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On a Sequence Related to the Factoradic Representation of an Integer

Maximiliano Sánchez Garza
Facultad de Ciencias Físico Matemáticas
Universidad Autónoma de Nuevo León
Av. Universidad
Ciudad Universitaria
San Nicolás de los Garza, Nuevo León 66451
México
maxsanchez1312@gmail.com

Enrique Treviño
Mathematics Department
Lake Forest College
555 N. Sheridan Road
Lake Forest, IL 60045
USA
trevino@lakeforest.edu

Abstract

For a positive integer r, define j_r to be the smallest positive integer n satisfying $n! > n^{r-1}$. In this paper we prove $j_{r+1} \in \{j_r + 1, j_r + 2\}$, which leads us to explore the set of positive integers r for which $j_{r+1} = j_r + 2$. We prove this set has the same density as the prime numbers. The sequence j_r was introduced by Carlson, Goedhart, and Harris in their work on factoradic happy numbers, and we prove some properties of j_r that lead to an improvement of one of their theorems.

1 Introduction

Let r be a positive integer and define j_r to be the smallest positive integer n satisfying

$$n! > n^{r-1}. \tag{1}$$

The first 20 values of j_r (A230319 in the *On-Line Encyclopedia of Integer Sequences* [9]) are

$$\{2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 27\}. \tag{2}$$

The sequence j_r made an appearance in work by Carlson, Goedhart, and Harris [2] on *factoradic happy numbers*. Our main goal in this paper is to prove some properties of j_r to be able to improve some of their results.

To get a better background, we need to define *happy numbers* (A007770) and *factoradic* expansion. Let n be a positive integer and $S_2(n)$ be the sum of the squares of its decimal digits. Consider the sequence of iterates of S_2 on n, i.e., $n, S_2(n), S_2^2(n), \ldots$. It is well known [7, pp. 74, 83–84] that eventually all the terms in the sequence are 1 or eventually the sequence becomes periodic with the cycle

$$4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4.$$

If the sequence reaches 1, we say n is *happy*. Many generalizations of happy numbers have been studied. For example, one can allow to change the base to $b \geq 2$ instead of 10, and one can replace sum of squares of digits, with the sum of r-th powers of the digits for some integer $r \geq 1$. Let $S_{r,b}(n)$ be the sum of r-th powers of the digits of n when n is written in base b. In depth analysis of the cases $r \in \{2, 3\}$, $b \in \{2, 3, \ldots, 10\}$ has been done by Grundman and Teeple [4]. The techniques developed by Grundman and Teeple [4] can easily be used to study other choices of r and b. Another generalization is to allow the base b to be a negative number. This has been done by Grundman and Harris [6] for $-2 \geq b \geq -10$ and $r = 2$. The authors also study in what cases there exist consecutive b-happy numbers in an arithmetic progression, generalizing work of El-Sedy and Siksek [3] (who do this for happy numbers) and the work of Grundman and Teeple [5] (who do this for b-happy r-power happy numbers). Bland, Cramer, de Castro, Domini, Edgar, Johnson, Klee, Koblitz, and Sundaresan [1] addressed a series of questions regarding a generalization of happy numbers to the fractional base 3/2, and Treviño and Zhylinski [10] addressed other fractional bases.

Every positive integer n can be written uniquely in the form

$$n = \sum_{i=1}^{k} a_i \cdot i!,$$

for some positive integer k satisfying $1 \leq a_k \leq k$, and $0 \leq a_i \leq i$ for $1 \leq i \leq k-1$. We call this the factoradic expansion of n. We use the notation $n = (a_k a_{k-1} \cdots a_1)_!$ to express a number written in its factoradic expansion. For example, $8 = 110_!$ because $8 = 0 \cdot 1! + 1 \cdot 2! + 1 \cdot 3!$.

Carlson, Goedhart, and Harris [2] generalized the concept of happy numbers to factoradic expansions as follows: let $S_{r,!}(n)$ be the sum of the r-th powers of the factoradic digits of a number n, then a positive number n is an r-power factoradic happy number if there exists an integer k such that $S^k_{r,!}(n) = 1$ (the k-iteration of $S_{r,!}$ is 1). Their main theorem is that for $r \in \{1, 2, 3, 4\}$, there exist arbitrarily long sequences of consecutive r-power factoradic happy numbers.

