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Combinatorial Properties of a Generalized Class of Laguerre Polynomials

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Abstract

In this paper, we consider various combinatorial aspects of a family of polynomials, denoted by $L_n^{(\alpha,\beta)}(x)$, whose coefficients $S_{\alpha,\beta}(n,k)$ correspond to a special case of the partial r-Bell polynomials. Among the particular cases of $L_n^{(\alpha,\beta)}(x)$ are the generalized Laguerre polynomials, associated Lah polynomials, and polynomials arising in the study of hyperbolic partial differential equations. Here we provide a combinatorial treatment of $L_n^{(\alpha,\beta)}(x)$ and its coefficients, which were studied previously strictly from an algebraic standpoint. In addition to providing combinatorial proofs of some prior identities, we derive several new relations using the combinatorial interpretations for $L_n^{(\alpha,\beta)}(x)$ and $S_{\alpha,\beta}(n,k)$. Our proofs make frequent use of sign-changing involutions on various weighted structures. Finally, we introduce a bivariate polynomial generalization arising as a distribution for a pair of statistics and establish some of its basic properties.

1 Introduction

The generalized sequence of polynomials $L_n^{(\alpha,\beta)}(x)$ are defined by Mihoubi and Sahari [17] as coefficients in the exponential generating function formula

$$\sum_{n\geq 0} L_n^{(\alpha,\beta)}(x)\frac{t^n}{n!} = (1-t)^\alpha \exp\left(x\left((1-t)^\beta - 1\right)\right), \tag{1}$$

where α and β are real numbers with $\beta \neq 0$. The special cases of $L_n^{(\alpha,\beta)}(x)$ when $(\alpha, \beta) = (-\frac{1}{2}, -\frac{1}{2})$ or $(-\frac{3}{2}, -\frac{1}{2})$ arise in the theory of hyperbolic partial differential equations, see, [6, pp. 391–398] and [10]. Other important special cases include the generalized Laguerre polynomials ($\alpha = -\lambda - 1, \beta = -1$) and the associated Lah polynomials ($\alpha = 0, \beta = -m$), see, e.g., [7] and [1], respectively. It is seen from (1) that $L_n^{(\alpha,\beta)}(x)$ may be expressed as

$$L_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} S_{\alpha,\beta}(n,k)x^k,$$

where

$$S_{\alpha,\beta}(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(-\alpha - \beta j)^{\overline{n}}, \qquad n, k \geq 0, \tag{2}$$

and $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ if $n \geq 1$, with $x^{\overline{0}} = 1$, denotes the rising factorial. Note that (2) implies $S_{\alpha,\beta}(n,k) = 0$ for $k > n \geq 0$, whence the $L_n^{(\alpha,\beta)}(x)$ are indeed polynomials.

Recall that the partial r-Bell polynomials [16] are defined by

$$\sum_{n\geq k} B_{n+r,k+r}^{(r)}(a_\ell; b_\ell)\frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{j\geq 1} a_j \frac{t^j}{j!}\right)^k \left(\sum_{j\geq 0} b_{j+1}\frac{t^j}{j!}\right)^r,$$

the $r = 0$ case of which corresponds to the classical partial Bell polynomials (see, e.g., [2]). It was shown [17] that

$$S_{r\alpha,\beta}(n,k) = B_{n+r,k+r}^{(r)}\left((-\beta)^{\overline{j}}, (-\alpha)^{\overline{j-1}}\right), \tag{3}$$

where r is a non-negative integer. Note that upon taking $\alpha = \beta = -1$ in (3) and replacing r with $2r$, one obtains the r-Lah numbers as a special case of $S_{\alpha,\beta}(n,k)$, which we denote by $\left\lfloor {n \atop k} \right\rfloor_r$ in accordance with [18]. The $r = 0$ case of $\left\lfloor {n \atop k} \right\rfloor_r$, which is written without a subscript, corresponds to the classical Lah numbers; see, e.g., [11] and A008297 in [23]. We remark that other polynomial generalizations of the Lah numbers related to $S_{\alpha,\beta}(n,k)$ have been considered previously [14, 21].

Here, we provide a unified combinatorial treatment of several of the previous algebraic results involving $L_n^{(\alpha,\beta)}(x)$ and $S_{\alpha,\beta}(n,k)$. To do so, we consider a slight variant, denoted by $L_r(n,k) = L_r^{(\alpha,\beta)}(n,k)$, of $S_{\alpha,\beta}(n,k)$ defined as $L_r(n,k) = S_{-r\alpha,-\beta}(n,k)$. We then let $L_{n,r}(x) = L_{n,r}^{(\alpha,\beta)}(x)$ be given by

$$L_{n,r}(x) = \sum_{k=0}^{n} L_r(n,k)x^k, \qquad n \geq 0. \tag{4}$$

We find these forms of $S_{\alpha,\beta}(n,k)$ and $L_n^{(\alpha,\beta)}(x)$ to be more convenient to deal with combinatorially. The extra parameter r is introduced here since in several of the subsequent identities, we consider integral increments of the parameter α. Also, in the final section, the non-negative integer r is replaced by its polynomial analogue $\frac{p^r-1}{p-1}$, where p is an indeterminate.

