Journal of Integer Sequences, Vol. 24 (2021),
Article 21.9.2

Enumeration of Flats of the Extended Catalan and Shi Arrangements with Species

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Abstract

The number of flats of a hyperplane arrangement is considered as a generalization of the Bell number and the Stirling number of the second kind. Robert Gill gave the exponential generating function of the number of flats of the extended Catalan arrangements, using species. In this article, we introduce the species of flats of the extended Catalan and Shi arrangements and they are given by iterated substitution of species of sets and lists. Moreover, we enumerate the flats of these arrangements in terms of infinite matrices.

1 Introduction

A *hyperplane arrangement* is a finite collection of affine subspaces of codimension 1 in an affine space over an arbitrary field $\mathbb{K}$. In spite of its simple definition, arrangements are investigated in a variety of ways, such as topological, algebrogeometric, and combinatorial aspects. A standard reference for hyperplane arrangements is the text written by Orlik and Terao [10].

Given an arrangement $\mathcal{A}$, let $L(\mathcal{A})$ denote the set of nonempty intersections of hyperplanes in $\mathcal{A}$. Note that the ambient space is a member of $L(\mathcal{A})$ since it is regarded as the intersection over the empty set. We call an element of $L(\mathcal{A})$ a *flat*. Define a partial order on $L(\mathcal{A})$ by the reverse inclusion, that is, $X \leq Y \Leftrightarrow X \supseteq Y$ for $X, Y \in L(\mathcal{A})$. We call $L(\mathcal{A})$ the *intersection poset* of $\mathcal{A}$. This poset plays an important role in the theory of hyperplane arrangements. For each nonnegative integer k, let

$$L_k(\mathcal{A}) := \{ \, X \in L(\mathcal{A}) \mid \dim X = k \, \} .$$

When $\mathcal{A}$ is *central*, that is, the intersection of all hyperplanes in $\mathcal{A}$ is nonempty, the poset $L(\mathcal{A})$ is a geometric lattice.

A *set partition* of a finite set V is a collection $\pi = \{B_1, \dots, B_k\}$ of nonempty subsets $B_i \subseteq V$ such that $B_i \cap B_j = \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^{k} B_i = V$. Each B_i is called a *block* of π. Let π and π' be set partitions of V. Define a partial order $\pi \leq \pi'$ if each block of π is a subset of some block of π'. We also say that π *refines* π' if $\pi \leq \pi'$. Then the collection of the set partitions of V forms a lattice.

For a positive integer n, let $[n]$ be the set $\{1, \dots, n\}$ and $[0]$ the empty set. The number of set partitions of $[n]$ is called the *Bell number*, denoted by $B(n)$, where $B(0) = 1$. The number of set partitions of $[n]$ into k blocks is called the *Stirling number of the second kind*, denoted by $S(n, k)$, where $S(0, 0) = 1$.

Let $x_1, \dots, x_n$ denote coordinates of $\mathbb{R}^n$ and $\mathcal{B}_n$ the n-dimensional *braid arrangement* (also known as the *Weyl arrangement of type* A_{n-1}), which consists of hyperplanes $\{x_i - x_j = 0\}$ in $\mathbb{R}^n$ for $1 \leq i < j \leq n$. It is well known that there exists an isomorphism from $L(\mathcal{B}_n)$ to the lattice of set partitions of $[n]$ which sends a k-dimensional flat to a set partition into k blocks. In other words, $|L(\mathcal{B}_n)| = B(n)$ and $|L_k(\mathcal{B}_n)| = S(n, k)$. The numbers of flats and k-dimensional flats of an arrangement is considered to be generalizations of the Bell numbers and the Stirling numbers.

Define the *extended Catalan arrangement* $\mathcal{C}_n^m$ and the *extended Shi arrangement* $\mathcal{S}_n^m$ in $\mathbb{R}^n$ as follows.

$$\begin{aligned} \mathcal{C}_n^m &:= \{ \, \{x_i - x_j = a\} \mid 1 \leq i < j \leq n, -m \leq a \leq m \, \} , \quad (m \geq 0), \\ \mathcal{S}_n^m &:= \{ \, \{x_i - x_j = a\} \mid 1 \leq i < j \leq n, 1 - m \leq a \leq m \, \} , \quad (m \geq 1). \end{aligned}$$

Note that $\mathcal{C}_n^0 = \mathcal{B}_n$ for every nonnegative integer n and $L_0(\mathcal{C}_n^m) = L_0(\mathcal{S}_n^m) = \varnothing$ unless $n = 0$. Gill [5] investigated the intersection posets of the extended Catalan arrangements. First Gill determined the number of maximal elements of the poset $L(\mathcal{C}_n^m)$ as follows.

Theorem 1 (Gill [5, Theorem 1]). *Let m be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} |L_1(\mathcal{C}_n^m)| \frac{x^n}{n!} = \frac{e^x - 1}{1 - m(e^x - 1)} .$$

In Gill's work [5], the use of species, which were initiated by Joyal [7], is a noteworthy point. Standard references for species are the texts written by Bergeron, Labelle, and Leroux [1, 2]. Let $\mathbb{B}$ denote the category of finite sets and bijections and $\mathbb{E}$ the category of finite sets and maps. A *species, or* $\mathbb{B}$*-species,* is a functor $\mathsf{F}\colon \mathbb{B} \to \mathbb{E}$. The value of a species F at a finite set V is denoted by $\mathsf{F}[V]$. Moreover, we write simply $\mathsf{F}[n]$ instead of $\mathsf{F}[[n]]$. The symbol $\mathsf{F}(x)$ is used for the exponential generating function

$$\mathsf{F}(x) \coloneqq \sum_{n=0}^{\infty} |\mathsf{F}[n]| \frac{x^n}{n!}.$$

Example 2. Let E denote the *species of sets*. Namely $\mathsf{E}[V] \coloneqq \{V\}$. Then we have $|\mathsf{E}[n]| = 1$ for every n and thus $\mathsf{E}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

Example 3. Let L denote the *species of lists*. Namely

$$\mathsf{L}[V] \coloneqq \{\, (v_1, \dots, v_n) \mid n = |V| \text{ and } V = \{v_1, \dots, v_n\} \,\}.$$

We have $|\mathsf{L}[n]| = n!$ for every n and hence $\mathsf{L}(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Making use of species enables us to calculate generating functions systematically. Namely we can define operations for species such as sum, product, and so on. These operations are compatible with the corresponding operations for the generating functions. Hence we may say that species is a refinement of the exponential generating function.

Definition 4. For species F and G, we define the *sum* $\mathsf{F} + \mathsf{G}$ by $(\mathsf{F} + \mathsf{G})[V] \coloneqq \mathsf{F}[V] \sqcup \mathsf{G}[V]$, where $\sqcup$ means the disjoint union.

Then we have $(\mathsf{F} + \mathsf{G})(x) = \mathsf{F}(x) + \mathsf{G}(x)$.

Definition 5. For every species F and a nonnegative integer k, define the species F_k by

$$\mathsf{F}_k[V] \coloneqq \begin{cases} \mathsf{F}[V], & \text{if } |V| = k; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Every species F has a canonical decomposition $\mathsf{F} = \sum_{k=0}^{\infty} \mathsf{F}_k$. Furthermore, if we write $\mathsf{F}_+ \coloneqq \sum_{k=1}^{\infty} \mathsf{F}_k$, then $\mathsf{F}_+[\varnothing] = \varnothing$ and $\mathsf{F}_+(x) = \mathsf{F}(x) - \mathsf{F}(0)$.

Example 6.

$$\mathsf{E}_+(x) = e^x - 1, \quad \mathsf{E}_k(x) = \frac{x^k}{k!}, \quad \text{and} \quad \mathsf{L}_+(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}.$$

In this article, we frequently use the operation $\mathsf{F} \circ \mathsf{G}$ of species, called *substitution* (or *composition*).

