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Some Results on Fundamental Gaps in Numerical Semigroups

Edgar Federico Elizeche and Amitabha Tripathi[1]
Department of Mathematics
Indian Institute of Technology
Hauz Khas
New Delhi 110016
India
maz188235@maths.iitd.ac.in
atripath@maths.iitd.ac.in

Abstract

A numerical semigroup is a submonoid of $\mathbb{Z}_{\geq 0}$ whose complement in $\mathbb{Z}_{\geq 0}$ is finite. The gap set $\mathrm{G}(S)$ of a numerical semigroup S is the finite set $\mathbb{Z}_{\geq 0} \setminus S$. A positive integer n is in the set $\mathrm{FG}(S)$ of fundamental gaps of S provided $n \notin S$ but $kn \in S$ for each $k \in \mathbb{Z}$, $k > 1$. We explore the set $\mathrm{FG}(S)$ mostly when S is generated by two or three integers, but also in some other special cases, including when S is generated by arithmetic progressions.

1 Introduction

A *numerical semigroup* S is a submonoid of $\mathbb{Z}_{\geq 0}$ whose complement $\mathbb{Z}_{\geq 0} \setminus S$ is finite. For the complement to be finite, it is necessary and sufficient that $\gcd(S) = 1$. For a given subset A of positive integers, we write

$$\langle A \rangle = \{a_1x_1 + \cdots + a_kx_k : a_i \in A, x_i \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}\}.$$

Note that $\langle A \rangle$ is a submonoid of $\mathbb{Z}_{\geq 0}$, and that $S = \langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

[1]Corresponding author.

We say that A is a set of generators of the semigroup S, or that the semigroup S is generated by the set A, when $S = \langle A \rangle$. Further, A is a minimal set of generators for S if A is a set of generators of S and no proper subset of A generates S. Every semigroup has a unique minimal set of generators. The embedding dimension $e(S)$ of S is the size of the minimal set of generators.

A very useful tool in the study of numerical semigroups is the determination of an Apéry set of the semigroup. Given a numerical semigroup S, and $a \in S$, the Apéry set of S corresponding to a is given by

$$\mathrm{Ap}(S, a) = \{\mathbf{m}_x : x \in \{0, 1, 2, \ldots, a-1\}\},$$

where $\mathbf{m}_x$ denotes the least positive integer in S congruent to x modulo a.

The finite set $\mathbb{Z}_{\geq 0} \setminus S$ is called the *gap set* of S, and is denoted by $\mathrm{G}(S)$. If $n \in \mathrm{G}(S)$ and $d \mid n$, $d \in \mathbb{N}$, then $d \in \mathrm{G}(S)$. This naturally leads to the *set* $\mathrm{FG}(S)$ *of fundamental gaps* of S, defined as

$$\mathrm{FG}(S) = \{n \in \mathrm{G}(S) : kn \in S \ \forall\ k > 1\};$$

see Rosales et al. [9] for more details.

We describe the set $\mathrm{FG}(S)$ in terms of the elements in $\mathrm{Ap}(S, a)$, for any $a \in S$, in Lemma 1.

Lemma 1. *Let A be any set of positive integers with $\gcd(A) = 1$, and let $S = \langle A \rangle$ be the numerical semigroup generated by A. Let $a \in A$, and let $\boldsymbol{m}_x$ denote the least positive integer in S congruent to x modulo a. Then $n \in \mathrm{FG}(S)$ if and only if*

$$n = \boldsymbol{m}_n - \lambda a, \quad 1 \leq \lambda \leq \frac{1}{a} \min\left\{\boldsymbol{m}_n - \tfrac{1}{2}\boldsymbol{m}_{2n}, \boldsymbol{m}_n - \tfrac{1}{3}\boldsymbol{m}_{3n}\right\}.$$

Proof. Note that $n \in \mathrm{FG}(S)$ if and only if $n < \mathbf{m}_n$, $2n \geq \mathbf{m}_{2n}$, and $3n \geq \mathbf{m}_{3n}$. Thus, $n = \mathbf{m}_n - \lambda a$ for some $\lambda \geq 1$. Using this in the other two constraints gives the upper bounds $\lambda \leq \frac{1}{a}\big(\mathbf{m}_n - \frac{1}{2}\mathbf{m}_{2n}\big)$ and $\lambda \leq \frac{1}{a}\big(\mathbf{m}_n - \frac{1}{3}\mathbf{m}_{3n}\big)$. □

We determine the set of fundamental gaps of some numerical semigroups in this article. We consider numerical semigroups with embedding dimension 2 in Section 2. The Apéry set in this case is well known, and we use this to determine the set of fundamental gaps, giving a simpler proof of the same result by Rosales [7]. We consider numerical semigroups with embedding dimension 3 in Section 3. The Apéry set in this case is in general difficult to compute. We consider several special cases where the Apéry set has been determined, and use that to determine the set of fundamental gaps in those cases. We consider numerical semigroups generated by arithmetic progressions in Section 4. Apéry sets for such semigroups have been determined, and we use these to determine the set of fundamental gaps.

2 The case of embedding dimension 2

Numerical semigroups with embedding dimension 2 are the simplest to study. The Apéry set of these semigroups is easy to see and part of basic number theory. Rosales [7] determined the fundamental gap of numerical semigroups $S = \langle a, b\rangle$, $\gcd(a, b) = 1$, by making use of the well known fact that $n \notin S$ if and only if $n = ab - ax - by$ with $x, y \in \mathbb{N}$. We use the Apéry set of S given in Lemma 2 to determine the set of fundamental gaps in S in Theorem 3.

Lemma 2. *Let $A = \{a, b\}$, where $\gcd(a, b) = 1$. Then the Apéry set for the semigroup $S = \langle a, b\rangle$ is given by*

$$\mathrm{Ap}(S, a) = \big\{bx : 0 \le x \le a - 1\big\}.$$

Theorem 3. [**7, Theorem 9**] *Let $S = \langle a, b\rangle$, where $\gcd(a, b) = 1$. The set of fundamental gaps is given by*

$$\mathrm{FG}(S) = \big\{bs - ar : 1 \le r \le \tfrac{b}{3}, \tfrac{a}{2} \le s < \tfrac{2a}{3}\big\} \bigcup \big\{bs - ar : 1 \le r \le \tfrac{b}{2}, \tfrac{2a}{3} \le s \le a - 1\big\}.$$

Proof. Recall that $n = bs - ar \notin \langle a, b\rangle$ if and only if $1 \le s \le a-1$ and $1 \le r < \frac{bs}{a}$, and that n is a fundamental gap if and only if $n \notin \langle a, b\rangle$ and $2n, 3n \in \langle a, b\rangle$.