The organization of this paper is as follows. In the next section, we find a combinatorial interpretation for $L_r(n,k)$ and $L_{n,r}(x)$ in terms of a pair of statistics on a structure closely related to the r-Lah distributions. We make use of this interpretation in the third section in finding some new relations involving $L_r(n,k)$ and $L_{n,r}(x)$. In the fourth section, we provide combinatorial proofs of some prior formulas for $S_{\alpha,\beta}(n,k)$ and $L_n^{(\alpha,\beta)}(x)$, rewritten in terms of $L_r(n,k)$ and $L_{n,r}(x)$, which were found previously by algebraic methods. Our proofs entail use of weight-preserving, sign-reversing involutions defined on certain weighted configurations involving various kinds of finite partitions whose blocks are contents-ordered. In the final section, we consider a (p,q)-generalization of $L_r(n,k)$ (and hence also of $L_{n,r}(x)$) by considering two further statistics (marked by p and q) on the underlying structure that is enumerated by $L_r(n,k)$ when $\alpha=\beta=1$. Some identities are found of the (p,q)-analogue, which extend earlier ones and the log-concavity is established for a range of p and q values.

2 Combinatorial definition and generating function

In this section, we provide a combinatorial interpretation for the sequences $L_r(n,k)$ and $L_{n,r}(x)$ and show how their (exponential) generating function formulas can be obtained using this definition. To do so, we first write a two-term recurrence for $L_r(n,k)$. Such a recurrence for $L_r(n,k)$ where $n,k\geq 1$ (along with initial conditions) may be derived from (2) using $L_r(n,k)=S_{-r\alpha,-\beta}(n,k)$ as follows:

$$\begin{aligned}
&\beta L_r(n-1,k-1)+(\alpha r+\beta k+n-1)L_r(n-1,k)\\
&\qquad=\frac{\beta}{(k-1)!}\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}(\alpha r+\beta j)^{\overline{n-1}}\\
&\qquad\quad+\frac{\alpha r+\beta k+n-1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\alpha r+\beta j)^{\overline{n-1}}\\
&\qquad=\frac{\beta}{k!}\sum_{j=0}^{k-1}(-1)^{k-1-j}(k-j)\binom{k}{j}(\alpha r+\beta j)^{\overline{n-1}}+\frac{(\alpha r+\beta k)^{\overline{n}}}{k!}\\
&\qquad\quad+\frac{\alpha r+\beta k+n-1}{k!}\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}(\alpha r+\beta j)^{\overline{n-1}}\\
&\qquad=\frac{1}{k!}\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}(\alpha r+\beta j)^{\overline{n-1}}\left(\beta(j-k)+\alpha r+\beta k+n-1\right)+\frac{(\alpha r+\beta k)^{\overline{n}}}{k!}
\end{aligned}$$

$$= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\alpha r + \beta j)^{\overline{n}} = L_r(n,k).$$

Thus, we have the recurrence

$$L_r(n,k) = \beta L_r(n-1,k-1) + (\alpha r + \beta k + n - 1)L_r(n-1,k), \qquad n,k \geq 1, \tag{5}$$

with initial conditions $L_r(n,0) = (\alpha r)^{\overline{n}}$ and $L_r(0,k) = \delta_{k,0}$ for all $n,k \geq 0$. Note that when $\alpha = \beta = 1$ and r is replaced by $2r$, it is apparent from (5) that $L_r(n,k)$ reduces to the r-Lah number. For a different generalization of the r-Lah numbers, see [4], where the r-Whitney-Lah numbers are introduced, and [9, 20], where they are studied further.

We now find a combinatorial interpretation for $L_r(n,k)$ in terms of a bivariate distribution. Given $n,k,r \geq 0$, let $\mathcal{P}_{n,k} = \mathcal{P}_{n,k}^{(r)}$ denote the set of partitions of $[n+r]$ into $k+r$ contents-ordered blocks such that the elements of $[r]$ belong to distinct blocks, with these elements first in their respective blocks. Within $\lambda \in \mathcal{P}_{n,k}$, we refer to blocks containing the elements of $[r]$ as *special*, with all other blocks being *non-special*, and at times refer to the elements of $[r]$ themselves as special. Note that by virtue of their association with a distinguished element, special blocks are labeled and in effect allowed to be empty, whereas non-special blocks are unlabeled and always non-empty.

We define a pair of statistics on $\mathcal{P}_{n,k}$ as follows. By a left-right minimum (lr min) within a non-special block $B = \{b_1 b_2 \cdots b_\ell\}$ of λ, we mean an element $b_i \in B$ such that $b_i < b_j$ for all $j < i$. The same definition applies to a left-right minimum in a special block of λ except that one considers only the sequence (possibly empty) of elements obtained by excluding the initial (special) element. Note that the first element of any non-special and the second element of any special block is vacuously an lr min. Let $\nu_1(\lambda)$ denote the total number of lr min in all of the special blocks of λ per the definition above and $\nu_2(\lambda)$ the number of lr min in its non-special blocks.

Define $T_r(n,k) = T_r^{(\alpha,\beta)}(n,k)$ to be the joint distribution on $\mathcal{P}_{n,k}^{(r)}$ given by

$$T_r(n,k) = \sum_{\lambda \in \mathcal{P}_{n,k}^{(r)}} \alpha^{\nu_1(\lambda)}\beta^{\nu_2(\lambda)}, \qquad 0 \leq k \leq n. \tag{6}$$

Note that $T_r(n,0) = (\alpha r)^{\overline{n}}$, upon considering the placement of the elements of $I = [r+1, r+n]$ within the special blocks, starting with $r+1$, and furthermore that $T_r(n,n) = \beta^n$, as each member of I in this case occupies its own (non-special) block and hence is an lr min. We take $T_r(n,k)$ to be zero if $k > n$ or $k < 0$.