When studying happy numbers, regardless of the setting, it is important to show that the relevant happy function S ($S = S_{r,b}$ or $S = S_{r,!}$) satisfies $S(n) < n$ for all $n > N$, for some integer N. One of the difficulties of the factoradic case is the bound for N is less easy to get that in the other happy number generalizations. Carlson, Goedhart, and Harris [2, Theorem 10] proved that for $2 \le r \le 30$, they can choose $N = M_r = \sum_{i=1}^{j_r} i \cdot i! = (j_r + 1)! - 1$. Our main motivation to study j_r is to be able to prove this result for all r, i.e., to prove

Theorem 1. *Let r be a positive integer. Write n in its factoradic expansion as $n = \sum_{i=1}^{k} a_i i!$ with $1 \le a_k \le k$, and $0 \le a_i \le i$ for $i \in \{1, 2, \ldots, k-1\}$. Let*

$$S_{r,!}(n) = \sum_{i=1}^{k} a_i^r.$$

Then for $n \ge (j_r + 1)!$,

$$S_{r,!}(n) < n.$$

In our quest to prove the above theorem, we need to study properties of the sequence j_r. In Section 2 we prove some properties of j_r that study how the sequence grows. For example, we prove that $j_{r+1} - j_r \in \{1, 2\}$. We also prove that the number of positive integers $r \le x$ for which $j_{r+1} - j_r = 2$ is asymptotic to $x/\log x$. Inspired by this, we say r is a *j-prime* if $j_{r+1} - j_r = 2$. In Section 3, we prove some lemmas that are necessary for our proof of Theorem 1, in particular we have a nice upper bound for sums of powers. Finally, in Section 4, we prove Theorem 1.

2 Studying the sequence j_r

For the purposes of proving Theorem 1, we need an upper bound on j_r. The following proposition is the main result from this section we need.

Proposition 2. *Let $\varepsilon > 0$ be a real number. Then there exists M such that, for integers $r > M$, we have that $j_r < (1 + \varepsilon)r$.*

Proof. It is enough to prove that there exists an M such that, for $r > M$,

$$\log(\lfloor (1 + \varepsilon)r \rfloor !) > (r - 1) \log\lfloor (1 + \varepsilon)r \rfloor.$$

By expanding log as a sum we have

$$\log(\lfloor(1+\varepsilon)r\rfloor!) > \int_1^{\lfloor(1+\varepsilon)r\rfloor} \log t \,\mathrm{d}t = \lfloor(1+\varepsilon)r\rfloor \log\lfloor(1+\varepsilon)r\rfloor - \lfloor(1+\varepsilon)r\rfloor + 1.$$

Now, we want to show that there exists M such that, for $r > M$,

$$\lfloor(1+\varepsilon)r\rfloor \log\lfloor(1+\varepsilon)r\rfloor - \lfloor(1+\varepsilon)r\rfloor + 1 > (r-1)\log\lfloor(1+\varepsilon)r\rfloor,$$

which is equivalent to

$$\log\lfloor(1+\varepsilon)r\rfloor > \frac{\lfloor(1+\varepsilon)r\rfloor - 1}{\lfloor(1+\varepsilon)r\rfloor - r + 1}.$$

Since $\lfloor(1+\varepsilon)r\rfloor > (1+\varepsilon)r - 1$, then

$$\log\lfloor(1+\varepsilon)r\rfloor > \log((1+\varepsilon)r - 1),$$

and

$$\frac{\lfloor(1+\varepsilon)r\rfloor - 1}{\lfloor(1+\varepsilon)r\rfloor - r + 1} = 1 + \frac{r-2}{\lfloor(1+\varepsilon)r\rfloor - r + 1} < 1 + \frac{r-2}{\varepsilon r}.$$