Since $2n = 2bs - 2ar \notin \langle a, b\rangle$ if $s < \frac{a}{2}$, we may henceforth assume $s \ge \frac{a}{2}$. If a is even and $s = \frac{a}{2}$, then $2n = a(b - 2r) \in \langle a, b\rangle$ and $3n = b \cdot \frac{a}{2} + a(b - 3r) \in \langle a, b\rangle$ if and only if $r \le \frac{b}{3}$. Hence $n = b \cdot \frac{a}{2} - ar$ is a fundamental gap if and only if $1 \le r \le \frac{b}{3}$.

For the rest of this proof, suppose $s > \frac{a}{2}$. Now $2n = b(2s - a) + a(b - 2r) \in \langle a, b\rangle$ if and only if $r \le \frac{b}{2}$, since $2s - a < a$. To decide whether or not $3n \in \langle a, b\rangle$, we consider the two cases (i) $s < \frac{2a}{3}$, and (ii) $\frac{2a}{3} \le s \le a - 1$. In case (i), $3n = b(3s - a) + a(b - 3r) \in \langle a, b\rangle$ if and only if $r \le \frac{b}{3}$, since $0 < 3s - a < a$. In case (ii), $3n = b(3s - 2a) + a(2b - 3r) \in \langle a, b\rangle$ if and only if $r \le \frac{2b}{3}$, since $0 < 3s - 2a < a$. Thus, in case (i), we have $r \le \min\big\{\frac{b}{2}, \frac{b}{3}\big\} = \frac{b}{3}$, while in case (ii), we have $r \le \min\big\{\frac{b}{2}, \frac{2b}{3}\big\} = \frac{b}{2}$. □

3 The case of embedding dimension 3

Numerical semigroups with embedding dimension 3 have received a lot of attention, primarily because these are the first class of numerical semigroups that pose a challenge. By contrast to the case of numerical semigroups with embedding dimension 2, Apéry sets of numerical semigroups with embedding dimension 3 are usually quite difficult to describe. The Frobenius number $\mathrm{F}(S) = \max(\mathbb{N} \setminus S)$ of a numerical semigroup S is easily computed from the Apéry set of S, since $\mathrm{F}(S) = \max(\mathrm{Ap}(S, a)) - a$ for each $a \in S$. Johnson [4] described an algorithm to compute $\mathrm{F}(S)$ in terms of minimum multiples of generators that are in the numerical semigroup generated by the other two elements, but without determining the Apéry set. Rosales and García-Sánchez [8] and independently, Tripathi and Vijay [16], were able to describe the Apéry set in terms of the constants first determined by Johnson [4]; see Proposition 4. However, the constants determined by Johnson [4] do not lead to an algebraic expression in terms of the generators of S and, in particular, to the determination of the Apéry set of S in the desired manner.

In this section, we consider three cases of numerical semigroups with embedding dimension 3 in which the Apéry sets can be computed explicitly in terms of their generators.

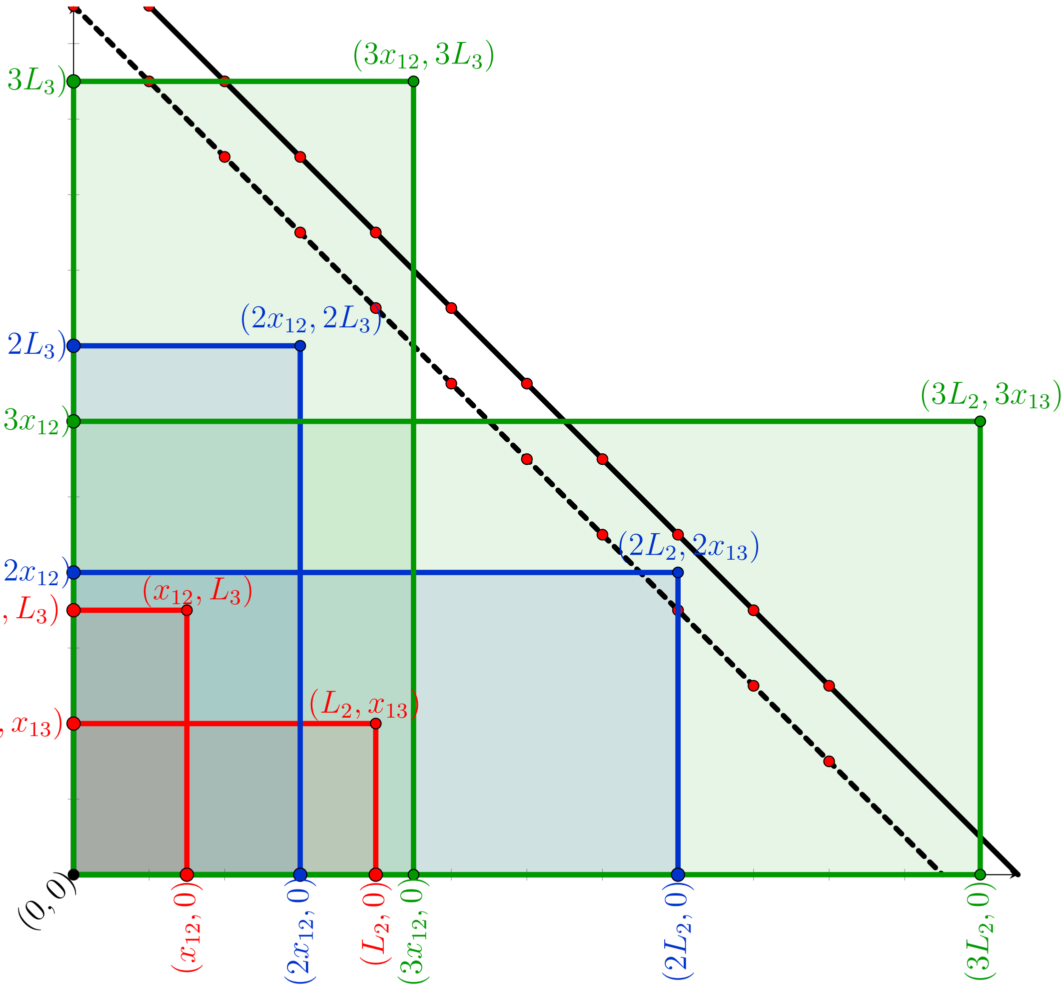

Figure 1: A geometric depiction of the set of fundamental gaps in numerical semigroups with embedding dimension 3. Each lattice point (x, y) represents the integer $n = a_2x + a_3y$. The integers in the Apéry set $\mathcal{A}$ (respectively, in $2\mathcal{A}$ and $3\mathcal{A}$) are those represented by lattice points inside the region given by the union of two rectangles enclosed by **red** lines (respectively, by **blue** lines and by **green** lines). The solid **black** line represents the equation $a_2x + a_3y = n$, whereas the dashed **black** line represents the equation $a_2x + a_3y = n - a_1$.