We have the following combinatorial interpretation for $L_r(n,k)$.

Theorem 1. *For all $n,k,r \geq 0$, $L_r(n,k) = T_r(n,k)$; i.e., $L_r(n,k)$ is the joint distribution for the statistics ν_1 and ν_2 on $\mathcal{P}_{n,k}^{(r)}$.*

Proof. The initial conditions of $T_r(n,k)$ when $n = 0$ or $k = 0$ are seen to agree with those of $L_r(n,k)$, so assume $n,k \geq 1$. Note that the weight of the members of $\mathcal{P}_{n,k}$ in which the element $n+r$ comprises its own block is given by $\beta T_r(n-1,k-1)$. On the other hand, if $n+r$ is placed just after a member of $[r]$ within one of the special blocks or at the beginning of a non-special block, then there are $\alpha r T_r(n-1,k)$ and $\beta k T_r(n-1,k)$ possibilities, respectively. Finally, members of $\mathcal{P}_{n,k}$ in which $n+r$ directly follows some element of $[r+1, r+n-1]$ contribute $(n-1)T_r(n-1,k)$ towards the weight. Combining the previous cases implies $T_r(n,k)$ satisfies recurrence (5), and hence $T_r(n,k) = L_r(n,k)$ for all n, k and r. □

Let $\mathcal{P}_n = \mathcal{P}_n^{(r)}$ be given by $\mathcal{P}_n = \cup_{k=0}^{n} \mathcal{P}_{n,k}$ for $n \geq 0$. Given $\lambda \in \mathcal{P}_n$, let $\mu(\lambda)$ denote the number of non-special blocks of λ. Then, from (4), it is seen that $L_{n,r}(x)$ gives the joint distribution of the ν_1, ν_2 and μ statistics on $\mathcal{P}_n$, where the μ statistic is marked by the x variable. Note that $L_{n,r}(x)$ reduces when $x = \alpha = \beta = 1$ to the n-th row sum of r-Lah numbers, which coincides with A000262 when $r = 0$.

The $L_n^{(\alpha,\beta)}(x)$, equivalently the $L_{n,r}(x)$, were defined in [17] as coefficients of a certain exponential generating function (egf), from which various algebraic properties are derived. Alternatively, starting with the combinatorial definition above for $L_{n,r}(x)$, it is possible to derive the corresponding egf formula.

Theorem 2. *We have*

$$\sum_{n\geq 0} L_{n,r}(x)\frac{t^n}{n!} = (1-t)^{-\alpha r}\exp\left(x\left((1-t)^{-\beta}-1\right)\right) \tag{7}$$

and

$$\sum_{n\geq k} L_r(n,k)\frac{t^n}{n!} = \frac{(1-t)^{-\alpha r}}{k!}\left((1-t)^{-\beta}-1\right)^k. \tag{8}$$

Proof. Since (8) follows from (7) and (4), we need only establish (7). To do so, first observe the identity

$$L_{n+1,r}(x) = (n+\alpha r+\beta x)L_{n,r}(x) + \beta x\frac{\partial}{\partial x}L_{n,r}(x), \qquad n \geq 0. \tag{9}$$

An equivalent form of (9) was shown in [17] algebraically. Using the interpretation given above for $L_{n,r}(x)$, one can give a quick combinatorial proof of (9) as follows. Note that the first term on the right side of (9) counts all $\rho \in \mathcal{P}_{n+1}$ where the element $n+r+1$ either directly follows a member of I, follows a special element or occurs as a singleton block. The second term is seen to count those ρ in which $n+r+1$ starts a non-singleton non-special block (and hence itself is an lr min). Observe that since no new block is created in this last case, no factor of x is introduced, which is witnessed with the multiplication of the x-partial derivative by x.

Let $f(t,x) = \sum_{n\geq 0} L_{n,r}(x)\frac{t^n}{n!}$. Multiplying both sides of (9) by $\frac{t^n}{n!}$, and summing over $n \geq 0$, gives

$$(1-t)\frac{\partial}{\partial t}f(t,x) - \beta x\frac{\partial}{\partial x}f(t,x) = (\alpha r+\beta x)f(t,x), \tag{10}$$

with initial condition $f(0,x)=1$. Solving explicitly the first-order linear partial differential equation (10) then yields

$$f(t,x)=(1-t)^{-\alpha r}\exp\left(x\left((1-t)^{-\beta}-1\right)\right),$$

as desired. □

Remark 3. Note that recurrence (5) also follows from equating like powers of x on both sides of (9).

