) is zero, the coefficient of t^{n+1} in the left-hand side of (19) is zero for $n \geq 0$. Therefore

$$zx \sum_{k=0}^{n} g_{k,k} g_{n-k,n-k} - g_{n+1,n+1} + (u - zx) g_{n,n} = 0,$$

and this equation gives us the recurrence relation

$$g_{n+1,n+1} = (u - zx)g_{n,n} + zx \sum_{k=0}^{n} g_{k,k} g_{n-k,n-k}.$$

This formula gives every central generalized Schröder number in terms of the preceding ones. It is an extension of a result obtained by Brualdi and Kirkland [4, Lemma 4.2] using combinatorial methods.

If z is not zero then $G_2 = D(z,u,x) - (x/z)^2 R^2 D(z,u,x) R^2$ is well defined and satisfies the recurrence relation (4). Furthermore, the entries of G_2 in the diagonal with indices $(n, n+2)$, for $n \geq 0$, are all equal to zero. If $D(z,u,x)$ is the classical Delannoy matrix then

$$G_2 = \begin{bmatrix} 1 & 1 & 0 & -4 & -12 & \dots \\ 1 & 3 & 4 & 0 & -16 & \dots \\ 1 & 5 & 12 & 16 & 0 & \dots \\ 1 & 7 & 24 & 52 & 68 & \dots \\ 1 & 9 & 40 & 116 & 236 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The part of this matrix to the left of the diagonal of zeros is a Hessenberg matrix. The sequence $1, 4, 16, 68, \dots$ in the diagonal to the right of the main diagonal is the sequence A006319, called royal paths in a lattice. The sequence in the main diagonal is A268208, and the sequence in the diagonal below the main one is A110190.

The construction of G_2 is generalized as follows. For $k \geq 1$ let

$$G_k = D(z,u,x) - (x/z)^k R^k D(z,u,x) R^k.$$

The matrix G_k satisfies the recurrence (4) and all its entries in the diagonal with indices $(n, n+k)$, for $n \geq 0$, are equal to zero. Such matrices can be seen as the solutions of the recurrence relation (4) with boundary values equal to zero on some diagonal to the right of the main diagonal. The rest of the boundary is the union of the 0-th column and the first $k-1$ positions of the 0-th row.

There are other ways to construct matrices with interesting properties using the generalized Delannoy matrices and the Toeplitz matrices $U(x,u,z)$. For example, forming products of the form $D(z,u,x)U(t,v,w)$, with $(z,u,x) \neq (t,v,w)$, or differences such as $D(z,u,x) - D(t,v,w)$, or $G(z,u,x) - G(t,v,w)$.

5 Conclusions

We studied some Pascal-like infinite matrices that contain all the generalized Delannoy and Schröder matrices. We described the structure of the matrices and how we obtain Schröder matrices from Delannoy matrices. We also obtained expressions for the entries of the Delannoy matrices in terms of Jacobi polynomials, and explicit formulas and generating functions for the central generalized Delannoy and Schröder numbers. We described a connection of our construction with certain simple Riordan matrices and a possible way to construct more general matrices using other Riordan matrices. Transforming our matrices to the lower triangular form we obtain a set of matrices that contains the Pascal-like matrices introduced recently by Barry [3].

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