Definition 7. Let F and G be species with $\mathsf{G}[\varnothing] = \varnothing$. Define

$$(\mathsf{F} \circ \mathsf{G})[V] := \bigsqcup_{\pi \in \Pi[V]} \left(\mathsf{F}[\pi] \times \prod_{B \in \pi} \mathsf{G}[B] \right),$$

where Π denotes the *species of set partitions*.

Substitution of species corresponds to substitution of generating functions, that is, $(\mathsf{F} \circ \mathsf{G})(x) = \mathsf{F}(\mathsf{G}(x))$. It is a functional tool for computing the generating function and explains why the exponential function e^x sometimes appears in the generating function.

Example 8. By definition, the exponential generating function of the Bell numbers is given by $\Pi(x)$. The species of set partitions Π coincides with the species $\mathsf{E} \circ \mathsf{E}_+$ (See Example 18 for details). Hence we have the following.

$$\Pi(x) = (\mathsf{E} \circ \mathsf{E}_+)(x) = \exp\left(e^x - 1\right).$$

Example 9. Given a species F, let $\mathsf{F}_+^{\circ m}$ be the m-times iterated self-substitution of F_+. For the species L of lists, we have

$$\mathsf{L}_+^{\circ 2}(x) = (\mathsf{L}_+ \circ \mathsf{L}_+)(x) = \frac{\frac{x}{1-x}}{1 - \frac{x}{1-x}} = \frac{x}{1-2x}.$$

More generally, one can show easily that

$$\mathsf{L}_+^{\circ m}(x) = \frac{x}{1 - mx}.$$

The construction of the extended Catalan arrangement $\mathcal{C}_n^m$ is functorial in n. Namely, for every finite set V, we may construct the corresponding Catalan arrangement in the vector space $\mathbb{R}^V = \mathrm{Map}(V, \mathbb{R})$ and hence there exist species $L\mathcal{C}^m$ and $L_k\mathcal{C}^m$ such that $L\mathcal{C}^m[n] = L(\mathcal{C}_n^m)$ and $L_k\mathcal{C}^m[n] = L_k(\mathcal{C}_n^m)$. Gill's second theorem is as follows.

Theorem 10 (Gill [5, Theorem 2])**.** *For every $m \geq 0$, the equality $L\mathcal{C}^m = \mathsf{E} \circ L_1\mathcal{C}^m$ holds. Moreover, the bivariate generating function of $|L_k(\mathcal{C}_n^m)|$ is given by*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} |L_k(\mathcal{C}_n^m)|\, t^k \frac{x^n}{n!} = \exp\left(t \frac{e^x - 1}{1 - m(e^x - 1)} \right).$$

We write $\mathsf{F} = \mathsf{G}$ if there exists a natural isomorphism between species F and G. In this paper, we will improve Gill's results in terms of species as follows.

Theorem 11. *Let m and k be nonnegative integers. Then*

$$L_k\mathcal{C}^m = \mathsf{E}_k \circ \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+ \quad \textit{and} \quad L\mathcal{C}^m = \mathsf{E} \circ \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+.$$

This theorem together with Example 6, Example 8, and Example 9 leads to the following corollary, which is equivalent to the generating functions in Theorem 1 and Theorem 10.

Corollary 12.

$$L_k\mathcal{C}^m(x) = \sum_{n=0}^{\infty} |L_k(\mathcal{C}_n^m)| \frac{x^n}{n!} = \frac{1}{k!}\left(\frac{e^x-1}{1-m(e^x-1)}\right)^k,$$
$$L\mathcal{C}^m(x) = \sum_{n=0}^{\infty} |L(\mathcal{C}_n^m)| \frac{x^n}{n!} = \exp\left(\frac{e^x-1}{1-m(e^x-1)}\right).$$

We will also give an analog for the extended Shi arrangement $\mathcal{S}_n^m$. However, the construction of $\mathcal{S}_n^m$ cannot be regarded as a functor on $\mathbb{B}$ since the construction requires the linear order on the set $[n]$. For this reason we consider $\mathbb{L}$-species, that is, a functor $\mathsf{F}\colon \mathbb{L} \to \mathbb{E}$, where $\mathbb{L}$ denotes the category of linearly ordered finite sets and order-preserving bijections. Note that every $\mathbb{B}$-species can be considered as an $\mathbb{L}$-species via the forgetful functor from $\mathbb{L}$ to $\mathbb{B}$.

Let $L\mathcal{S}^m$ and $L_k\mathcal{S}^m$ denote the $\mathbb{L}$-species such that $L\mathcal{S}^m[n] = L(\mathcal{S}_n^m)$ and $L_k\mathcal{S}^m[n] = L_k(\mathcal{S}_n^m)$. The following is another main result of this article.

Theorem 13. *Let m and k be nonnegative integers. Then*

$$L_k\mathcal{S}^m = \mathsf{E}_k \circ \mathsf{L}_+^{\circ m} \quad \text{and} \quad L\mathcal{S}^m = \mathsf{E} \circ \mathsf{L}_+^{\circ m}.$$

Corollary 14.

$$L_k\mathcal{S}^m(x) = \sum_{n=0}^{\infty} |L_k(\mathcal{S}_n^m)| \frac{x^n}{n!} = \frac{1}{k!}\left(\frac{x}{1-mx}\right)^k,$$
$$L\mathcal{S}^m(x) = \sum_{n=0}^{\infty} |L(\mathcal{S}_n^m)| \frac{x^n}{n!} = \exp\left(\frac{x}{1-mx}\right).$$

Moreover, we will give explicit formulas for the numbers $|L_k(\mathcal{S}_n^m)|$ and $|L_k(\mathcal{C}_n^m)|$ of k-dimensional flats of n-dimensional extended Catalan and extended Shi arrangements with infinite matrices. Let $[a_{ij}]$ denote the infinite matrix whose entry in the i-th row and the j-th column is a_{ij}, where i and j run over all positive integers. Let

$$c \coloneqq [c(j,i)] \quad \text{and} \quad S \coloneqq [S(j,i)],$$

where $c(j,i)$ denote the *unsigned Stirling number of the first kind*, that is, the number of ways to partition a j-element set into i cycles. Note that most of the tables, including Tables 5–6, consisting of such numbers are lower triangular. However, our infinite matrices c and

S are transposed and hence upper triangular. Namely,

$$c = \begin{bmatrix} 1 & 1 & 2 & 6 & 24 & \cdots \\ 0 & 1 & 3 & 11 & 50 & \cdots \\ 0 & 0 & 1 & 6 & 35 & \cdots \\ 0 & 0 & 0 & 1 & 10 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 3 & 7 & 15 & \cdots \\ 0 & 0 & 1 & 6 & 25 & \cdots \\ 0 & 0 & 0 & 1 & 10 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 15.

$$\Big[\big| L_i(\mathcal{C}_j^m) \big| \Big] = (Sc)^m S \quad \text{and} \quad \Big[\big| L_i(\mathcal{S}_j^m) \big| \Big] = (Sc)^m.$$

From Theorem 15, we may also calculate the matrices recursively as follows.

$$\begin{array}{ccccccccccc} \mathcal{C}^0 & \xrightarrow{c} & \mathcal{S}^1 & \xrightarrow{S} & \mathcal{C}^1 & \xrightarrow{c} & \mathcal{S}^2 & \xrightarrow{S} & \mathcal{C}^2 & \xrightarrow{c} & \cdots \\ S & & Sc & & ScS & & ScSc & & ScScS & & \end{array}$$

The *Lah number* is the number of ways to partition an n-element set into k nonempty lists, which is equal to the cardinality of $(\mathsf{E}_k \circ \mathsf{L}_+)[n]$ (See Example 18 for details). It is well known that the Lah number is given by the following formula.

Proposition 16 (See [11, p.44] for example)**.** *Let k and n be nonnegative integers. Then*

$$\big| (\mathsf{E}_k \circ \mathsf{L}_+)[n] \big| = \frac{n!(n-1)!}{k!(k-1)!(n-k)!}.$$

Using the Lah numbers, we give an explicit formula for the number of flats of the extended Shi arrangement.