3 New identities for $L_r(n,k)$ and $L_{n,r}(x)$

In this section, we derive some new identities involving $L_r(n,k)$ and $L_{n,r}(x)$. Applying recurrence (5) repeatedly yields the following formula for $n\geq k\geq 1$:

$$L_r(n,k)=(\alpha r)^{\overline{n-k}}\beta^k+\sum_{j=0}^{k-1}(\alpha r+\beta(k-j)+n-j-1)\beta^j L_r(n-j-1,k-j), \tag{11}$$

which may also be shown by considering the largest element $n-j+r$ that either goes in a special block or in a non-singleton non-special block. Multiplying both sides of (11) by x^k, and summing over $1\leq k\leq n$, gives after simplification the following recurrence:

$$L_{n,r}(x)=(\beta x)^n+\sum_{j=0}^{n-1}(\beta x)^{n-j-1}\left((\alpha r+j)L_{j,r}(x)+\beta x\frac{\partial}{\partial x}L_{j,r}(x)\right),\qquad n\geq 1. \tag{12}$$

From (2), $L_r(n,k)$ is given explicitly by

$$L_r(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\alpha r+\beta j)^{\overline{n}},\qquad n,k\geq 0. \tag{13}$$

Let $x^{\underline{n}}=x(x-1)\cdots(x-n+1)$ for $n\geq 1$, with $x^{\underline{0}}=1$, denote the falling factorial. We have the following further recurrences for $L_r(n,k)$ that can be obtained from (13).

Theorem 4. *If $n,k\geq 0$ and $0\leq s\leq r$, then*

$$L_r^{(\alpha,\beta)}(n,k)=\sum_{j=k}^{n}\binom{n}{j}(\alpha(r-s))^{\overline{n-j}}L_s^{(\alpha,\beta)}(j,k) \tag{14}$$

and

$$L_r^{(\alpha,\alpha)}(n,k)=\sum_{j=k}^{n}\binom{j}{k}(r-s)^{\underline{j-k}}L_s^{(\alpha,\alpha)}(n,j). \tag{15}$$

Proof. By (13), we have

$$\begin{aligned}
&\sum_{j=k}^{n}\binom{n}{j}(\alpha(r-s))^{\overline{n-j}}L_s^{(\alpha,\beta)}(j,k)\\
&=\sum_{j=0}^{n}\binom{n}{j}(\alpha(r-s))^{\overline{n-j}}\cdot\frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}(\alpha s+\beta\ell)^{\overline{j}}\\
&=\frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\sum_{j=0}^{n}\binom{n}{j}(\alpha(r-s))^{\overline{n-j}}(\alpha s+\beta\ell)^{\overline{j}}\\
&=\frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}(\alpha r+\beta\ell)^{\overline{n}}=L_r^{(\alpha,\beta)}(n,k),
\end{aligned}$$

where we have used the fact $(x+y)^{\overline{n}}=\sum_{j=0}^{n}\binom{n}{j}x^{\overline{j}}y^{\overline{n-j}}$ in the penultimate equality. For (15), first observe that for a fixed n, k and s where $n\geq k$, both sides of (15) may be viewed as polynomials in r of degree $n-k$. Thus, it suffices to show (15) for all $r\geq n-k+s$. Let $m=k+r-s$. By (13), we then have

$$\begin{aligned}
&\sum_{j=k}^{n}\binom{j}{k}(r-s)^{\underline{j-k}}L_s^{(\alpha,\alpha)}(n,j)=\sum_{j=k}^{m}\binom{j}{k}(r-s)^{\underline{j-k}}\cdot\frac{1}{j!}\sum_{\ell=0}^{j}(-1)^{j-\ell}\binom{j}{\ell}(\alpha(s+\ell))^{\overline{n}}\\
&=\sum_{\ell=0}^{m}(\alpha(s+\ell))^{\overline{n}}\sum_{j=k}^{m}\frac{(-1)^{j-\ell}}{j!}\binom{j}{k}\binom{j}{\ell}(r-s)^{\underline{j-k}}\\
&=\frac{1}{k!}\sum_{\ell=0}^{m}(\alpha(s+\ell))^{\overline{n}}\sum_{j=k}^{m}(-1)^{j-\ell}\binom{r-s}{j-k}\binom{j}{\ell}\\
&=\frac{1}{k!}\sum_{\ell=0}^{m}(\alpha(s+\ell))^{\overline{n}}\cdot(-1)^{r-s+k-\ell}\binom{k}{\ell-r+s}\\
&=\frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}(\alpha(r+\ell))^{\overline{n}}=L_r^{(\alpha,\alpha)}(n,k),
\end{aligned}$$

where we have used [8, Formula 5.24] in the antepenultimate equality □

Remark 5. Identities (14) and (15), in the case when $\alpha=\beta=1$ and r and s are even, were shown in [18] by a different method. Furthermore, identity (14) may be given a combinatorial proof by considering the number $n-j$ of elements of I in the final $r-s$ special blocks within a member of $\mathcal{P}_n^{(r)}$. Identity (15) may also be obtained combinatorially by first arranging the elements of $[n+s]$ according to a member of $\mathcal{P}_{n,j}^{(s)}$ where $j\geq k$ and then selecting $j-k$ of the non-special blocks whose contents to be transferred to $r-s$ additional special blocks. We leave the details of this argument to the interested reader.

The combinatorial interpretation for $L_r(n,k)$ given in Theorem 1 above yields further recurrence formulas for $L_r(n,k)$ as follows.

Theorem 6. *If $n,m,k \geq 0$ and $0 \leq s \leq r$, then*

$$L_r(n+m,k) = \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}(\alpha(r-s))^{\overline{i+j}}L_s(n+m-i-j,k) \tag{16}$$

and

$$L_r(n+m+1,k) = \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\big(r\alpha^{\overline{i+j+1}}L_{r-1}(n+m-i-j,k) + \beta^{\overline{i+j+1}}L_r(n+m-i-j,k-1)\big). \tag{17}$$

Proof. To show (16), consider the number i of elements of I and the number j of elements of $[r+n+1,r+n+m]$ that go in the final $r-s$ special blocks within a member of $\mathcal{P}_{n+m,k}$. There are $\binom{n}{i}\binom{m}{j}(\alpha(r-s))^{\overline{i+j}}$ ways in which to choose and arrange these elements and then $L_s(n+m-i-j,k)$ ways to arrange the remaining members of $[r+1,r+n+m]$, together with the first s special elements. Summing over all possible i and j gives (16).