If we find an M such that, for $r > M$, we have

$$\log((1+\varepsilon)r - 1) > 1 + \frac{r-2}{\varepsilon r},$$

then we are done. Note that

$$\lim_{r\to\infty} \log((1+\varepsilon)r - 1) = \infty \quad \text{and} \quad \lim_{r\to\infty} \left(1 + \frac{r-2}{\varepsilon r}\right) = 1 + \frac{1}{\varepsilon}$$

since $\varepsilon > 0$. This clearly implies the existence of the desired M. □

Remark 3. Given ε as in Proposition 2, we can take M to be

$$M = \left\lfloor \frac{e^{1+\frac{1}{\varepsilon}} + 1}{1+\varepsilon} \right\rfloor.$$

Then, for $r > M$, we have that $r > \frac{e^{1+\frac{1}{\varepsilon}}+1}{1+\varepsilon}$. Thus,

$$\log\left((1+\varepsilon)r - 1\right) > 1 + \frac{1}{\varepsilon} > 1 + \frac{1}{\varepsilon} - \frac{2}{\varepsilon r} = 1 + \frac{r-2}{\varepsilon r},$$

where the last inequality holds since $\varepsilon > 0$.

The following theorem provides an asymptotic for j_r that improves the upper bound from Proposition 2 and provides a strong lower bound. We do not need this result to prove our main theorem, but the result might be of independent interest.

Theorem 4. *For a positive integer r, there exists a real number θ_r such that*

$$j_r = r + \frac{r}{\log r} + \theta_r \left(\frac{r}{\log r} \right),$$

with $\theta_r \to 0$ as $r \to \infty$.

Proof. We first prove the upper bound. Let $\varepsilon > 0$ be a real number. We show that

$$j_r < r + \frac{(1+\varepsilon)r}{\log r}$$

for r sufficiently large. It suffices to prove that

$$\sum_{i=1}^{r+\left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor} \log i > (r-1)\log\left(r + \left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor\right) \tag{3}$$

for r sufficiently large. Let $n = r + \left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor$. We can bound the left side of (3) using the following explicit version of the lower bound in Stirling's formula [8], which is valid for every positive integer n:

$$\sum_{i=1}^{n} \log i > \frac{1}{2}\log(2\pi) + \left(n + \frac{1}{2}\right)\log n - n + \frac{1}{12n+1}. \tag{4}$$

Applying (4) to (3), it suffices to show

$$\log n > 1 + \frac{r - \frac{3}{2}}{\left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor + \frac{3}{2}} - \frac{\log(2\pi)}{2\left(\left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor + \frac{3}{2}\right)} - \frac{1}{(12n+1)\left(\left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor + \frac{3}{2}\right)}.$$

Since $\frac{(1+\varepsilon)r}{\log r} \geq \left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor > \frac{(1+\varepsilon)r}{\log r} - 1$, we need only prove

$$\log\left(r + \frac{(1+\varepsilon)r}{\log r} - 1\right) > 1 + \left(\frac{1}{1+\varepsilon}\right)\log r.$$

But the left side is greater than $\log r$ which is larger than $1 + \frac{1}{1+\varepsilon}\log r$ for r sufficiently large.

For the lower bound, we prove that

$$j_r > r + \frac{r}{\log r}$$

for r sufficiently large. It suffices to prove that

$$\sum_{i=1}^{r+\left\lfloor \frac{r}{\log r} \right\rfloor} \log i \leq (r-1)\log\left(r + \left\lfloor \frac{r}{\log r} \right\rfloor\right) \tag{5}$$

for r sufficiently large.