Theorem 17.

$$\big| L_k(\mathcal{S}_n^m) \big| = m^{n-k} \frac{n!(n-1)!}{k!(k-1)!(n-k)!}.$$

The organization of this paper is as follows.

In §2, we give some examples of substitutions of species. Also, we introduce tree notation for the species $\mathsf{L}_+^{\circ m}$. Although actually this notation is not required for the proofs of our main theorems, it helps us to recognize elements of $\mathsf{L}_+^{\circ m}$.

In §3, we review the theory of gain graphs developed by Zaslavsky. Since the extended Catalan and Shi arrangements are expressed by using gain graphs, the intersection posets of these arrangements can be represented by a kind of partitions of the vertices of the corresponding gain graphs. This guarantees that it suffices to know the 1-dimensional flats in order to know all flats, that is, $L\mathcal{C}^m = \mathsf{E} \circ L_1\mathcal{C}^m$ and $L\mathcal{S}^m = \mathsf{E} \circ L_1\mathcal{S}^m$.

Finally in §4, we will give proofs of Theorem 11, Theorem 13, Theorem 15, and Theorem 17.

In §5, as an appendix we give numerical tables for the number of flats with ID numbers in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [9].

Note that, throughout of this paper, we will construct several natural transformations. However, we will omit proofs of commutativity of diagrams for the natural transformations since they are obvious.

2 Substitution of species and tree notation

2.1 Examples

Let F and G be species with $\mathsf{G}[\varnothing] = \varnothing$. Recall the definition of the substitution $\mathsf{F} \circ \mathsf{G}$.

$$(\mathsf{F} \circ \mathsf{G})[V] \coloneqq \bigsqcup_{\pi \in \Pi[V]} \left(\mathsf{F}[\pi] \times \prod_{B \in \pi} \mathsf{G}[B] \right),$$

where Π denotes the species of set partitions.

The substitution $\mathsf{F} \circ \mathsf{G}$ has the external and internal structure. Namely, for each set partition π, we take an element of $\mathsf{F}[\pi]$, which is the external structure. For every block B of π, we choose an element of $\mathsf{G}[B]$, which is the internal structure. In the usual case, a constituent of the external structure is labeled with blocks of a partition and if we "substitute" the labels with internal structures, then we obtain an element of $(\mathsf{F} \circ \mathsf{G})[V]$.

Example 18. Let G be a species such that if $g \in \mathsf{G}[B], g' \in \mathsf{G}[B']$, and $g = g'$, then $B = B'$. We will see that the species $\mathsf{E} \circ \mathsf{G}$ may be considered as a species of set partitions consisting of G-structures. By definition,

$$(\mathsf{E} \circ \mathsf{G})[V] = \bigsqcup_{\pi \in \Pi[V]} \left(\mathsf{E}[\pi] \times \prod_{B \in \pi} \mathsf{G}[B] \right) = \bigsqcup_{\pi \in \Pi[V]} \left(\{\pi\} \times \prod_{B \in \pi} \mathsf{G}[B] \right).$$

For any set partition $\pi = \{B_1, \dots, B_k\}$, every element $(\pi, (g_{B_1}, \dots, g_{B_k})) \in \{\pi\} \times \prod_{B \in \pi} \mathsf{G}[B]$ can be identified with the set $\{g_{B_1}, \dots, g_{B_k}\}$. Then the set $(\mathsf{E} \circ \mathsf{G})[V]$ is identified with

$$\{ \{g_{B_1}, \dots, g_{B_k}\} \mid \{B_1, \dots, B_k\} \in \Pi[V],\ g_{B_i} \in \mathsf{G}[B_i] \}.$$

In particular, $\mathsf{E} \circ \mathsf{E}_+$ gives set partitions consisting of sets, that is, ordinary set partitions. Namely $\mathsf{E} \circ \mathsf{E}_+ = \Pi$. Therefore $\left|(\mathsf{E}_k \circ \mathsf{E}_+)[n]\right|$ is the Stirling number of the second kind. For the same reason $\left|(\mathsf{E}_k \circ \mathsf{L}_+)[n]\right|$ yields the Lah number.

We sometimes omit commas in sets, lists, and so on. For example, we write $\{123\}$ instead of $\{1, 2, 3\}$.

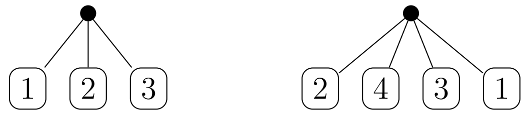


Figure 1: (123) and (2431)

Example 19. Let G be as above. By the similar discussion, we may say that the species $\mathsf{L}\circ\mathsf{G}$ may be considered as a species of lists consisting of G-structures. Namely the set $(\mathsf{L}\circ\mathsf{G})[V]$ can be identified naturally with

$$\{\ (g_{B_1},\cdots,g_{B_k})\mid\{B_1,\ldots,B_k\}\in\Pi[V],\ \ g_{B_i}\in\mathsf{G}[B_i]\ \}\,.$$

The species $\mathsf{L}\circ\mathsf{E}_+$ is known as a species of *set compositions* (or *ordered set partitions*). The cardinality $|(\mathsf{L}\circ\mathsf{E}_+)[n]|$ is called the *ordered Bell number* (or *Fubini number*). For instance $|(\mathsf{L}\circ\mathsf{E}_+)[3]|=13$ since it consists of the following set compositions.

$$\begin{aligned}&(\{123\}),(\{12\}\{3\}),(\{13\}\{2\}),(\{23\}\{1\}),(\{3\}\{12\}),(\{2\}\{13\}),(\{1\}\{23\}),\\&(\{1\}\{2\}\{3\}),(\{1\}\{3\}\{2\}),(\{2\}\{1\}\{3\}),(\{2\}\{3\}\{1\}),(\{3\}\{1\}\{2\}),(\{3\}\{2\}\{1\}).\end{aligned}$$

Example 20. The species E_1 of singletons behaves as the identity element with respect to substitution. Namely, $\mathsf{E}_1\circ\mathsf{G}=\mathsf{G}$ and $\mathsf{F}\circ\mathsf{E}_1=\mathsf{F}$.

A lot of researchers have been studied iterated substitutions of species of sets and lists. For example, Motzkin [8] investigated several structures including, "sets of sets" $\mathsf{E}\circ\mathsf{E}_+$, "sets of lists" $\mathsf{E}\circ\mathsf{L}_+$, "lists of sets" $\mathsf{L}\circ\mathsf{E}_+$, and "lists of lists" $\mathsf{L}\circ\mathsf{L}_+$. Sloane and Wieder [12] call an element of $(\mathsf{E}\circ\mathsf{L}_+\circ\mathsf{E}_+)[n]$ a *hierarchical ordering* (or *society*). Callan [3, Section 2] gave a bijection between lists of noncrossing sets and sets of lists $\mathsf{E}\circ\mathsf{L}_+$. Hedmark [6, Subsection 5.2] introduced an α-colored partition lattice for a positive integer α and stated that it can be regarded as $\mathsf{L}_+^{\circ\alpha}\circ\mathsf{E}_+$.

2.2 Tree notation for $\mathsf{L}_+^{\circ m}$

The species $\mathsf{L}_+^{\circ m}$ can be considered as the species of "m-dimensional lists". For example $(((49)(5))((3)(71)(6))((82)))$ is an element of $\mathsf{L}_+^{\circ 3}[9]$. However, it is difficult to understand the structure of this at first glance. Hence we introduce tree notation for $\mathsf{L}_+^{\circ m}$.

The idea is very simple. We just regard a list as an ordered rooted tree of height one with labeled leaves, where an *ordered rooted tree* means a rooted tree whose sibling sets are linearly ordered as lists. For example the lists (123) and (2413) are expressed as in Figure 1.