For (17), we consider the number i of elements of $[r+2,r+n+1]$ and the number j of elements of $[r+n+2,r+n+m+1]$ that go in the same block as $r+1$ within a member of $\mathcal{P}_{n+m+1,k}$. If $r+1$ is to go in one of the r special blocks, then there are $\alpha^{\overline{i+j+1}}$ ways in which to order the elements in this block and $L_{r-1}(n+m-i-j,k)$ ways in which to arrange the remaining members of $[r+n+m+1]$. If $r+1$ goes in a non-special block, then there are $\beta^{\overline{i+j+1}}$ ways to arrange the elements in this block and $L_r(n+m-i-j,k-1)$ ways in which to arrange the remaining elements. Combining the two previous cases gives the generic term in the sum on the right side of (17), which implies the result. □

A similar argument to that given for (16) above implies

$$L_{n+m,r}(x) = \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}(\alpha(r-s))^{\overline{i+j}}L_{n+m-i-j,s}(x), \qquad 0 \leq s \leq r,$$

which may also be obtained by multiplying both sides of (16) by x^k and summing over all k. A formula comparable to (17) may be given for $L_{n+m+1,r}(x)$ as well. The $m=r=0$ case of (17) may be written equivalently as

$$L_0(n+1,k) = \sum_{i=0}^{n}\binom{n}{i}\beta^{\overline{i+1}}L_0(n-i,k-1), \qquad n,k \geq 0,$$

which can also be obtained by considering the number of additional elements in the block containing 1. When $\beta = 1$, note that this is a standard recurrence for $\left\lfloor {n \atop k} \right\rfloor$.

Theorem 7. *If $n, m, j \geq 0$ and $0 \leq s \leq r$, then*

$$L_r(n+1, m+j+1) = \beta \sum_{i=0}^{n-m-j} \sum_{k=m}^{n-i} \sum_{\ell=j}^{n-i-k} \binom{n+1}{i} \binom{n-i-k}{\ell} (\alpha s + \beta(m+1) + k + 1)^{\overline{n-i-k-\ell}} \times (\alpha(r-s))^{\overline{i}} L_s(k,m) L_0(\ell, j). \tag{18}$$

Proof. We enumerate the members of $\mathcal{P}_{n+1,m+j+1}$ according to i, k and ℓ defined as follows. Given $\lambda \in \mathcal{P}_{n+1,m+j+1}$, let i denote the number of elements of $[r+1, r+n+1]$ going in the final $r-s$ special blocks of λ, and let R denote the subset of elements so selected. Let $T = [r+1, r+n+1] - R$ be given by $T = \{t_1, \ldots, t_{n-i+1}\}$ where $t_1 < \cdots < t_{n-i+1}$, with k such that t_{k+1} is the smallest element of the $(m+1)$-st non-special block of λ (where non-special blocks are arranged from left to right in increasing order of smallest elements). Finally, let ℓ denote the number of elements of $K = \{t_{k+2}, t_{k+3}, \ldots, t_{n-i+1}\}$ that belong to the final j non-special blocks of λ and denote by L the corresponding subset of K. Note that by the definitions, $m \leq k \leq n-i$ and $j \leq \ell \leq n-i-k$, and hence $0 \leq i \leq n-m-j$, in order for such λ to exist.

To enumerate λ, first note that there are $\binom{n+1}{i}(\alpha(r-s))^{\overline{i}}$ ways in which to choose and position the elements of R. Once this has been done, there are $L_s(k,m)$ ways in which to arrange the elements of $\{t_1, \ldots, t_k\} \cup [s]$ such that there are exactly m non-special blocks. We then put t_{k+1} into an additional non-special block by itself (which accounts for the extra factor of β) and insert the $n-i-k-\ell$ elements of $K-L$ into the first s special and first $m+1$ non-special blocks. Note that there are $\binom{n-i-k}{\ell}(\alpha s + \beta(m+1) + k + 1)^{\overline{n-i-k-\ell}}$ possibilities concerning the selection and placement of the elements of $K-L$, as there are already $k+1$ non-special elements altogether within these blocks. Finally, there are $L_0(\ell, j)$ ways in which to arrange the members of L since none of these elements go in special blocks. Thus, the generic summand on the right side of (18) enumerates all $\lambda \in \mathcal{P}_{n+1,m+j+1}$ meeting the restrictions described in the first paragraph. Considering all possible i, k and ℓ then gives (18). □

Remark 8. Taking $s = r$ in (18) yields

$$L_r(n+1, m+j+1) = \beta \sum_{k=m}^{n} \sum_{\ell=j}^{n-k} \binom{n-k}{\ell} (\alpha r + \beta(m+1) + k + 1)^{\overline{n-k-\ell}} L_r(k,m) L_0(\ell, j). \tag{19}$$

Note that when $\alpha = \beta = 1$ in (19), one obtains an r-Lah number analogue of formula (7) from [5].