Let $n = r + \left\lfloor \frac{r}{\log r} \right\rfloor$. We can bound the left side of (5) using the following explicit version of the upper bound in Stirling's formula [8], which is valid for every positive integer n:

$$\sum_{i=1}^{n} \log i < \frac{1}{2} \log(2\pi) + \left(n + \frac{1}{2}\right) \log n - n + \frac{1}{12n}. \tag{6}$$

Applying (6) to (5), we need only show

$$\log n \leq 1 + \frac{r - \frac{3}{2}}{\left\lfloor \frac{r}{\log r} \right\rfloor + \frac{3}{2}} - \frac{\log(2\pi)}{2\left(\left\lfloor \frac{r}{\log r} \right\rfloor + \frac{3}{2}\right)} - \frac{1}{12\left(r + \left\lfloor \frac{r}{\log r} \right\rfloor\right)\left(\left\lfloor \frac{r}{\log r} \right\rfloor + \frac{3}{2}\right)}.$$

Since $\frac{r}{\log r} \geq \left\lfloor \frac{r}{\log r} \right\rfloor > \frac{r}{\log r} - 1$, letting $x = r + \frac{r}{\log r}$, we want to show that

$$\log x \leq 1 + \frac{r - \frac{3}{2}}{\frac{r}{\log r} + \frac{3}{2}} - \frac{\log(2\pi)}{2\left(\frac{r}{\log r} + \frac{1}{2}\right)} - \frac{1}{12(x-1)\left(\frac{r}{\log r} + \frac{1}{2}\right)}$$

for r sufficiently large; or, after subtracting by $\log r$ on both sides,

$$\log\left(1 + \frac{1}{\log r}\right) \leq 1 - \frac{3(\log^2 r + \log r)}{2r + 3\log r} - \frac{\log(2\pi)}{2\left(\frac{r}{\log r} + \frac{1}{2}\right)} - \frac{1}{12(x-1)\left(\frac{r}{\log r} + \frac{1}{2}\right)}.$$

Note that the left side goes to 0 as $r \to \infty$ while the right side goes to 1 as $r \to \infty$. Therefore the inequality is true for r sufficiently large. $\square$

In the interest of exploring j_r further, let's analyze the relationship between j_r and j_{r+1}.

Proposition 5. *For all positive integers r we have that $j_r + 1 \leq j_{r+1} \leq j_r + 2$.*

Proof. It is easy to see that $j_{r+1} > j_r$. So, it suffices to prove that

$$(j_r + 2)! > (j_r + 2)^r.$$

Assume, for the sake of contradiction, that $(j_r + 2)! \leq (j_r + 2)^r$. Then,

$$\frac{1}{(j_r + 2)!} \geq \frac{1}{(j_r + 2)^r}. \tag{7}$$

Multiplying inequalities (1) and (7),

$$\frac{j_r!}{(j_r + 2)!} > \frac{j_r^{r-1}}{(j_r + 2)^r}.$$

It follows that,

$$\frac{1}{j_r+1} > \frac{j_r^{r-1}}{(j_r+2)^{r-1}} = \left(\frac{j_r}{j_r+2}\right)^{r-1},$$

which implies

$$\left(1+\frac{2}{j_r}\right)^{r-1} > j_r+1. \tag{8}$$

Since $r^{r-1} \geq r(r-1)\cdots(2) = r!$, we know $j_r \geq r$. Using this in (8),

$$\left(1+\frac{2}{r}\right)^{r-1} \geq \left(1+\frac{2}{j_r}\right)^{r-1} > j_r+1 \geq r+1.$$

Therefore

$$\left(1+\frac{2}{r}\right)^{r-1} > r+1. \tag{9}$$

But $\left(1+\frac{2}{r}\right)^{r-1} \leq e^2 < 8$. This contradicts (9) for $r \geq 7$. We can use (2) to confirm that (9) is not satisfied for $1 \leq r \leq 6$. Thus (9) fails for $r \geq 1$. The result follows. □

Therefore, for some positive integers r we have $j_{r+1} = j_r + 1$ and for others we have $j_{r+1} = j_r + 2$. Let S be the set of positive integers such that $j_{r+1} = j_r + 2$. That is

$$S = \{r \in \mathbb{N} \,|\, j_{r+1} = j_r + 2\}. \tag{10}$$

The first few values in S can be easily determined from (2) to be 3, 6, 9, 12, 15, 19. This is sequence A336803. From Theorem 4 and Proposition 5 we can prove the following theorem which inspired us to name the elements of S as j-primes.