We then use this to determine the set of fundamental gaps in each case. We deal with 3-term compound sequences in Subsection 3.1. These are given by $c_0 = a_1a_2$, $c_1 = b_1a_2$, $c_2 = b_1b_2$, where a_1, a_2 and b_1, b_2 are pairs of positive integers such that $a_i < b_i$ for each i and $\gcd(a_i, b_j) = 1$ for each pair i, j, $i \geq j$. We consider numerical semigroups generated by a, b, c, where $a \mid (b + c)$, in Subsection 3.2. We consider numerical semigroups generated by a, b, c, where $a \mid \operatorname{lcm}(b, c)$, in Subsection 3.3.

Proposition 4. [4, 8, 16] *Let* $A = \{a_1, a_2, a_3\}$ *be a set of positive integers, with* $\gcd(a_1, a_2, a_3) = 1$*. Define* L_1, L_2, L_3 *by*

$$L_1 = \min\left\{k_1 \in \mathbb{N} : k_1 a_1 = v_{12}a_2 + v_{13}a_3,\ v_{12} \geq 0, v_{13} \geq 0\right\},$$
$$L_2 = \min\left\{k_2 \in \mathbb{N} : k_2 a_2 = v_{21}a_1 + v_{23}a_3,\ v_{21} \geq 0, v_{23} \geq 0\right\},$$
$$L_3 = \min\left\{k_3 \in \mathbb{N} : k_3 a_3 = v_{31}a_1 + v_{32}a_2,\ v_{31} \geq 0, v_{32} \geq 0\right\}.$$

Then there exist nonnegative integers $x_{12}, x_{13}, x_{21}, x_{23}, x_{31}, x_{32}$ *such that*

$$L_1 a_1 = x_{12}a_2 + x_{13}a_3, \quad L_2 a_2 = x_{21}a_1 + x_{23}a_3, \quad L_3 a_3 = x_{31}a_1 + x_{32}a_2.$$

Moreover, if no element in A *is dependent on the other two, then each* $x_{ij} \geq 1$ *and each* $L_i = x_{ji} + x_{ki}$*. We have*

$$\mathrm{Ap}(S, a_1) = \left\{a_2 x + a_3 y : 0 \leq x \leq x_{12} - 1, 0 \leq y \leq L_3 - 1 \ \textit{ or } \ 0 \leq x \leq L_2 - 1, 0 \leq y \leq x_{13} - 1\right\}.$$

3.1 The case of compound sequences

Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be two sequences of positive integers such that $a_i < b_i$ for each i and $\gcd(a_i, b_j) = 1$ for each pair i, j, $i \geq j$. The *compound sequence* formed from these two sequences is

$$c_0 = a_1 a_2 a_3 \cdots a_k, c_1 = b_1 a_2 a_3 \cdots a_k, c_2 = b_1 b_2 a_3 \cdots a_k, \ldots, c_k = b_1 b_2 b_3 \cdots b_k.$$

Note that $\gcd(c_0, c_1, c_2, \ldots, c_k) = 1$. Two important special cases are

- The compound sequence for $a_1 = \cdots = a_k = a$ and $b_1 = \cdots = b_k = b$, $\gcd(a, b) = 1$ is the geometric sequence
$$a^k, a^{k-1}b, a^{k-2}b^2, \ldots, b^k.$$
- For pairwise coprime positive integers $a_1, \ldots, a_k$, the compound sequence for $a_2, a_3, \ldots, a_k$ and $a_1, a_2, \ldots, a_{k-1}$ is the supersymmetric sequence
$$\frac{P}{a_1}, \frac{P}{a_2}, \ldots, \frac{P}{a_k},$$
where $P = a_1 a_2 \cdots a_k$.

Numerical semigroups generated by compound sequences were studied by Kiers et al. [5]. In their study, Kiers et al. determined an Apéry set, the Frobenius number, Betti elements, and catenary degree, and also computed bounds on the delta set. We use their result on the Apéry set of numerical semigroups to determine the set of fundamental gaps of these semigroups for the case where the embedding dimension equals 3.

Lemma 5. [5, **Theorem 15**] *Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be two sequences of positive integers such that $a_i < b_i$ for each i and $\gcd(a_i, b_j) = 1$ for each pair i, j, $i \geq j$. Then an Apéry set for the numerical semigroup S generated by the compound sequence $c_0, c_1, \ldots, c_k$ of these two sequences is given by*

$$\mathrm{Ap}(S, c_0) = \left\{ \sum_{i=1}^{k} c_i x_i : 0 \leq x_i \leq a_i - 1, i = 1, \ldots, k \right\}.$$

The compound sequence of a_1, a_2 and b_1, b_2 is a 3-term sequence c_0, c_1, c_2. We study the set of fundamental gaps of numerical semigroups generated by c_0, c_1, c_2.

Theorem 6. *Let a_1, a_2 and b_1, b_2 be two sequences of positive integers such that $a_i < b_i$ for each i and $\gcd(a_i, b_j) = 1$ for each pair i, j, $i \geq j$. Let $c_0 = a_1 a_2$, $c_1 = b_1 a_2$, and $c_2 = b_1 b_2$, and let $S = \langle c_0, c_1, c_2 \rangle$. Let $b_2 = q a_1 + r$, $0 \leq r < a_1$, and $\delta = \left\lfloor \frac{2r}{a_1} \right\rfloor$. Let $n = c_1 x + c_2 y - c_0 z \in G(S)$.*

(I)

$$2n \in S \iff \begin{cases} \frac{a_1}{2} \leq x < a_1, & 0 \leq y < \frac{a_2}{2}, \quad 1 \leq z \leq \frac{b_1}{2}, \quad \text{or} \\ 0 \leq x < \frac{a_1 - r}{2}, & \frac{a_2}{2} \leq y < a_2, \quad 1 \leq z \leq \frac{q b_1}{2}, \quad \text{or} \\ \frac{a_1 - r}{2} \leq x < \frac{2a_1 - r}{2}, & \frac{a_2}{2} \leq y < a_2, \quad 1 \leq z \leq \frac{(q+1) b_1}{2}, \quad \text{or} \\ \frac{2a_1 - r}{2} \leq x < a_1, & \frac{a_2}{2} \leq y < a_2, \quad 1 \leq z \leq \frac{(q+2) b_1}{2}. \end{cases}$$