We have the further recurrence relation for $L_{n,r}(x)$ which follows from its combinatorial interpretation.

Theorem 9. *If $m, n \geq 0$ and $0 \leq s \leq r$, then*

$$L_{m+n,r}(x) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{n}{j} x^i (\alpha s + \beta i + m)^{\overline{n-j}} L_r(m,i) L_{j,r-s}(x). \tag{20}$$

Proof. Given $\lambda \in \mathcal{P}_{m+n}$, suppose that there are exactly i non-special blocks of λ containing at least one member of $R = [r+1, r+m]$ where $0 \le i \le m$. Suppose further that within λ there are exactly $n-j$ members of $T = [r+m+1, r+m+n]$ where $0 \le j \le n$ that either (i) occur as an lr min in one of the first s special blocks, (ii) occur as an lr min in one of the non-special blocks containing a member of R, or (iii) directly follow a member of R or some other member of T either within any of the non-special blocks mentioned in (ii) or within any of the special blocks. Note that (i) and (iii) preclude the possibility of an element of T enumerated by $n-j$ from directly following the special element in any one of the final $r-s$ special blocks.

We now enumerate all $\lambda \in \mathcal{P}_{m+n}$ subject to the parameters i and j and show that it is given by the generic summand in (20). To do so, first observe that there are $x^i L_r(m,i)$ ways in which to arrange the elements of $[r+m]$ in their blocks. Let $X = \{x_1, \ldots, x_{n-j}\}$ denote the subset of T whose elements (written in increasing order) each satisfy one of the conditions (i)–(iii) above. Then there are $\binom{n}{j}$ ways in which to select the elements of X and, once this selection has been made, we consider adding $x_1, x_2, \ldots,$ sequentially, to the blocks already containing the elements of $[r+m]$. Note that there are $\alpha s + \beta i + m$ possibilities for the placement of x_1, upon considering separately (i)–(iii). In general, for $i > 1$, it is seen that there are $\alpha s + \beta i + m + i - 1$ ways in which to insert x_i, since in addition to the possibilities mentioned for x_1, the element x_i may be placed directly after any member of $x_1, \ldots, x_{i-1}$.

Thus, there are $(\alpha s + \beta i + m)^{\overline{n-j}}$ ways in which to insert the elements of X into the previous blocks. Further, a member of $T - X$ can go either in a non-special block containing no members of R or in any one of the final $r-s$ special blocks such that no element of R occurs between it and the special element in that block. There are then $L_{j,r-s}(x)$ ways in which to arrange the elements of $T - X$ in their blocks as there is no restriction on the number of additional non-special blocks that are to be occupied. Note that members of T belonging to one of the final $r-s$ special blocks and occurring to the right of the leftmost member of R in the block (if it exists) all belong to X, while those occurring to the left belong to $T - X$ (with all members belonging to $T - X$ if it is the case that the block contains no member of R). Therefore, given $\lambda \in \mathcal{P}_{m+n}$, the elements of $T - X$, and hence of X, may be retrieved by considering the contents of the final $r-s$ special blocks together with any non-special blocks that fail to contain an element of R. Considering all possible values of i and j thus implies the result. □

Remark 10. The case of (20) where $\alpha = \beta = 1$ and r, s are both even corresponds to [19, Theorem 3.3], which was shown algebraically by finding two different expansions of $(x+2r)^{\overline{m+n}}$ and equating like coefficients of $x^{\underline{k}}$.

Proceeding as in the proof of [22, Theorem 3.1] yields the following further formula, where

$n,k,r\geq 0$ and x_i and y_i for $i\geq 0$ denote arbitrary sequences:

$$\sum_{j=0}^{n+k}\binom{n+k}{j}x_jy_{n+k-j}L_{j,r}(x)=\sum_{p=0}^{k}\sum_{\ell=0}^{k}\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{j}\binom{k}{\ell}\binom{j}{i}x_{n+\ell+i-j}y_{k+j-\ell-i}$$
$$\times\, x^p(\alpha r+\beta p+\ell)^{\overline{n-j}}L_r(\ell,p)L_{i,0}(x). \quad (21)$$

We note an important special case of (21). Let $U_{n,r}(x)=\sum_{m=0}^{n}\left\lfloor{n \atop m}\right\rfloor_r x^m$ be the r-Lah polynomial of order n, the $r=0$ case of which being denoted simply by $U_n(x)$. Taking $x_i=1$ and $y_i=\delta_{i,0}$ for all $i\geq 0$ in (21), with $\alpha=\beta=1$, gives the formula

$$U_{n+k,r}(x)=\sum_{p=0}^{k}\sum_{j=0}^{n}\binom{n}{j}x^p(k+p+2r)^{\overline{n-j}}\left\lfloor{k \atop p}\right\rfloor_r U_j(x), \quad (22)$$

which is an r-Lah polynomial version of Spivey's formula [24] for the classical Bell numbers A000110.

4 Combinatorial proofs of prior identities

In this section, we provide combinatorial proofs of some prior formulas involving $L_r(n,k)$ and $L_{n,r}(x)$ that were shown previously by various algebraic methods.

We first prove a couple of relations involving $L_{n,r}(x)$ and the two kinds of Stirling numbers that occur in a slightly different form as [17, Proposition 5]. Let $\left[{n \atop k}\right]$ denote the (signless) Stirling number of the first kind A008275 and $\left\{{n \atop k}\right\}$ the Stirling number of the second kind A008277. Let $B_n(x)=\sum_{k=0}^{n}\left\{{n \atop k}\right\}x^k$ be the n-th Bell polynomial (see, e.g., [13] or [15]).