Then the species $\mathsf{L}_+^{\circ m}$ can be regarded as a rooted tree of height m. For example,

$$(((49)(5))((3)(71)(6))((82)))\in\mathsf{L}_+^{\circ 3}[9]$$

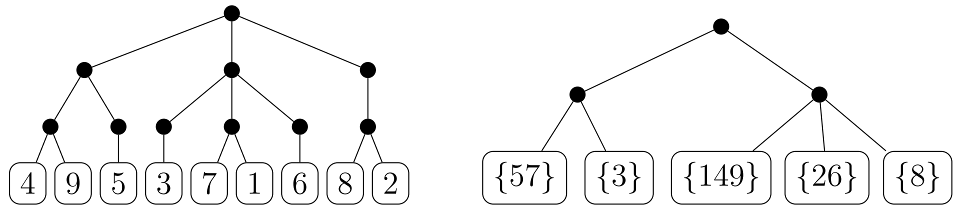


Figure 2: Elements in $\mathsf{L}_+^{\circ 3}[9]$ and $(\mathsf{L}_+^{\circ 2} \circ \mathsf{E}_+)[9]$

is expressed as the left in Figure 2. We also can express elements of $\mathsf{L}_+^{\circ m} \circ \mathsf{E}_+$ by taking labels consisting of sets. For example

$$((\{57\}\{3\})(\{149\}\{26\}\{8\})) \in (\mathsf{L}_+^{\circ 2} \circ \mathsf{E}_+)[9]$$

is as the right in Figure 2.

3 Gain graphs and the associated posets

3.1 Review of graphic arrangements

First we recall graphic arrangements and their intersection lattices. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a simple graph on vertex set $V_\Gamma = [n]$. We can associate Γ with a hyperplane arrangement $\mathcal{A}_\Gamma$ in $\mathbb{R}^n$, called the *graphic arrangement*, consisting of hyperplanes defined by $\{x_i - x_j = 0\}$ with $\{i, j\} \in E_\Gamma$, where $x_1, \dots, x_\ell$ denote coordinates of $\mathbb{R}^\ell$.

It is well known that the intersection lattice $L(\mathcal{A}_\Gamma)$ can be represented by using set partitions as explained below. A *connected partition* of Γ is a set partition of V_Γ whose every block induces a connected subgraph of Γ. Let $L(\Gamma)$ be the set of all connected partitions of Γ with the partial order defined by refinement. Namely, $\pi \leq \pi'$ if each block of π' is the union of some blocks of π. We call $L(\Gamma)$ the *lattice of connected partitions* (or the *lattice of contractions*), which is naturally isomorphic to $L(\mathcal{A}_\Gamma)$.

Note that the braid arrangement $\mathcal{B}_n$ can be regarded as a graphic arrangement with the complete graph K_n. Since $L(K_n)$ consists of all set partitions of $[n]$, the number of flats of the braid arrangement is associated with the Bell number and the Stirling number of the second kind.

3.2 Integral gain graphs and affinographic arrangements

The extended Catalan and Shi arrangements are represented by using gain graphs. Gain graphs were introduced by Zaslavsky [14] for abstraction of linear independence of hyperplanes of the form $\{x_i - x_j = a\}$ with $a \in \mathbb{Z}$. Roughly speaking, a gain graph is a graph with labeled edges (i, j, a), which corresponds to the $\{x_i - x_j = a\}$. However, since $\{x_i - x_j = a\} = \{x_j - x_i = -a\}$, we must identify (i, j, a) with $(j, i, -a)$. Here we give a formal definition of gain graphs which are required in this article. See the paper by Zaslavsky [14] for a general treatment.

Definition 21. An *integral gain graph* is a pair $\Gamma = (V_\Gamma, E_\Gamma)$ satisfying the following conditions.

(i) V_Γ is a finite set.

(ii) E_Γ is a finite subset of $\{\, (u,v,a) \in V_\Gamma \times V_\Gamma \times \mathbb{Z} \mid u \neq v \,\}$ divided by the equivalence relation $\sim$ generated by $(u,v,a) \sim (v,u,-a)$.

Let $\{u,v\}_a$ denote the equivalence class containing (u,v,a). Then $\{u,v\}_a = \{v,u\}_{-a}$. Elements in V_Γ and E_Γ are called vertices and edges of the gain graph Γ.

Definition 22. Suppose that Γ is an integral gain graph on $[n]$. Define an affine arrangement $\mathcal{A}_\Gamma$ in $\mathbb{R}^n$ by

$$\mathcal{A}_\Gamma \coloneqq \{\, \{x_i - x_j = a\} \mid \{i,j\}_a \in E_\Gamma \,\}.$$

We call $\mathcal{A}_\Gamma$ the *affinographic arrangement* of Γ.

Note that every simple graph is regarded as a gain graph by regarding $\{u,v\}$ as $\{u,v\}_0$. So every graphic arrangement is considered to be an affinographic arrangement. Hence there is no confusion to use the same symbol $\mathcal{A}_\Gamma$ for the graphic and affinographic arrangements.

Definition 23. Let A be a finite subset of $\mathbb{Z}$. For every positive integer n, define K_n^A as an integral gain graph on $[n]$ with edges

$$E_{K_n^A} \coloneqq \{\, \{i,j\}_a \mid 1 \leq i < j \leq n,\ a \in A \,\}.$$

We call K_n^A the *complete gain graph with gain* A.

For integers a, b with $a \leq b$, let $[a,b] \coloneqq \{a, a+1, \dots, b\} \subseteq \mathbb{Z}$.

Example 24. The extended Catalan arrangements and the extended Shi arrangements are obtained by

$$\mathcal{C}_n^m = \mathcal{A}_{K_n^{[-m,m]}} \quad \text{and} \quad \mathcal{S}_n^m = \mathcal{A}_{K_n^{[1-m,m]}}.$$

3.3 Poset of connected partitions

A *height function* on a finite set B is a map $h\colon B \to \mathbb{Z}$ such that $\min(h(B)) = 0$.

Let Γ be an integral gain graph. Consider a pair (B,h), where $B \subseteq V_\Gamma$ and h is a height function on B. Define $\Gamma[B,h]$ as the integral gain graph on B with edges

$$\begin{aligned} E_{\Gamma[B,h]} &\coloneqq \{\, \{u,v\}_a \in E_\Gamma \mid u,v \in B,\ h(u) + a = h(v) \,\} \\ &= \left\{\, \{u,v\}_{-h(u)+h(v)} \in E_\Gamma \;\middle|\; u,v \in B \,\right\}. \end{aligned}$$

Let $v_1, v_2, \ldots, v_r$ be distinct vertices of an integral gain graph Γ. A *path* on the vertices $v_1, \ldots, v_r$ is a set of edges $\{v_1, v_2\}_{a_1}, \{v_2, v_3\}_{a_2}, \ldots, \{v_{r-1}, v_r\}_{a_{r-1}}$. We say that Γ is *connected* if there exists a path joining any two distinct vertices. A *connected partition* of Γ is a collection $\pi = \{(B_1, h_1), \ldots, (B_k, h_k)\}$ such that $\{B_1, \ldots, B_k\}$ is a set partition of V_Γ and each $\Gamma[B_i, h_i]$ is connected. An element of π is called a *block*.

Let π and π' be connected partitions. We say that π *refines* π', denoted by $\pi \leq \pi'$ if for any $(B, h) \in \pi$ there exist $(B', h') \in \pi'$ and $g \in \mathbb{Z}$ such that $B \subseteq B'$ and $h(v) = g + h'(v)$ for any $v \in B$. Let $L(\Gamma)$ denote the set of all connected partitions, which forms a poset together with the refinement. Call $L(\Gamma)$ the *poset of connected partitions* of Γ.

Note that when a simple graph Γ is viewed as a gain graph, connected partitions coincide with the usual connected partitions of the simple graph Γ. Therefore the poset of connected partitions of a gain graph is a generalization of the lattice of connected partitions of a simple graph.