(II)

$$3n \in S \iff \begin{cases} \frac{a_1}{3} \leq x < \frac{2a_1}{3}, & 0 \leq y < \frac{a_2}{3}, \quad 1 \leq z \leq \frac{b_1}{3}, \quad \text{or} \\ \frac{2a_1}{3} \leq x < a_1, & 0 \leq y < \frac{a_2}{3}, \quad 1 \leq z \leq \frac{2b_1}{3}, \quad \text{or} \\ 0 \leq x < \frac{a_1 - r}{3}, & \frac{a_2}{3} \leq y < \frac{2a_2}{3}, \quad 1 \leq z \leq \frac{q b_1}{3}, \quad \text{or} \\ \frac{a_1 - r}{3} \leq x < \frac{2a_1 - r}{3}, & \frac{a_2}{3} \leq y < \frac{2a_2}{3}, \quad 1 \leq z \leq \frac{(q+1) b_1}{3}, \quad \text{or} \\ \frac{2a_1 - r}{3} \leq x < \frac{3a_1 - r}{3}, & \frac{a_2}{3} \leq y < \frac{2a_2}{3}, \quad 1 \leq z \leq \frac{(q+2) b_1}{3}, \quad \text{or} \\ \frac{3a_1 - r}{3} \leq x < a_1, & \frac{a_2}{3} \leq y < \frac{2a_2}{3}, \quad 1 \leq z \leq \frac{(q+3) b_1}{3}, \quad \text{or} \\ 0 \leq x < \frac{(1+\delta)a_1 - 2r}{3}, & \frac{2a_2}{3} \leq y < a_2, \quad 1 \leq z \leq \frac{(2q+\delta) b_1}{3}, \quad \text{or} \\ \frac{(1+\delta)a_1 - 2r}{3} \leq x < \frac{(2+\delta)a_1 - 2r}{3}, & \frac{2a_2}{3} \leq y < a_2, \quad 1 \leq z \leq \frac{(2q+\delta+1) b_1}{3}, \quad \text{or} \\ \frac{(2+\delta)a_1 - 2r}{3} \leq x < \frac{(3+\delta)a_1 - 2r}{3}, & \frac{2a_2}{3} \leq y < a_2, \quad 1 \leq z \leq \frac{(2q+\delta+2) b_1}{3}, \quad \text{or} \\ \frac{(3+\delta)a_1 - 2r}{3} \leq x < a_1, & \frac{2a_2}{3} \leq y < a_2, \quad 1 \leq z \leq \frac{(2q+\delta+3) b_1}{3}. \end{cases}$$

Proof. By Lemma 5, $n \in \mathrm{G}(S)$ if and only if $n = c_1 x + c_2 y - c_0 z$, where $0 \leq x < a_1$, $0 \leq y < a_2$, and $z \in \mathbb{N}$.

(I) We derive conditions on n for which $2n \in S$ by considering five cases. The argument in each case is similar, so we present the first two cases in detail and give only the result in the remaining three cases.

(i) For $0 \le x < \frac{a_1}{2}$ and $0 \le y < \frac{a_2}{2}$, $2n = 2c_1x + 2c_2y - 2c_0z \notin S$, since $2c_1x + 2c_2y \in \mathrm{Ap}(S, c_0)$.

(ii) For $\frac{a_1}{2} \le x < a_1$ and $0 \le y < \frac{a_2}{2}$, $2n = c_1(2x - a_1) + 2c_2y - c_0(2z - b_1)$ and $c_1(2x - a_1) + 2c_2y \in \mathrm{Ap}(S, c_0)$. Therefore $2n \in S \Leftrightarrow z \le \frac{b_1}{2}$.

(iii) For $0 \le x < \frac{a_1-r}{2}$ and $\frac{a_2}{2} \le y < a_2$, we have $2n \in S \Leftrightarrow z \le \frac{qb_1}{2}$.

(iv) For $\frac{a_1-r}{2} \le x < \frac{2a_1-r}{2}$ and $\frac{a_2}{2} \le y < a_2$, we have $2n \in S \Leftrightarrow z \le \frac{(q+1)b_1}{2}$.

(v) For $\frac{2a_1-r}{2} \le x < a_1$ and $\frac{a_2}{2} \le y < a_2$, we have $2n \in S \Leftrightarrow z \le \frac{(q+2)b_1}{2}$.

(II) We derive conditions on n for which $3n \in S$ by considering eleven cases. Again, the argument in each case is similar, so we present the first two cases in detail and give only the result in the remaining nine cases.

(i) For $0 \le x < \frac{a_1}{3}$ and $0 \le y < \frac{a_2}{3}$, $3n = 3xc_1 + 3yc_2 - 3zc_0 \notin S$, because $3c_1x + 3c_2y \in \mathrm{Ap}(S, c_0)$.

(ii) For $\frac{a_1}{3} \le x < \frac{2a_1}{3}$ and $0 \le y < \frac{a_2}{3}$, $3n = c_1(3x - a_1) + 3c_2y - c_0(3z - b_1)$ and $c_1(3x - a_1) + 3c_2y \in \mathrm{Ap}(S, c_0)$. Therefore $3n \in S \Leftrightarrow z \le \frac{b_1}{3}$.

(iii) For $\frac{2a_1}{3} \le x < a_1$ and $0 \le y < \frac{a_2}{3}$, we have $3n \in S \Leftrightarrow z \le \frac{2b_1}{3}$.

(iv) For $0 \le x < \frac{a_1-r}{3}$ and $\frac{a_2}{3} \le y < \frac{2a_2}{3}$, we have $3n \in S \Leftrightarrow z \le \frac{qb_1}{3}$.

(v) For $\frac{a_1-r}{3} \le x < \frac{2a_1-r}{3}$ and $\frac{a_2}{3} \le y < \frac{2a_2}{3}$, we have $3n \in S \Leftrightarrow z \le \frac{(q+1)b_1}{3}$.

(vi) For $\frac{2a_1-r}{3} \le x < \frac{3a_1-r}{3}$ and $\frac{a_2}{3} \le y < \frac{2a_2}{3}$, we have $3n \in S \Leftrightarrow z \le \frac{(q+2)b_1}{3}$.

(vii) For $\frac{3a_1-r}{3} \le x < a_1$ and $\frac{a_2}{3} \le y < \frac{2a_2}{3}$, we have $3n \in S \Leftrightarrow z \le \frac{(q+3)b_1}{3}$.