Theorem 11. *If $n,r\geq 0$, then*

$$L_{n,r}(x)=\sum_{k=0}^{n}\sum_{j=0}^{k}\left[{n \atop k}\right]\binom{k}{j}(\alpha r)^{k-j}\beta^jB_j(x) \quad (23)$$

and

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}L_{k,r}(x)=\sum_{k=0}^{n}\binom{n}{k}(\alpha r)^{n-k}\beta^kB_k(x). \quad (24)$$

Proof. To show (23), we form members $\lambda\in\mathcal{P}_n$ as follows. First arrange the elements of I according to a permutation of $[n]$ having exactly k cycles expressed in standard cycle form (i.e., smallest element first within each cycle, with cycles arranged from left to right in increasing order of first elements). Let $C_1,C_2,\ldots,C_k$ denote the cycles so obtained. We then select $k-j$ of the C_i, and for each chosen C_i, we write the elements contained therein in the same order within one of the special blocks (after the special element). If two or more C_i are selected for the same special block, say $C_{i_1},C_{i_2},\ldots,$ where $i_1>i_2>\cdots$, then we write all of the element contained in C_{i_1} first, followed by those in C_{i_2} and so on. Then there are $\binom{k}{j}(\alpha r)^{k-j}$ ways in which to select the $k-j$ cycles and arrange the elements contained therein. Note that the first element of each of the chosen C_i corresponds to a special block lr min and hence contributes a factor of α towards the weight. The remaining j cycles are then arranged according to an arbitrary partition of $[j]$ such that the contents of cycles going in the same block are written in descending order of their indices. Thus each cycle starter in this case corresponds to a non-special block lr min, which accounts for the factor of β^j. This aside, there are then $B_j(x)$ possibilities concerning the relative positions of the cycles within the various (non-special) blocks. Considering all k and j then yields uniquely all possible λ, which implies (23).

To show (24), first let $\mathcal{A}_{n,k}$ for $1 \leq k \leq n$ denote the set of ordered pairs (π, ρ), where π is a partition of I with k blocks and ρ is a member of $\mathcal{P}_k$ using the blocks of π as its non-special elements (together with the members of $[r]$), where blocks are ordered by the relative sizes of their smallest elements. Define the (signed) weight of $(\pi, \rho) \in \mathcal{A}_{n,k}$ as $(-1)^{n-k}\alpha^{\nu_1(\rho)}\beta^{\nu_2(\rho)}x^j$, where j denotes the number of non-special blocks of ρ. Then the left side of (24) is seen to give the sum of the weights of all members of $\mathcal{A}_n = \cup_{k=1}^n \mathcal{A}_{n,k}$, where we may assume $n \geq 1$.

Let $\mathcal{A}'_n$ denote the subset of $\mathcal{A}_n$ consisting of those (π, ρ) such that $\pi = \{r+1\}, \{r+2\}, \ldots, \{r+n\}$ and within all blocks of ρ, the singletons of π occur in descending order. To define an involution of $\mathcal{A}_n - \mathcal{A}'_n$, consider the largest $j \in I$, which we denote by j^*, such that either (i) the singleton $\{j\}$ occurs in π, with $\{j\}$ directly following another block B of π within some block of ρ such that j is greater than the largest element of B, or (ii) j is the largest element of some non-singleton block of π. Replacing option (i) with (ii) by removing the singleton $\{j^*\}$ from π and adding j^* to the block B, or vice versa if (ii) occurs, is seen to define an involution of $\mathcal{A}_n - \mathcal{A}'_n$. It always preserves the weight (since the number of blocks of ρ does not change and neither does its number of lr min), while it reverses the sign (the number of blocks of π changing by one). Upon considering the number k of singleton blocks of π occurring in the non-special blocks of ρ, the sum of the weights of all members of $\mathcal{A}'_n$ is seen to be given by the right side of (24), which completes the proof. □

The $L_r(n,k)$ serve as connection constants between the polynomial bases $((\alpha r + \beta x)^{\overline{n}})_{n \geq 0}$ and $(x^{\underline{n}})_{n \geq 0}$, as pointed out in [17, Corollary 4]. Here, we provide a combinatorial explanation as to why, which makes use of a sign-changing involution.

Theorem 12. *If $n, r \geq 0$, then*

$$(\alpha r + \beta x)^{\overline{n}} = \sum_{j=0}^{n} L_r(n,j) x^{\underline{j}}. \tag{25}$$

Proof. Given $\pi \in \mathcal{P}_{n,j}$, let π^* denote a permutation of the non-special blocks of π. Let $m(\pi^*)$ be the number of left-right minima of π^*, where it is understood that the blocks permuted by π^* (which are construed as the elements in a permutation) are ordered by the relative sizes of their respective smallest elements. Let $\widetilde{\mathcal{P}}_{n,j} = \{(\pi, \pi^*) : \pi \in \mathcal{P}_{n,j}\}$, where it is understood that π^* can range over all possible permutations of the non-special blocks of π for each π. Given $\widetilde{\pi} = (\pi, \pi^*) \in \widetilde{\mathcal{P}}_{n,j}$, define the (signed) weight $w(\widetilde{\pi})$ by

$$w(\widetilde{\pi}) = (-1)^{j-m(\pi^*)}\alpha^{\nu_1(\pi)}\beta^{\nu_2(\pi)}x^{m(\pi^*)}.$$

It is seen from the definitions that the right-hand side of (25) gives the sum of the weights of all $\widetilde{\pi} \in \widetilde{\mathcal{P}}_n$, where $\widetilde{\mathcal{P}}_n = \cup_{j=0}^{n}\widetilde{\mathcal{P}}_{n,j}$.