Remark 25. For a general gain graph with gain group G, we need notion of G-labeled set to define connected partitions. However, for integral gain graphs, we need only height functions, which were introduced by Corteel, Forge, and Ventos [4].

Theorem 26. *Let Γ be an integral gain graph on $[n]$.*

(1) If (B, h) is a block of a connected partition of Γ with $B = \{i_1, \ldots, i_r\}$, then

$$X_{(B,h)} \coloneqq \bigcap_{\{i,j\}_a \in E_{\Gamma[B,h]}} \{x_i - x_j = a\} = \{x_{i_1} + h(i_1) = \cdots = x_{i_r} + h(i_r)\}.$$

(2) The following map is an isomorphism of posets.

$$\begin{array}{ccc} L(\Gamma) & \longrightarrow & L(\mathcal{A}_\Gamma) \\ \pi & \longmapsto & \displaystyle\bigcap_{(B,h)\in\pi} X_{(B,h)}. \end{array}$$

(3) For any nonnegative integer k, the isomorphism in (2) induces a bijection between $L_k(\Gamma)$ and $L_k(\mathcal{A}_\Gamma)$, where

$$L_k(\Gamma) \coloneqq \{\ \pi \in L(\Gamma) \mid \ |\pi| = k\ \}$$

and $|\pi|$ denotes the number of blocks of π.

Proof. (1) Recall $E_{\Gamma[B,h]} = \left\{ \{i,j\}_{-h(i)+h(j)} \in E_\Gamma \ \middle|\ i, j \in B \right\}$. The hyperplane corresponding to the edge $\{i,j\}_{-h(i)+h(j)}$ is

$$\{x_i - x_j = -h(i) + h(j)\} = \{x_i + h(i) = x_j + h(j)\}.$$

Since $\Gamma[B, h]$ is connected, the intersection coincides with the right hand side.

(2) This follows from theorems by Zaslavsky [15, Corollary 4.5(a)], [13, Lemma 3.1A and 3.1B].

(3) Obvious from (2). $\square$

3.4 Partitional decompositions

Let A be a finite subset of $\mathbb{Z}$. The construction of the complete gain graph with gain A is considered to be an $\mathbb{L}$-species. Namely, for each finite linearly ordered set V, we may construct the complete gain graph K_V^A on V with gain A. Hence, there exist $\mathbb{L}$-species LK^A and L_kK^A such that $LK^A[n] = L(K_n^A)$ and $L_kK^A[n] = L_k(K_n^A)$. In particular, if $-A = A$, then the $\mathbb{L}$-species LK^A and L_kK^A may be considered to be species.

Lemma 27. *Let A be a finite subset of $\mathbb{Z}$. Then*

$$L_kK^A = \mathsf{E}_k \circ L_1K^A \quad \textit{and} \quad LK^A = \mathsf{E} \circ L_1K^A$$

In particular,

$$L_k\mathcal{C}^m = L_kK^{[-m,m]} = \mathsf{E}_k \circ L_1K^{[-m.m]}, \qquad L\mathcal{C}^m = LK^{[-m,m]} = \mathsf{E} \circ L_1K^{[-m.m]},$$

$$L_k\mathcal{S}^m = L_kK^{[1-m,m]} = \mathsf{E}_k \circ L_1K^{[1-m.m]}, \qquad L\mathcal{S}^m = LK^{[1-m,m]} = \mathsf{E} \circ L_1K^{[1-m.m]}.$$

Proof. Let (B,h) be a block of a connected partition of K_n^A. Then the block (B,h) may be regarded as an element of $L_1K^A[B]$. This identification leads to the natural isomorphisms $L_kK^A = \mathsf{E}_k \circ L_1K^A$ and $LK^A = \mathsf{E} \circ L_1K^A$. The rest follows by Theorem 26. □

4 Proofs

4.1 Proof of Theorem 11

In order to describe $L\mathcal{C}^m$, it is sufficient to determine the species $L_1K^{[-m,m]}$ by Lemma 27. Given a finite set V, an element of $L_1K^{[-m,m]}[V] = L_1(K_V^{[-m,m]})$ is identified with a height function h on V such that $K_V^{[-m,m]}[V,h]$ is connected. In order to characterize such functions, define gap(h) and level(h) for each height functions as follows.

Suppose that $h(V) = \{a_1, \ldots, a_k\}$ with $0 = a_1 < a_2 < \cdots < a_k$. Then we define

$$\begin{aligned} \operatorname{level}(h) &:= \left(h^{-1}(a_1), h^{-1}(a_2), \ldots, h^{-1}(a_k)\right), \\ \operatorname{gap}(h) &:= (a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}). \end{aligned}$$

Let $\max(\operatorname{gap}(h))$ denote the maximum of entries of gap(h).

Lemma 28. *Let V be a finite set and h a height function on V. For every nonnegative integer m, $K_V^{[-m,m]}[V,h]$ is connected if and only if* $\max(\operatorname{gap}(h)) \leq m$.

Proof. Let level$(h) = (B_1, \ldots, B_k)$ and gap$(h) = (\alpha_1, \ldots, \alpha_{k-1})$. Suppose that $K_V^{[-m,m]}[V,h]$ is connected. Then for every $i \in \{1, \ldots, k-1\}$ there exists an edge $\{u,v\}_a$, where $u \in B_1 \cup \cdots \cup B_i$ and $v \in B_{i+1} \cup \cdots \cup B_k$. Then $\alpha_i \leq h(v) - h(u) = a \leq m$. Thus $\max(\operatorname{gap}(h)) \leq m$.

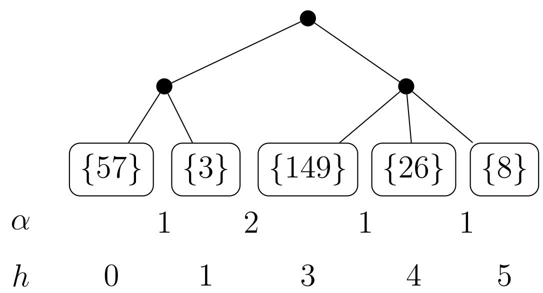


Figure 3: An example of the correspondence for the extended Catalan arrangement

To prove the converse, suppose that $\max(\mathrm{gap}(h)) \leq m$. Suppose that $u, v \in B_i$ for some $i \in \{1, \ldots, k\}$. Then $\{u, v\}_0$ is an edge of $K_V^{[-m,m]}[V, h]$. Now, let $i \in \{1, \ldots, k-1\}$ and take vertices $u \in B_i$ and $v \in B_{i+1}$. Then $h(v) - h(u) = \alpha_i \leq \max(\mathrm{gap}(h)) \leq m$. Therefore $\{u, v\}_{\alpha_i}$ is an edge of $K_V^{[-m,m]}[V, h]$. Hence we can deduce $K_V^{[-m,m]}[V, h]$ is connected. □

Lemma 29. *Let m be a nonnegative integer. Then $L_1 K^{[-m,m]} = \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+$.*

Proof. We proceed by induction on m. First suppose that $m = 0$. Let V be a finite set. Then $K_V^{\{0\}}[V, h]$ is connected if and only if h is identically 0 by Lemma 28. Hence the identification $(V, 0)$ with V yields the natural isomorphism $L_1 K^{\{0\}} = \mathsf{E}_+$.