(viii) For $0 \le x < \frac{(1+\delta)a_1-2r}{3}$ and $\frac{2a_2}{3} \le y < a_2$, we have $3n \in S \Leftrightarrow z \le \frac{(2q+\delta)b_1}{3}$.

(ix) For $\frac{(1+\delta)a_1-2r}{3} \le x < \frac{(2+\delta)a_1-2r}{3}$ and $\frac{2a_2}{3} \le y < a_2$, we have $3n \in S \Leftrightarrow z \le \frac{(2q+\delta+1)b_1}{3}$.

(x) For $\frac{(2+\delta)a_1-2r}{3} \le x < \frac{(3+\delta)a_1-2r}{3}$ and $\frac{2a_2}{3} \le y < a_2$, we have $3n \in S \Leftrightarrow z \le \frac{(2q+\delta+2)b_1}{3}$.

(xi) For $\frac{(3+\delta)a_1-2r}{3} \le x < a_1$ and $\frac{2a_2}{3} \le y < a_2$, we have $3n \in S \Leftrightarrow z \le \frac{(2q+\delta+3)b_1}{3}$.

□

3.2 The case $A = \{a, b, c\}$, where $a \mid (b + c)$

Lemma 7. [**14, Theorem 1**] *Let $A = \{a, b, c\}$, where $\gcd(a, b, c) = 1$ and $a \mid (b + c)$. Then* *the Apéry set for the semigroup $S = \langle a, b, c\rangle$ is given by*

$$\begin{aligned}\mathrm{Ap}(S, a) &= \left\{ \min\{bx, c(a-x)\} : 0 \le x \le a-1 \right\} \\ &= \begin{cases} bx, & \text{if } 0 \le x \le \frac{ac}{b+c}; \\ c(a-x), & \text{if } \frac{ac}{b+c} < x \le a-1. \end{cases}\end{aligned}$$

Theorem 8. *Let $S = \langle a, b, c\rangle$, where $\gcd(a, b, c) = 1$. If $a \mid (b+c)$ and $q = \left\lfloor \frac{ac}{b+c} \right\rfloor$, then $n \in \mathrm{FG}(S)$ if and only if*

(I) $q < \frac{2a}{3}$ *and*

$$\begin{aligned} & n = bx - az, && \tfrac{q}{2} < x < \tfrac{a}{3}, && 1 \le z \le \tfrac{b+c}{a}x - \tfrac{c}{2}, \\ \text{or } & n = bx - az, && \tfrac{a}{3} \le x < \tfrac{a}{2}, && 1 \le z \le \min\left\{\tfrac{b+c}{a}x - \tfrac{c}{2}, \tfrac{b}{3}\right\}, \\ \text{or } & n = bx - az, && \tfrac{a}{2} \le x < \tfrac{a+q}{3}, && 1 \le z \le \tfrac{b}{3}, \\ \text{or } & n = bx - az, && \tfrac{a+q}{3} \le x \le q, && 1 \le z \le \min\left\{\tfrac{b+c}{a}x - \tfrac{2c}{3}, \tfrac{b}{2}\right\}, \\ \text{or } & n = c(a-x) - az, && q < x < \tfrac{2a}{3}, && 1 \le z \le \min\left\{\tfrac{b+2c}{2} - \tfrac{b+c}{a}x, \tfrac{c}{3}\right\}, \\ \text{or } & n = c(a-x) - az, && \tfrac{2a}{3} \le x \le \tfrac{a+q}{2}, && 1 \le z \le \tfrac{b+2c}{2} - \tfrac{b+c}{a}x. \end{aligned}$$

(II) $q \ge \frac{2a}{3}$ *and*

$$\begin{aligned} & n = bx - az, && \tfrac{q}{2} < x < \tfrac{a}{2}, && 1 \le z \le \min\left\{\tfrac{b+c}{a}x - \tfrac{c}{2}, \tfrac{b}{3}\right\}, \\ \text{or } & n = bx - az, && \tfrac{a}{2} \le x < \tfrac{a+q}{3}, && 1 \le z \le \tfrac{b}{3}, \\ \text{or } & n = bx - az, && \tfrac{a+q}{3} < x < \tfrac{2a}{3}, && 1 \le z \le \min\left\{\tfrac{b+c}{a}x - \tfrac{2c}{3}, \tfrac{b}{2}\right\}, \\ \text{or } & n = bx - az, && \tfrac{2a}{3} \le x \le q, && 1 \le z \le \tfrac{b}{2}, \\ \text{or } & n = c(a-x) - az, && q < x \le \tfrac{a+q}{2}, && 1 \le z \le \tfrac{b+2c}{2} - \tfrac{b+c}{a}x. \end{aligned}$$

Proof. We assume, without loss of generality, that $b < c$. Let $q = \left\lfloor \frac{ac}{b+c} \right\rfloor$. Since $\left\lfloor \frac{ab}{b+c} \right\rfloor + \left\lfloor \frac{ac}{b+c} \right\rfloor = a-1$ or a according as $(b+c) \nmid ac$ or $(b+c) \mid ac$, we have $q \ge \frac{a-1}{2}$ if $(b+c) \nmid ac$ and $q > \frac{a}{2}$ if $(b+c) \mid ac$. By Lemma 7, $n \notin S$ if and only if $n = bx - az > 0$, $1 \le x \le q$, $z \ge 1$ or $n = c(a-x) - az > 0$, $q+1 \le x \le a-1$, $z \ge 1$.

Case I. $\left(q < \frac{2a}{3}\right)$

(i) For $1 \le x \le \frac{q}{2}$, $\mathbf{m}_{bx} = bx$ and $\mathbf{m}_{2bx} = 2bx$. Hence $n = bx - az$, so that $2n < \mathbf{m}_{2bx}$. Therefore $n \notin \mathrm{FG}(S)$ in this case.

(ii) For $\frac{q}{2} < x < \frac{a}{3}$, $\mathbf{m}_{bx} = bx$, $\mathbf{m}_{2bx} = c(a-2x)$, and $\mathbf{m}_{3bx} = c(a-3x)$. Hence $n = bx - az$, and $2n \ge \mathbf{m}_{2bx}$ and $3n \ge \mathbf{m}_{3bx}$ translate to $z \le \frac{b+c}{a}x - \frac{c}{2}$ and $z \le \frac{b+c}{a}x - \frac{c}{3}$, respectively. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \frac{b+c}{a}x - \frac{c}{2}$.