Theorem 6. *For a positive real number x, let $S(x) = \{r \leq x \,|\, j_{r+1} = j_r + 2\}$. Then*

$$S(x) \sim \frac{x}{\log x}.$$

Proof. From Proposition 5 we know $j_r \in \{j_{r-1}+1, j_{r-1}+2\}$. Therefore

$$j_r = j_1 + r - 1 + \sum_{\substack{i \in S \\ i \leq r}} 1 = r + 1 + S(r).$$

From Theorem 4 we have $j_r - r \sim \frac{r}{\log r}$. Hence $S(r) \sim \frac{r}{\log r}$. □

3 Some useful lemmas

We are interested in the sum of the first $n = j_r$ r-powers. There is an easy way to get a bound that works for all n:

$$\sum_{i=1}^{n} i^r \leq \int_1^{n+1} t^r \,\mathrm{d}t = \frac{(n+1)^{r+1} - 1}{r+1}. \tag{11}$$

For large n, (11) is great, but for n close to r, which is more relevant when $n = j_r$, we need a tighter bound. First, we recall some known properties of the Bernoulli numbers (A027641), where we take $B_1 = \frac{1}{2}$ instead of the usual $B_1 = -\frac{1}{2}$ to simplify (14).

- For $k > 0$, $B_k < 0$ if and only if $k \equiv 0 \pmod 4$.
- We have
$$\sum_{k=0}^{\infty} \frac{B_k}{k!} = \frac{e}{e-1}. \tag{12}$$
- For $k > 0$, we have
$$B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k). \tag{13}$$
- $B_{2k+1} = 0$ for $k \geq 1$.
- *(Bernoulli's formula)* For positive integers n and r, we have
$$\sum_{i=1}^{n} i^r = \frac{1}{r+1} \sum_{k=0}^{r} \binom{r+1}{k} B_k n^{r+1-k}. \tag{14}$$

Lemma 7. *Let $r \geq 2$ be an integer. Then, for an integer $n \geq r+1$,*

$$\sum_{i=1}^{n} i^r \leq \frac{C n^{r+1}}{r+1}$$

where $C = \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4 - 1)} = 1.583\cdots$.

Proof. For $n \geq r+1$, observe that $\binom{r+1}{k} \leq \frac{(r+1)^k}{k!} \leq \frac{n^k}{k!}$. Using that $B_{2k} < 0$ for even $k \geq 1$ and $B_k = 0$ for odd $k > 1$, by Bernoulli's formula (14) we have

$$\sum_{i=1}^{n} i^r \leq \frac{1}{r+1} \left(1 + \sum_{\substack{1 \leq k \leq r \\ k \not\equiv 0 \pmod 4}} \frac{n^k}{k!} B_k n^{r+1-k} \right) \leq \frac{n^{r+1}}{r+1} \left(1 + \sum_{\substack{1 \leq k \leq r \\ k \not\equiv 0 \pmod 4}} \frac{B_k}{k!} \right).$$

Now, using (12), (13), and using $\zeta(4k) \le \zeta(4) = \frac{\pi^4}{90}$, we have

$$\begin{aligned} 1 + \sum_{\substack{1\le k\le r \\ k \not\equiv 0 \pmod 4}} \frac{B_k}{k!} &\le \sum_{k=0}^{\infty} \frac{B_k}{k!} - \sum_{k=1}^{\infty} \frac{B_{4k}}{(4k)!} \\ &= \frac{e}{e-1} + 2\sum_{k=1}^{\infty} \frac{1}{(2\pi)^{4k}} \zeta(4k) \\ &\le \frac{e}{e-1} + 2\zeta(4) \frac{1}{(2\pi)^4 \left(1 - \frac{1}{(2\pi)^4}\right)} \\ &= \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4 - 1)}. \end{aligned}$$

Therefore, for $n \ge r+1$,

$$\sum_{i=1}^{n} i^r \le C \frac{n^{r+1}}{r+1},$$

where $C = \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4-1)}$, as desired. □

What follows is a simple inequality we need to use to prove our main theorem.