Now suppose that $m \geq 1$. We will prove $L_1 K^{[-m,m]} = \mathsf{L}_+ \circ L_1 K^{[-(m-1),m-1]}$. Define a natural transformation $\eta \colon L_1 K^{[-m,m]} \to \mathsf{L}_+ \circ L_1 K^{[-(m-1),m-1]}$ as follows. Let V be a finite set and h be a height function on V such that $K_V^{[-m,m]}[V, h]$ is connected. Let $\mathrm{level}(h) = (B_1, \ldots, B_k)$ and $\mathrm{gap}(h) = (\alpha_1, \ldots, \alpha_{k-1})$ and suppose that $\{\, j \mid \alpha_j = m \,\} = \{j_1, \ldots, j_{s-1}\}$ with $j_1 < \cdots < j_{s-1}$. For each $i \in \{1, \ldots, s\}$ define $C_i \coloneqq B_{j_{i-1}+1} \cup B_{j_{i-1}+2} \cup \cdots \cup B_{j_i}$, where $j_0 \coloneqq 0$ and $j_s \coloneqq k$. Define the height function h_i on C_i by $h_i(v) \coloneqq h(v) - \min(h(C_i)) \quad (v \in C_i)$. Then $\max(\mathrm{gap}(h_i)) \leq m - 1$ and hence (C_i, h_i) is an element of $L_1 K^{[-(m-1),m-1]}[C_i]$. Therefore we define η_V by $\eta_V\left((V, h)\right) \coloneqq ((C_1, h_1), \ldots, (C_s, h_s))$.

We will construct another natural transformation $\xi \colon \mathsf{L}_+ \circ L_1 K^{[-(m-1),m-1]} \to L_1 K^{[-m,m]}$ as follows. Let V be a finite set and take an element $((C_1, h_1), \ldots, (C_s, h_s))$ from $\mathsf{L}_+ \circ L_1 K^{[-(m-1),m-1]}$. Then $\{C_1, \ldots, C_s\}$ is a set partition of V and $(C_i, h_i) \in L_1 K^{[-(m-1),m-1]}[C_i]$ for every $i \in \{1, \ldots, s\}$. By Lemma 28 $\max(\mathrm{gap}(h_i)) \leq m - 1$. Let h be the height function on V defined by $h(v) \coloneqq h_i(v) + \sum_{j=1}^{i-1} \left(m + \max(h_j(C_j))\right)$ for $v \in C_i$. Since $\max(\mathrm{gap}(h)) \leq m$, $(V, h) \in L_1 K^{[-m,m]}[V]$. Thus we may define ξ_V by $\xi_V\left(((C_1, h_1), \ldots, (C_s, h_s))\right) \coloneqq (V, h)$.

It is obvious that η and ξ are inverse to each other. Therefore $L_1 K^{[-m,m]} = \mathsf{L}_+ \circ L_1 K^{[-(m-1),m-1]}$. By induction hypothesis, we can conclude that $L_1 K^{[-m,m]} = \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+$. □

Proof of Theorem 11. Use Lemma 27 and Lemma 29. □

Example 30. Consider $((B_1 B_2)(B_3 B_4 B_5)) = ((\{57\}\{3\})(\{149\}\{26\}\{8\})) \in (\mathsf{L}_+^{\circ 2} \circ \mathsf{E}_+)[9]$ (See Figure 3). We construct the corresponding flat of the extended Catalan arrangement $\mathcal{C}_9^2$. First, for each i, let α_i denote the height of the minimal tree containing leaves B_i and

B_{i+1}. In this case we have the integer composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 2, 1, 1)$. By taking the partial sum $\sum_{j=1}^{i-1} \alpha_j$ for each i, we obtain the sequence of heights $(0, 1, 3, 4, 5)$. The height function $h : [9] \to \mathbb{Z}$ is obtained by the following table.

v	5	7	3	1	4	9	2	6	8
$h(v)$	0	0	1	3	3	3	4	4	5

The corresponding flat in $\mathcal{C}_9^2$ is

$$\{x_5 = x_7 = x_3 + 1 = x_1 + 3 = x_4 + 3 = x_9 + 3 = x_2 + 4 = x_6 + 4 = x_8 + 5\}.$$

4.2 Proof of Theorem 13

We assume that all species in this subsection are $\mathbb{L}$-species and V denotes a finite linearly ordered set. The symbols $\min(B)$ and $\max(B)$ stand for the minimum and the maximum elements of a subset $B \subseteq V$.

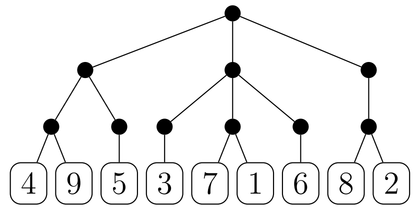
Lemma 31. *Let V be a finite linearly ordered set and h a height function on V. Let* $\operatorname{level}(h) = (B_1, \dots, B_k)$ *and* $\operatorname{gap}(h) = (\alpha_1, \dots, \alpha_{k-1})$. *For every positive integer m, the graph $K_V^{[1-m,m]}[V, h]$ is connected if and only if* $\max(\operatorname{gap}(h)) \leq m$ *holds and $\alpha_i = m$ implies* $\min(B_i) < \max(B_{i+1})$.

Proof. Suppose that $K_V^{[1-m,m]}[V, h]$ is connected. Then $K_V^{[-m,m]}[V, h]$ is also connected since $K_V^{[1-m,m]}[V, h]$ is a subgraph of $K_V^{[-m,m]}[V, h]$. Therefore $\max(\operatorname{gap}(h)) \leq m$ by Lemma 28. Suppose that $\alpha_i = m$. Then there exist vertices $u \in B_i$ and $v \in B_{i+1}$ such that $\{u, v\}_m$ is an edge of $K_V^{[1-m,m]}[V, h]$ since $K_V^{[1-m,m]}[V, h]$ is connected, which implies $u < v$ in V. Thus it follows that $\min(B_i) < \max(B_{i+1})$. The proof of the converse is similar to the proof of Lemma 28. □

Lemma 32. *Let m be a positive integer. Then $L_1 K^{[1-m,m]} = \mathsf{L}_+^{\circ m}$.*

Proof. We proceed by induction on m. Suppose that $m = 1$. We will construct a natural transformation $\eta\colon L_1 K^{[0,1]} \to \mathsf{L}_+$. Let V be a finite linearly ordered set and h a height function on V such that $K_V^{[0,1]}[V, h]$ is connected. Let $\operatorname{level}(h) = (B_1, \dots, B_k)$. Make the list $\beta_i = (v_{i1}, \dots, v_{ij_i})$ such that $B_i = \{v_{i1}, \dots, v_{ij_i}\}$ with $v_{i1} > \cdots > v_{ij_i}$. Define η_V as the list obtaining by concatenating $\beta_1, \dots, \beta_k$.

Next we will construct another natural transformation $\xi\colon \mathsf{L}_+ \to L_1 K^{[0,1]}$. Every list in $\mathsf{L}_+[V]$ can be expressed as $(v_{11}, \dots, v_{1j_1}, v_{21}, \dots, v_{2j_2}, \dots, v_{k1}, \dots, v_{kj_k})$, where $v_{i1} > \cdots > v_{ij_i}$ and $v_{ij_i} < v_{i+1\,1}$ for each $i \in \{1, \dots k-1\}$. Let $B_i \coloneqq \{v_{i1}, \dots, v_{ij_i}\}$ and define the height function on V by $h(v) \coloneqq i - 1$ for $v \in B_i$. Then $\max(\operatorname{gap}(h)) \leq 1$ and $\min(B_i) = v_{ij_i} < v_{i+1\,1} = \max(B_{i+1})$. Therefore by Lemma 31 $(V, h) \in L_1 K^{[0,1]}[V]$. Thus we may define ξ_V by the correspondence above. One can show that η and ξ are inverse to each other and hence $L_1 K^{[0,1]} = \mathsf{L}_+$.