(iii) For $\frac{a}{3} \le x < \frac{a}{2}$, $\mathbf{m}_{bx} = bx$, $\mathbf{m}_{2bx} = c(a-2x)$, and $\mathbf{m}_{3bx} = b(3x-a)$. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \min\left\{\frac{b+c}{a}x - \frac{c}{2}, \frac{b}{3}\right\}$.

(iv) For $\frac{a}{2} \le x < \frac{a+q}{3}$, $\mathbf{m}_{bx} = bx$, $\mathbf{m}_{2bx} = b(2x-a)$, and $\mathbf{m}_{3bx} = b(3x-a)$. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \frac{b}{3}$.

(v) For $\frac{a+q}{3} \le x \le q$, $\mathbf{m}_{bx} = bx$, $\mathbf{m}_{2bx} = b(2x-a)$, and $\mathbf{m}_{3bx} = c\big(a-(3x-a)\big)$. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \min\left\{\frac{b+c}{a}x - \frac{2c}{3}, \frac{b}{2}\right\}$.

(vi) For $q < x < \frac{2a}{3}$, $\mathbf{m}_{bx} = c(a-x)$, $\mathbf{m}_{2bx} = b(2x-a)$, and $\mathbf{m}_{3bx} = c\big(a-(3x-a)\big)$. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \min\left\{\frac{b+2c}{2} - \frac{b+c}{a}x, \frac{c}{3}\right\}$.

(vii) For $\frac{2a}{3} \le x \le \frac{a+q}{2}$, $\mathbf{m}_{bx} = c(a-x)$, $\mathbf{m}_{2bx} = b(2x-a)$, and $\mathbf{m}_{3bx} = b(3x-2a)$. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \frac{b+2c}{2} - \frac{b+c}{a}x$.

(viii) For $\frac{a+q}{2} < x \le a-1$, $\mathbf{m}_{bx} = c(a-x)$ and $\mathbf{m}_{2bx} = c\big(a-(2x-a)\big)$. Hence $n = c(a-x)-az$, so that $2n < \mathbf{m}_{2bx}$. Therefore $n \notin \mathrm{FG}(S)$ in this case.

Case II. $\left(q \ge \frac{2a}{3}\right)$ There are seven subcases to consider, of which subcases (i), (iii), and (vii) are the subcases (i), (iv), and (viii) from Case I. We only consider the four remaining subcases, listed as (ii), (iv), (v), and (vi).

(ii) For $\frac{q}{2} < x < \frac{a}{2}$, $\mathbf{m}_{bx} = bx$, $\mathbf{m}_{2bx} = c(a-2x)$ (since $q < 2x < a$), and $\mathbf{m}_{3bx} = b(3x-a)$ (since $a < 3x < a+q$). Hence $n = bx - az$, and $2n \ge \mathbf{m}_{2bx}$ and $3n \ge \mathbf{m}_{3bx}$ translate to $z \le \frac{b+c}{a}x - \frac{c}{2}$ and $z \le \frac{b}{3}$, respectively. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \min\left\{\frac{b+c}{a}x - \frac{c}{2}, \frac{b}{3}\right\}$.

(iv) For $\frac{a+q}{3} < x < \frac{2a}{3}$, $\mathbf{m}_{bx} = bx$ (since $\frac{2a}{3} \le q$), $\mathbf{m}_{2bx} = b(2x-a)$ (since $a < 2x < a+q$), and $\mathbf{m}_{3bx} = c\big(a-(3x-a)\big)$ (since $a+q < 3x < 2a$). Hence $n = bx - az$, and $2n \ge \mathbf{m}_{2bx}$ and $3n \ge \mathbf{m}_{3bx}$ translate to $z \le \frac{b}{2}$ and $z \le \frac{b+c}{a}x - \frac{2c}{3}$, respectively. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \min\left\{\frac{b+c}{a}x - \frac{2c}{3}, \frac{b}{2}\right\}$.

(v) For $\frac{2a}{3} \le x \le q$, $\mathbf{m}_{bx} = bx$, $\mathbf{m}_{2bx} = b(2x-a)$, and $\mathbf{m}_{3bx} = b(3x-2a)$. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \frac{b}{2}$.

(vi) For $q < x \le \frac{a+q}{2}$, $\mathbf{m}_{bx} = c(a-x)$, $\mathbf{m}_{2bx} = b(2x-a)$, and $\mathbf{m}_{3bx} = b(3x-2a)$. Thus, $n \in \mathrm{FG}(S) \Leftrightarrow z \le \frac{b+2c}{2} - \frac{b+c}{a}x$.

□

3.3 The case $A = \{a, b, c\}$, where $a \mid \mathrm{lcm}(b, c)$

Lemma 9. [**15**, **Theorem 8**] *Let $A = \{a, b, c\}$, where $\gcd(a, b, c) = 1$ and $a \mid \mathrm{lcm}(b, c)$. Then the Apéry set for the semigroup $S = \langle a, b, c\rangle$ is given by*

$$\mathrm{Ap}(S, a) = \big\{bx + cy : 0 \le x \le s-1, 0 \le y \le r-1\big\},$$

where $r = \gcd(a, b)$ and $s = \gcd(a, c)$.

Theorem 10. *Let $S = \langle a, b, c\rangle$, where $\gcd(a, b, c) = 1$ and $a \mid \operatorname{lcm}(b, c)$. Let $r = \gcd(a, b)$ and $s = \gcd(a, c)$. Then $bx + cy - az \in \mathrm{FG}(S)$ if and only if $z \geq 1$ and*

$$\begin{array}{lll}
0 \leq x < \frac{s}{3}, & \frac{r}{2} \leq y < \frac{2r}{3}, & z \leq \frac{c}{3s}, \\
\textit{or } \frac{s}{3} \leq x < \frac{s}{2}, & \frac{r}{2} \leq y < \frac{2r}{3}, & z \leq \min\{\frac{c}{2s}, \frac{b}{3r} + \frac{c}{3s}\}, \\
\textit{or } 0 \leq x < \frac{s}{2}, & \frac{2r}{3} \leq y < r, & z \leq \frac{c}{2s}, \\
\textit{or } \frac{s}{2} \leq x < \frac{2s}{3}, & 0 \leq y < \frac{r}{3}, & z \leq \frac{b}{3r}, \\
\textit{or } \frac{s}{2} \leq x < \frac{2s}{3}, & \frac{r}{3} \leq y < \frac{r}{2}, & z \leq \min\{\frac{b}{2r}, \frac{b}{3r} + \frac{c}{3s}\} \\
\textit{or } \frac{2s}{3} \leq x < s, & 0 \leq y < \frac{r}{2}, & z \leq \frac{b}{2r}, \\
\textit{or } \frac{s}{2} \leq x < \frac{2s}{3}, & \frac{r}{2} \leq y < \frac{2r}{3}, & z \leq \frac{b}{3r} + \frac{c}{3s}, \\
\textit{or } \frac{s}{2} \leq x < \frac{2s}{3}, & \frac{2r}{3} \leq y < r, & z \leq \min\{\frac{b}{2r} + \frac{c}{2s}, \frac{b}{3r} + \frac{2c}{3s}\}, \\
\textit{or } \frac{2s}{3} \leq x < s, & \frac{r}{2} \leq y < \frac{2r}{3}, & z \leq \min\{\frac{b}{2r} + \frac{c}{2s}, \frac{2b}{3r} + \frac{c}{3s}\}, \\
\textit{or } \frac{2s}{3} \leq x < s, & \frac{2r}{3} \leq y < r, & z \leq \frac{b}{2r} + \frac{c}{2s}.
\end{array}$$