Lemma 8. *Let $0 < x < r$ be a real number. Then, for all integers $r \ge 1$, we have that*

$$\left(1 - \frac{x}{r}\right)^r < e^{-x}.$$

Proof. Using the Taylor series expansion of $\log(1 - x/r)$, we have that

$$r \log\left(1 - \frac{x}{r}\right) = -r\left(\frac{x}{r} + \frac{1}{2}\left(\frac{x}{r}\right)^2 + \frac{1}{3}\left(\frac{x}{r}\right)^3 + \cdots\right) = -x - \frac{x^2}{2r} - \frac{x^3}{3r^2} - \cdots < -x.$$

□

Lemma 9. *Let $0 < \varepsilon < 2$. Then there exists a positive integer M such that, for all integers $r > M$, we have that*

$$\left(1 - \frac{1}{j_r}\right)^{r+1} < e^{-1+\varepsilon}.$$

Proof. Let $\varepsilon_1 > 0$ be a real number. First, by Proposition 2, there exists M_1 positive integer such that, for $r > M_1$, we have that $j_r < (1+\varepsilon_1)r$. Then, for $r > M_1$, we have

$$\left(1 - \frac{1}{j_r}\right)^{r+1} < \left(1 - \frac{1}{(1+\varepsilon_1)r}\right)^{r+1}. \tag{15}$$

Observe that

$$\lim_{r\to\infty}\left(1-\frac{1}{(1+\varepsilon_1)r}\right)^{r+1} = e^{-\frac{1}{1+\varepsilon_1}}. \tag{16}$$

On the other hand, we know that $j_r > r$ because $j_1 > 1$ and $j_{r+1} \geq j_r + 1$. It follows that

$$\left(1-\frac{1}{j_r}\right)^{r+1} > \left(1-\frac{1}{r}\right)^{r+1}, \tag{17}$$

and we have that

$$\lim_{r\to\infty}\left(1-\frac{1}{r}\right)^{r+1} = e^{-1}. \tag{18}$$

Since (15) and (16) hold for all $\varepsilon_1 > 0$, from (15), (16), (17), and (18) we conclude that

$$\lim_{r\to\infty}\left(1-\frac{1}{j_r}\right)^{r+1} = e^{-1}.$$

Thus, there exists M such that, for $r > M$, we have that

$$\left(1-\frac{1}{j_r}\right)^{r+1} < e^{-1+\varepsilon}$$

as desired. □

Remark 10. From Remark 3, we know that

$$j_r < \left(\frac{1}{1-\frac{\varepsilon}{2}}\right) r$$

for $r > \left(e^{\frac{2}{\varepsilon}}+1\right)\left(1-\frac{\varepsilon}{2}\right)$. Then,

$$\left(1-\frac{1}{j_r}\right)^{r+1} < \left(1-\frac{1-\frac{\varepsilon}{2}}{r}\right)^{r+1} < \left(1-\frac{1-\frac{\varepsilon}{2}}{r}\right)^{r}$$

where

$$\left(1-\frac{1-\frac{\varepsilon}{2}}{r}\right)^{r} < e^{-1+\varepsilon}$$

for all integers $r \geq 1$ (by Lemma 8 with $x = 1-\varepsilon$). Therefore, we can take M in Lemma 9 as

$$M = \left\lfloor \left(e^{\frac{2}{\varepsilon}}+1\right)\left(1-\frac{\varepsilon}{2}\right)\right\rfloor.$$

4 Proof of Theorem 1

Carlson, Goedhart, and Harris [2, Theorem 10] reduced the proof of Theorem 1 to showing the inequality[1]

$$(j_r+1)! - (j_r+1)^{r-1} + j_r! - 1 - \sum_{i=1}^{j_r-1} i^r > 0. \tag{19}$$

They proved it via computation [2, Table 1] for $2 \le r \le 30$. We prove that (19) works for all positive integers r in the next proposition.