α		1	2	3	2	1	2	3	1
α'		1	1	2	2	0	2	3	0
h	0	1	2	4	6	6	8	11	11

Figure 4: An example of the correspondence for the extended Shi arrangement

Assume that $m \geq 2$ and we will prove that $L_1K^{[1-m,m]} = \mathsf{L}_+ \circ L_1K^{[2-m,m-1]}$. We will construct a natural transformation $\eta\colon L_1K^{[1-m,m]} \to \mathsf{L}_+ \circ L_1K^{[2-m,m-1]}$. Let h be a height function on a finite linearly ordered set V such that $K_V^{[1-m,m]}[V,h]$ is connected. Let $\operatorname{level}(h) = (B_1, \ldots, B_k)$ and $\operatorname{gap}(h) = (\alpha_1, \ldots, \alpha_{k-1})$. Suppose that $\{\, j \mid \alpha_j = m \,\} \cup \{\, j \mid \alpha_j = m-1,\ \min(B_j) > \max(B_{j+1}) \,\} = \{j_1, \ldots, j_{s-1}\}$ with $j_1 < \cdots < j_{s-1}$. For each $i \in \{1, \ldots, s\}$, put $C_i \coloneqq B_{j_{i-1}+1} \cup \cdots \cup B_{j_i}$, where $j_0 = 0$ and $j_s \coloneqq k$. Define the height function h_i on C_i by $h_i(v) \coloneqq h(v) - \min(h(C_i))$ for $v \in C_i$. Then $\max(\operatorname{gap}(h_i)) \leq m-1$ and $\operatorname{level}(h_i) = (B_{j_{i-1}+1}, \ldots, B_{j_i})$. Moreover for $i \in \{j_{i-1}+1, \ldots, j_i\}$, if $\alpha_i = m-1$, then $\min(B_i) < \max(B_{i+1})$. Therefore $(C_i, h_i) \in L_1K^{[2-m,m-1]}[C_i]$ by Lemma 31. Thus we may define η_V by $\eta_V((V,h)) \coloneqq ((C_1,h_1), \ldots, (C_s,h_s))$.

To construct another natural transformation $\xi\colon \mathsf{L}_+ \circ L_1K^{[2-m,m-1]} \to L_1K^{[1-m,m]}$, take an element $((C_1,h_1), \ldots, (C_s,h_s)) \in \mathsf{L}_+ \circ L_1K^{[2-m,m-1]}[V]$. For each $i \in \{1, \ldots, s-1\}$, define the integer μ_i as follows. If the minimum of the terminal block of $\operatorname{level}(h_i)$ is less than the maximum of the initial block of $\operatorname{level}(h_{i+1})$, then $\mu_i \coloneqq m$. Otherwise let $\mu_i \coloneqq m-1$. Define the height function h on V by $h(v) \coloneqq h_i(v) + \sum_{j=1}^{i-1} (\mu_j + \max(h_j(C_j)))$ for $v \in C_i$. Then one can deduce that $(V,h) \in L_1K^{[1-m,m]}[V]$ by Lemma 31. Hence we may define ξ_V by $\xi_V(((C_1,h_1), \ldots, (C_s,h_s))) \coloneqq (V,h)$.

It is easy to show that η and ξ are inverse to each other and hence $L_1K^{[1-m,m]} = \mathsf{L}_+ \circ L_1K^{[2-m,m-1]}$. Finally by the induction hypothesis, we have $L_1K^{[1-m,m]} = \mathsf{L}_+^{\circ m}$. □

Proof of Theorem 13. Use Lemma 27 and Lemma 32. □

Example 33. Consider

$$(((v_1v_2)(v_3))((v_4)(v_5v_6)(v_7))((v_8v_9))) = (((49)(5))((3)(71)(6))((82))) \in \mathsf{L}_+^{\circ 3}[9]$$

(See Figure 4). We construct the corresponding flat of the extended Shi arrangement $\mathcal{S}_9^3$. First let α be the integer composition obtained in a similar way in Example 30. In this case $\alpha = (1,2,3,2,1,2,3,1)$. Next define the integer composition α' by

$$\alpha'_i \coloneqq \begin{cases} \alpha_i, & \text{if } v_i < v_{i+1}; \\ \alpha_i - 1, & \text{if } v_i > v_{i+1}. \end{cases}$$

In this case $\alpha' = (1,1,2,2,0,2,3,0)$. By taking the partial sum $\sum_{j=1}^{i-1} \alpha_j$ for each i, we obtain the sequence of heights $(0,1,2,4,6,6,8,11,11)$. The height function h is obtained by the following table.

v	4	9	5	3	7	1	6	8	2
$h(v)$	0	1	2	4	6	6	8	11	11

The corresponding flat is

$$\{x_4 = x_9 + 1 = x_5 + 2 = x_3 + 4 = x_7 + 6 = x_1 + 6 = x_6 + 8 = x_8 + 11 = x_2 + 11\}.$$

4.3 Proof of Theorem 15

Let F be a species with $\mathsf{F}[\varnothing] = \varnothing$. We consider the infinite matrix $\left[\left|(\mathsf{E}_i \circ \mathsf{F})[j]\right|\right]$. Note that almost all entries of each column of the matrix are 0 since

$$\sum_{i=1}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[j]\right| = \left|(\mathsf{E} \circ \mathsf{F})[j]\right| < \infty.$$

We show that substitution of species is compatible with product of the infinite matrices.

Proposition 34. *Let* F *and* G *be species with* $\mathsf{F}[\varnothing] = \mathsf{G}[\varnothing] = \varnothing$*. Then*

$$\left[\left|(\mathsf{E}_i \circ \mathsf{F})[j]\right|\right]\left[\left|(\mathsf{E}_i \circ \mathsf{G})[j]\right|\right] = \left[\left|(\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G})[j]\right|\right].$$

Proof. Fix a positive integer i. By definition,

$$(\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G})(x) = \sum_{j=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G})[j]\right| \frac{x^j}{j!}.$$

We give another calculation of the series as follows.

$$\begin{aligned}
(\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G})(x) &= (\mathsf{E}_i \circ \mathsf{F})(\mathsf{G}(x)) \\
&= \sum_{k=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[k]\right| \frac{\mathsf{G}(x)^k}{k!} \\
&= \sum_{k=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[k]\right| (\mathsf{E}_k \circ \mathsf{G})(x) \\
&= \sum_{k=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[k]\right| \sum_{j=0}^{\infty} \left|(\mathsf{E}_k \circ \mathsf{G})[j]\right| \frac{x^j}{j!} \\
&= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[k]\right| \left|\mathsf{E}_k \circ \mathsf{G}[j]\right| \right) \frac{x^j}{j!}.
\end{aligned}$$

Therefore we have

$$\left|\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G}[j]\right| = \sum_{k=0}^{\infty} \left|\mathsf{E}_i \circ \mathsf{F}[k]\right| \left|\mathsf{E}_k \circ \mathsf{G}[j]\right|$$

for any positive integers i and j. Hence the assertion holds. □

Example 35. Let C denote the *species of cyclic permutations*. Then the substitution $\mathsf{E} \circ \mathsf{C}_+$ coincides with the *species of permutations*. As mentioned in [1, p. 346], we have that $\mathsf{L} = \mathsf{E} \circ \mathsf{C}_+$ as $\mathbb{L}$-species. Indeed let each permutation $\sigma \in (\mathsf{E} \circ \mathsf{C}_+)[n]$ correspond to the list $(\sigma(1), \sigma(2), \cdots, \sigma(n)) \in \mathsf{L}[n]$. It is easy to see that this correspondence is bijective. Note that L and $\mathsf{E} \circ \mathsf{C}_+$ are not isomorphic as $\mathbb{B}$-species. Recall that c, S, and $\left[\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|\right]$ is the infinite upper triangular matrix consisting of Stirling numbers of the first and second kind, and Lah numbers. We can recover the following well-known equality.

$$\left[\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|\right] = \left[\left|(\mathsf{E}_i \circ \mathsf{E}_+ \circ \mathsf{C}_+)[j]\right|\right] = \left[\left|(\mathsf{E}_i \circ \mathsf{E}_+)[j]\right|\right]\left[\left|(\mathsf{E}_i \circ \mathsf{C}_+)[j]\right|\right] = Sc.$$

Proof of Theorem 15. From Theorem 11, Proposition 34, and Example 35

$$\begin{aligned}\left[\,\left|L_i(\mathcal{C}_j^m)\right|\,\right] &= \left[\,\left|L_i\mathcal{C}^m[j]\right|\,\right] = \left[\,\left|(\mathsf{E}_i \circ \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+)[j]\right|\,\right] \\ &= \left[\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|\right]^m \left[\left|(\mathsf{E}_i \circ \mathsf{E}_+)[j]\right|\right] = (Sc)^m S.\end{aligned}$$

Similarly from Theorem 13, Proposition 34, and Example 35

$$\begin{aligned}\left[\,\left|L_i(\mathcal{S}_j^m)\right|\,\right] &= \left[\,\left|L_i\mathcal{S}^m[j]\right|\,\right] = \left[\,\left|(\mathsf{E}_i \circ \mathsf{L}_+^{\circ m})[j]\right|\,\right] \\ &= \left[\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|\right]^m = (Sc)^m.\end{aligned}$$

□

4.4 Proof of Theorem 17

Let a_{ij} denote the Lah number $\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|$. By Proposition 16, $a_{ij} = \frac{j!(j-1)!}{i!(i-1)!(j-i)!}$. Note that if $i > j$, then $a_{ij} = 0$.