Proof. We make repeated use of Lemma 9. Note that $a = rs$ and that $n \notin S$ if and only if $n = bx + cy - az > 0$, where $0 \leq x \leq s - 1$, $0 \leq y \leq r - 1$ and $z \geq 1$.

To determine conditions on x, y, z under which $2n \in S$, we consider nine cases with $(x, y) \in \left[\frac{\lambda s}{2}, \frac{(\lambda+1)s}{2}\right) \times \left[\frac{\mu s}{2}, \frac{(\mu+1)s}{2}\right)$, with $\lambda, \mu \in \{0, 1\}$. Fix the ordered pair (λ, μ). Then

$$2n = b(2x - \lambda s) + c(2y - \mu r) + a\left(\frac{\lambda b}{r} + \frac{\mu c}{s} - 2z\right) \in S \text{ if and only if } 2z \leq \frac{\lambda b}{r} + \frac{\mu c}{s}. \quad (1)$$

To determine conditions on x, y, z under which $3n \in S$, we consider nine cases with $(x, y) \in \left[\frac{\lambda s}{3}, \frac{(\lambda+1)s}{3}\right) \times \left[\frac{\mu s}{3}, \frac{(\mu+1)s}{3}\right)$, with $\lambda, \mu \in \{0, 1, 2\}$. Fix the ordered pair (λ, μ). Then

$$3n = b(3x - \lambda s) + c(3y - \mu r) + a\left(\frac{\lambda b}{r} + \frac{\mu c}{s} - 3z\right) \in S \text{ if and only if } 3z \leq \frac{\lambda b}{r} + \frac{\mu c}{s}. \quad (2)$$

We must consider only those pairs $(\lambda, \mu) \in \{0, 1\} \times \{0, 1\}$ in eqn. (1) for which $\frac{\lambda b}{r} + \frac{\mu c}{s} \geq 2$ and simultaneously only those pairs $(\lambda, \mu) \in \{0, 1, 2\} \times \{0, 1, 2\}$ in eqn. (2) for which $\frac{\lambda b}{r} + \frac{\mu c}{s} \geq 3$. Therefore $n = bx + cy - az \in \mathrm{FG}(S)$ according to Table 1 below.

$\frac{2r}{3} \leq y < r$	$\min\left\{\frac{c}{2s}, \frac{2c}{3s}\right\} = \frac{c}{2s}$	$\min\left\{\frac{c}{2s}, \frac{b}{3r} + \frac{2c}{3s}\right\} = \frac{c}{2s}$	$\min\left\{\frac{b}{2r} + \frac{c}{2s}, \frac{b}{3r} + \frac{2c}{3s}\right\}$	$\min\left\{\frac{b}{2r} + \frac{c}{2s}, \frac{2b}{3r} + \frac{2c}{3s}\right\} = \frac{b}{2r} + \frac{c}{2s}$
$\frac{r}{2} \leq y < \frac{2r}{3}$	$\min\left\{\frac{c}{2s}, \frac{c}{3s}\right\} = \frac{c}{3s}$	$\min\left\{\frac{c}{2s}, \frac{b}{3r} + \frac{c}{3s}\right\}$	$\min\left\{\frac{b}{2r} + \frac{c}{2s}, \frac{b}{3r} + \frac{c}{3s}\right\} = \frac{b}{3r} + \frac{c}{3s}$	$\min\left\{\frac{b}{2r} + \frac{c}{2s}, \frac{2b}{3r} + \frac{c}{3s}\right\}$
$\frac{r}{3} \leq y < \frac{r}{2}$	**X**	**X**	$\min\left\{\frac{b}{2r}, \frac{b}{3r} + \frac{c}{3s}\right\}$	$\min\left\{\frac{b}{2r}, \frac{2b}{3r} + \frac{c}{3s}\right\} = \frac{b}{2r}$
$0 \leq y < \frac{r}{3}$	**X**	**X**	$\min\left\{\frac{b}{2r}, \frac{b}{3r}\right\} = \frac{b}{3r}$	$\min\left\{\frac{b}{2r}, \frac{2b}{3r}\right\} = \frac{b}{2r}$
	$0 \leq x < \frac{s}{3}$	$\frac{s}{3} \leq x < \frac{s}{2}$	$\frac{s}{2} \leq x < \frac{2s}{3}$	$\frac{2s}{3} \leq x < s$

Table 1: Upper bounds for z

□

4 The case of arithmetic progressions

Numerical semigroups generated by arithmetic progressions have been extensively studied; see [1, 3, 6, 10, 11, 12], for instance. By $\mathrm{AP}(a,d;k)$, we mean the k-term arithmetic progression $\{a, a+d, \ldots, a+(k-1)d\}$, with $\gcd(a,d)=1$ and $k \geq 2$. The Apéry set for the semigroup $S = \langle \mathrm{AP}(a,d;k) \rangle$ is given as Lemma 11. We use this to determine the set of fundamental gaps in S in Theorem 12.