Proposition 11. *Let r be a positive integer. Then*

$$(j_r+1)! - (j_r+1)^{r-1} + j_r! - 1 - \sum_{i=1}^{j_r-1} i^r > 0.$$

Proof. Let $C = \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4-1)}$ and let L_r be the left side of (19). We want to show $L_r > 0$. We know that $j_r > r$. Also, as observed by Carlson, Goedhart, and Harris [2], we have that

$$(j_r+1)^{r-1} = j_r^{r-1}\left(1+\frac{1}{j_r}\right)^{r-1} < j_r^{r-1}\left(1+\frac{1}{r}\right)^{r} < 3j_r^{r-1}.$$

From this and from Lemma 7, we have that for $r \ge 2$,

$$\begin{aligned}
L_r &> j_r!(j_r+2) - 3j_r^{r-1} - 1 - \frac{C(j_r-1)^{r+1}}{r+1} \\
&= j_r!(j_r+2) - 3j_r^{r-1} - 1 - j_r^{r-1}\left(\frac{j_r^2}{r+1}\right)\left(C\left(\frac{j_r-1}{j_r}\right)^{r+1}\right) \\
&> j_r^{r-1}\left(r-1-\left(\frac{j_r^2}{r+1}\right)\left(C\left(1-\frac{1}{j_r}\right)^{r+1}\right)\right) - 1.
\end{aligned}$$

Let $0 < \varepsilon < 1 - \log(C)$ be a real number. By Lemma 9, there exists a positive integer M_1 such that, for $r > M_1$,

$$\left(1-\frac{1}{j_r}\right)^{r+1} < e^{-1+\varepsilon},$$

which leads to

$$C\left(1-\frac{1}{j_r}\right)^{r+1} < Ce^{-1+\varepsilon}.$$

[1] They incorrectly omitted the "−1" term from the inequality.

It follows that, for $r > \max(2, M_1)$,

$$\begin{aligned} L_r &> j_r^{r-1}\left(r - 1 - Ce^{-1+\varepsilon}\left(\frac{j_r^2}{r+1}\right)\right) - 1 \\ &= Ce^{-1+\varepsilon} j_r^{r-1}\left(\left(\frac{1}{Ce^{-1+\varepsilon}}\right)(r-1) - \frac{j_r^2}{r+1}\right) - 1. \end{aligned}$$

Let $k > 1$ be a constant such that $k^2 < \frac{1}{Ce^{-1+\varepsilon}}$. By Proposition 2, there exists a positive integer M_2 such that, for $r > M_2$, we have that

$$j_r < k(r+1).$$

Then, for $r > \max(2, M_1, M_2)$,

$$L_r > Ce^{-1+\varepsilon} j_r^{r-1}\left(\left(\frac{1}{Ce^{-1+\varepsilon}} - k^2\right) r - \frac{1}{Ce^{-1+\varepsilon}} - k^2\right) - 1$$

which is greater than -1 whenever $r > \frac{1+Ck^2e^{-1+\varepsilon}}{1-Ck^2e^{-1+\varepsilon}}$. Since the expression on the left is an integer, we conclude the result for $r > \max\left(2, M_1, M_2, \left\lfloor \frac{1+Ck^2e^{-1+\varepsilon}}{1-Ck^2e^{-1+\varepsilon}} \right\rfloor\right)$.

From Remarks 3 and 10 we know that we can choose M_1, M_2 to be

$$\begin{aligned} M_1 &= \left\lfloor \left(e^{\frac{2}{\varepsilon}} + 1\right)\left(1 - \frac{\varepsilon}{2}\right) \right\rfloor, \\ M_2 &= \left\lfloor \frac{e^{1+\frac{1}{k-1}} + 1}{k} - 1 \right\rfloor. \end{aligned}$$

The idea is to minimize the value of $\max\left(2, M_1, M_2, \left\lfloor \frac{1+Ck^2e^{-1+\varepsilon}}{1-Ck^2e^{-1+\varepsilon}} \right\rfloor\right)$. Choosing $\varepsilon = 0.26$ and $k = 1.15$, we have that

$$\max\left(2, M_1, M_2, \left\lfloor \frac{1+Ck^2e^{-1+\varepsilon}}{1-Ck^2e^{-1+\varepsilon}} \right\rfloor\right) = \max(2, 1907, 1857, 2167) = 2167.$$

This proves the result for $r > 2167$. The cases $r \leq 2167$ can be easily checked with a computer. □

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