Lemma 36. *For every positive integer m, $[a_{ij}]^m = [m^{j-i}a_{ij}]$.*

Proof. We proceed by induction on m. If $m = 1$, then it is trivial. Assume that $m \geq 2$. By the induction hypothesis, the (i,j)-entry of the matrix $[a_{ij}]^m = [a_{ij}][a_{ij}]^{m-1}$ is

$$\begin{aligned}\sum_{k=i}^{j} a_{ik}(m-1)^{j-k}a_{kj} &= \sum_{k=i}^{j} \frac{k!(k-1)!}{i!(i-1)!(k-i)!}(m-1)^{j-k}\frac{j!(j-1)!}{k!(k-1)!(j-k)!} \\ &= \frac{j!(j-1)!}{i!(i-1)!(j-i)!}\sum_{k=i}^{j}\frac{(j-i)!}{(j-k)!(k-i)!}(m-1)^{j-k} \\ &= a_{ij}\sum_{k=i}^{j}\binom{j-i}{k-i}(m-1)^{j-k} = a_{ij}\sum_{k=0}^{j-i}\binom{j-i}{k}(m-1)^{j-i-k} = m^{j-i}a_{ij}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 17. The assertion holds immediately from Proposition 16, Theorem 15 and Lemma 36. □

5 Numerical tables

n	1	2	3	4	5	6	7	OEIS
$L(\mathcal{B}_n)$	1	2	5	15	52	203	877	A000110
$L(\mathcal{C}_n^1)$	1	4	23	173	1602	17575	222497	A075729
$L(\mathcal{C}_n^2)$	1	6	53	619	8972	155067	3109269	A109092
$L(\mathcal{C}_n^3)$	1	8	95	1497	29362	688439	18766393	None
$L(\mathcal{C}_n^4)$	1	10	149	2951	72852	2152651	74031869	None

Table 1: The numbers of flats of the extended Catalan arrangements

n	1	2	3	4	5	6	7	OEIS
$L(\mathcal{S}_n^1)$	1	3	13	73	501	4051	37633	A000262
$L(\mathcal{S}_n^2)$	1	5	37	361	4361	62701	1044205	A025168
$L(\mathcal{S}_n^3)$	1	7	73	1009	17341	355951	8488117	A321837
$L(\mathcal{S}_n^4)$	1	9	121	2161	48081	1279801	39631369	A321847
$L(\mathcal{S}_n^5)$	1	11	181	3961	108101	3532651	134415961	A321848

Table 2: The numbers of flats of the extended Shi arrangements

n	1	2	3	4	5	6	7	OEIS
$L_1(\mathcal{B}_n)$	1	1	1	1	1	1	1	A000012
$L_1(\mathcal{C}_n^1)$	1	3	13	75	541	4683	47293	A000670
$L_1(\mathcal{C}_n^2)$	1	5	37	365	4501	66605	1149877	A050351
$L_1(\mathcal{C}_n^3)$	1	7	73	1015	17641	367927	8952553	A050352
$L_1(\mathcal{C}_n^4)$	1	9	121	2169	48601	1306809	40994521	A050353

Table 3: The numbers of 1-dimensional flats of the extended Catalan arrangements

n	1	2	3	4	5	6	7	OEIS
$L_1(\mathcal{S}_n^1)$	1	2	6	24	120	720	5040	A000142
$L_1(\mathcal{S}_n^2)$	1	4	24	192	1920	23040	322560	A002866
$L_1(\mathcal{S}_n^3)$	1	6	54	648	9720	174960	3674160	A034001
$L_1(\mathcal{S}_n^4)$	1	8	96	1536	30720	737280	20643840	A034177
$L_1(\mathcal{S}_n^5)$	1	10	150	3000	75000	2250000	78750000	A034325

Table 4: The numbers of 1-dimensional flats of the extended Shi arrangements

$L_k(\mathcal{B}_n)$					A008277
$n\backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$L_k(\mathcal{S}_n^1)$					A105278
$n\backslash k$	1	2	3	4	5
1	1				
2	2	1			
3	6	6	1		
4	24	36	12	1	
5	120	240	120	20	1

$L_k(\mathcal{C}_n^1)$					A088729
$n\backslash k$	1	2	3	4	5
1	1				
2	3	1			
3	13	9	1		
4	75	79	18	1	
5	541	765	265	30	1

$L_k(\mathcal{S}_n^2)$					A079621
$n\backslash k$	1	2	3	4	5
1	1				
2	4	1			
3	24	12	1		
4	192	144	24	1	
5	1920	1920	480	40	1

Table 5: Triangles of the number of k-dimensional flats, part 1

6 Acknowledgments

This work was supported by JSPS KAKENHI Grant Number JP20K14282. The authors are very grateful to the anonymous referee for helpful comments.

$L_k(\mathcal{C}_n^2)$					A308440
$n\backslash k$	1	2	3	4	5
1	1				
2	5	1			
3	37	15	1		
4	365	223	30	1	
5	4501	3675	745	50	1

$L_k(\mathcal{S}_n^3)$					A308281
$n\backslash k$	1	2	3	4	5
1	1				
2	6	1			
3	54	18	1		
4	648	324	36	1	
5	9720	6480	1080	60	1

$L_k(\mathcal{C}_n^3)$					NONE
$n\backslash k$	1	2	3	4	5
1	1				
2	7	1			
3	73	21	1		
4	1015	439	42	1	
5	17641	10185	1465	70	1

$L_k(\mathcal{S}_n^4)$					A048786
$n\backslash k$	1	2	3	4	5
1	1				
2	8	1			
3	96	24	1		
4	1536	576	48	1	
5	30720	15360	1920	80	1

$L_k(\mathcal{C}_n^4)$					NONE
$n\backslash k$	1	2	3	4	5
1	1				
2	9	1			
3	121	27	1		
4	2169	727	54	1	
5	48601	21735	2425	90	1

$L_k(\mathcal{S}_n^5)$					A308282
$n\backslash k$	1	2	3	4	5
1	1				
2	10	1			
3	150	30	1		
4	3000	900	60	1	
5	75000	30000	3000	100	1

Table 6: Triangles of the number of k-dimensional flats, part 2

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2020 *Mathematics Subject Classification*: Primary 05A19; Secondary 05A18, 05C22, 52C35.
Keywords: hyperplane arrangement, Shi arrangement, Catalan arrangement, intersection poset, species, gain graph, set partition, Bell number, Stirling number, Lah number.

(Concerned with sequences A000012, A000110, A000142, A000262, A000670, A002866, A008277, A025168, A034001, A034177, A034325, A048786, A050351, A050352, A050353, A075729, A079621, A088729, A105278, A109092, A308281, A308282, A308440, A321837, A321847, and A321848.)

Received June 18 2021; revised version received September 29 2021. Published in *Journal of Integer Sequences*, September 30 2021.

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