Lemma 11. [**11**, **Lemma 2**] *Let a,d,k be positive integers, with $\gcd(a,d)=1$. Let $\mathrm{AP}(a,d;k) = \{a+id : 0 \leq i \leq k-1\}$. Then the Apéry set for the semigroup $S = \langle \mathrm{AP}(a,d;k) \rangle$ is given by*

$$\mathrm{Ap}(S,a) = \left\{ a\left(1 + \left\lfloor \tfrac{x-1}{k-1} \right\rfloor\right) + dx : 0 \leq x \leq a-1 \right\}.$$

Theorem 12. *Let a,d,k be positive integers, with $\gcd(a,d)=1$. Let $\mathrm{AP}(a,d;k) = \{a+id : 0 \leq i \leq k-1\}$. Then $ax+dy \in \mathrm{FG}\left(\mathrm{AP}(a,d;k)\right)$ if and only if*

$$\tfrac{1}{3}\left(\left\lfloor \tfrac{3y-a}{k-1} \right\rfloor - (d-1)\right) \leq x \leq \left\lfloor \tfrac{y-1}{k-1} \right\rfloor, \qquad \tfrac{a}{2} \leq y < \tfrac{2a}{3},$$

or

$$\tfrac{1}{2}\left(\left\lfloor \tfrac{2y-a}{k-1} \right\rfloor - (d-1)\right) \leq x \leq \left\lfloor \tfrac{y-1}{k-1} \right\rfloor, \qquad \tfrac{2a}{3} \leq y \leq a-1.$$

Proof. Recall that n is a fundamental gap of the numerical semigroup S if and only if $n \notin S$ and $2n, 3n \in S$. Let $S = \langle \mathrm{AP}(a,d;k) \rangle$. We make repeated use of Lemma 11.

Fix $y \in \{1, \ldots, a-1\}$, and suppose $n = ax+dy > 0$. Then $n \notin S$ if and only if $x \leq \left\lfloor \frac{y-1}{k-1} \right\rfloor$.

To determine conditions on x under which $2n \in S$, we consider two cases: (i) $y \leq \frac{a-1}{2}$, and (ii) $y > \frac{a-1}{2}$. In case (i), $2n = 2ax + 2dy \in S$ if and only if $2x \geq 1 + \left\lfloor \frac{2y-1}{k-1} \right\rfloor$. Since $1 + \left\lfloor \frac{2y-1}{k-1} \right\rfloor > 2\left\lfloor \frac{y-1}{k-1} \right\rfloor$, the necessary and sufficient condition requires $x > \left\lfloor \frac{y-1}{k-1} \right\rfloor$, in contradiction to the requirement for $n \notin S$. Hence there is no fundamental gap in case (i). In case (ii), $y \geq \frac{a}{2}$, so $2n = a(2x+d) + d(2y-a) \in S$ if and only if $2x + d \geq 1 + \left\lfloor \frac{2y-a}{k-1} \right\rfloor$.

To determine conditions on x under which $3n \in S$, we consider three cases: (i) $y \leq \frac{a-1}{3}$, (ii) $\frac{a-1}{3} < y < \frac{2a}{3}$, and (iii) $\frac{2a}{3} \leq y \leq a-1$. In order that n be a fundamental gap, we must have $y \geq \frac{a}{2}$. Hence there is no fundamental gap in case (i). In case (ii), $3n = a(3x+d) + d(3y-a) \in S$ if and only if $3x+d \geq 1 + \left\lfloor \frac{3y-a}{k-1} \right\rfloor$. In case (iii), $3n = a(3x+2d) + d(3y-2a) \in S$ if and only if $3x + 2d \geq 1 + \left\lfloor \frac{3y-2a}{k-1} \right\rfloor$.

Therefore $n = ax+dy > 0$ is a fundamental gap for the set S if and only if one of the following holds:

- for $\frac{a}{2} \leq y < \frac{2a}{3}$,

$$x \geq \max\left\{ \tfrac{1}{2}\left(\left\lfloor \tfrac{2y-a}{k-1} \right\rfloor - (d-1)\right), \tfrac{1}{3}\left(\left\lfloor \tfrac{3y-a}{k-1} \right\rfloor - (d-1)\right) \right\},$$

- for $\frac{2a}{3} \leq y \leq a-1$,

$$x \geq \max\left\{ \tfrac{1}{2}\left(\left\lfloor \tfrac{2y-a}{k-1} \right\rfloor - (d-1)\right), \tfrac{1}{3}\left(\left\lfloor \tfrac{3y-2a}{k-1} \right\rfloor - (2d-1)\right) \right\}.$$

In the first case, we have

$$\begin{aligned}
2\left(\left\lfloor \tfrac{3y-a}{k-1}\right\rfloor-(d-1)\right)-3\left(\left\lfloor \tfrac{2y-a}{k-1}\right\rfloor-(d-1)\right) &= 2\left\lfloor \tfrac{3y-a}{k-1}\right\rfloor-3\left\lfloor \tfrac{2y-a}{k-1}\right\rfloor+(d-1)\\
&\geq \left(\left\lfloor \tfrac{2(3y-a)}{k-1}\right\rfloor-1\right)-\left\lfloor \tfrac{3(2y-a)}{k-1}\right\rfloor+(d-1)\\
&\geq \left\lfloor \tfrac{a}{k-1}\right\rfloor+(d-2) \text{ since } \lfloor x\rfloor-\lfloor y\rfloor \geq \lfloor x-y\rfloor\\
&\geq 0.
\end{aligned}$$

In the second case, we have

$$\begin{aligned}
3\left(\left\lfloor \tfrac{2y-a}{k-1}\right\rfloor-(d-1)\right)-2\left(\left\lfloor \tfrac{3y-2a}{k-1}\right\rfloor-(2d-1)\right) &= 3\left\lfloor \tfrac{2y-a}{k-1}\right\rfloor-2\left\lfloor \tfrac{3y-2a}{k-1}\right\rfloor+(d+1)\\
&\geq \left(\left\lfloor \tfrac{3(2y-a)}{k-1}\right\rfloor-2\right)-\left\lfloor \tfrac{2(3y-2a)}{k-1}\right\rfloor+(d+1)\\
&\geq \left\lfloor \tfrac{3(2y-a)}{k-1}\right\rfloor-\left\lfloor \tfrac{2(3y-2a)}{k-1}\right\rfloor\\
&\geq 0.
\end{aligned}$$

Thus, $n = ax + dy > 0$ is a fundamental gap if and only if

$$\begin{array}{llr}
 & \tfrac{1}{3}\left(\left\lfloor \tfrac{3y-a}{k-1}\right\rfloor-(d-1)\right) \leq x \leq \left\lfloor \tfrac{y-1}{k-1}\right\rfloor, & \tfrac{a}{2} \leq y < \tfrac{2a}{3},\\
\text{or} & \tfrac{1}{2}\left(\left\lfloor \tfrac{2y-a}{k-1}\right\rfloor-(d-1)\right) \leq x \leq \left\lfloor \tfrac{y-1}{k-1}\right\rfloor, & \tfrac{2a}{3} \leq y \leq a-1.
\end{array}$$

□

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