To complete the proof, we define a sign-reversing involution on $\widetilde{\mathcal{P}}_n$ as follows. In each block permuted by π^* within $\widetilde{\pi} = (\pi, \pi^*) \in \widetilde{\mathcal{P}}_{n,j}$, consider the ordering of the elements of I that it contains. Decompose the ordering s_ℓ of the elements contained in the ℓ-th block of π^* (from left to right) by

$$s_\ell = s_1^{(\ell)}\alpha_1^{(\ell)}s_2^{(\ell)}\alpha_2^{(\ell)}\cdots s_{r_\ell}^{(\ell)}\alpha_{r_\ell}^{(\ell)}, \qquad 1 \le \ell \le j,$$

where $s_1^{(\ell)} > s_2^{(\ell)} > \cdots > s_{r_\ell}^{(\ell)}$ denote the lr min of s_ℓ and the $\alpha_i^{(\ell)}$ are possibly empty. We call a sequence s_ℓ for some $\ell \in [j]$ *disqualifying* if (i) $r_\ell \ge 2$ (i.e., block ℓ contains at least two lr min), or (ii) $r_\ell = 1$ where $\ell > 1$, with the ℓ-th block of π^* (when viewed from left to right) having larger first element than its predecessor.

Let ℓ' denote the largest ℓ such that s_ℓ is disqualifying. If $s_{\ell'}$ is disqualifying via (i), then break off the initial segment $s_1^{(\ell')}\alpha_1^{(\ell')}$ of $s_{\ell'}$ and form a separate (contents-ordered) block with it to directly follow the remaining part of the parent block. Otherwise, reverse this operation if (ii) applies. One may verify that this defines a sign-reversing involution on $\widetilde{\mathcal{P}}_n$ since j always changes by one, and hence the sign, with the other factors in the definition of $w(\widetilde{\pi})$ unchanged. In particular, note that $m(\pi^*)$ does not change since blocks are ordered by the size of their smallest elements and thus an lr min is neither introduced nor removed when performing the operations above. The survivors of the involution are those members of $\widetilde{\mathcal{P}}_n$ in which the smallest element is first in each non-special block of π, with these blocks arranged in decreasing order of smallest elements from left to right (i.e., π^* corresponds to the permutation $j(j-1)\cdots 1$). Each additional non-special block within such members of $\widetilde{\mathcal{P}}_n$ then yields a factor of βx (as it corresponds to an lr min of π^*). Thus, the sum of the weights of the survivors is given by $(\alpha r+\beta x)(\alpha r+\beta x+1)\cdots(\alpha r+\beta x+n-1) = (\alpha r+\beta x)^{\overline{n}}$, since for each $i \in I$, one can either insert i as an lr min in a special block, as a non-special block starter, or as a direct successor of some member of $[r+1, r+i-1]$. □

The following reciprocity result was shown in [17] using generating functions.

Theorem 13. *If $n, r \ge 0$, then*

$$L_{n,r}^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}(-1)^k \left\lfloor {n \atop k} \right\rfloor L_{k,r}^{(-\alpha,-\beta)}(x). \tag{26}$$

Proof. Let $\mathcal{U}_{n,k}$ denote the set of ordered pairs $\rho = (\rho_1, \rho_2)$, where ρ_1 is a partition of $[n]$ into k contents-ordered blocks and $\rho_2 \in \mathcal{P}_k$, with the weight of ρ taken to be that of ρ_2, but with α and β replaced respectively by $-\alpha$ and $-\beta$. Let members of $\mathcal{U}_{n,k}$ have sign $(-1)^k$ and $\mathcal{U}_n = \cup_{k=0}^n \mathcal{U}_{n,k}$. Then the right side of (26) gives the sum of the (signed) weights of all members of $\mathcal{U}_n$. To show that the left side of (26) also achieves this, we first regard members ρ of $\mathcal{U}_n$ as follows. Let ρ' be obtained from ρ by arranging the blocks of ρ_1, ordered by relative size of their smallest elements, according to the partition ρ_2 (where r extra special elements are added that are distinct from the blocks of ρ_1). We let ρ' have sign and weight equal to that of ρ. Thus, the right side of (26) gives the sum of the weights of all configurations ρ', the set of which we again denote by $\mathcal{U}_n$.

We define an involution on $\mathcal{U}_n$ as follows. Let us refer to the blocks of ρ' whose elements are themselves blocks of ρ_1 as *superblocks*. In a left-to-right scan of the superblocks of $\rho' \in \mathcal{U}_n$, where the special superblocks are written before the non-special, consider the first, denoted by $\mathcal{B}$, which contains within its blocks at least two elements of $[n]$ altogether. Let B denote the rightmost block within $\mathcal{B}$. If B is not a singleton, then break off the final element of B and form a singleton block with it to follow B. If B is a singleton, then we add its element to the block that directly precedes it at the end. Note that this operation always reverses the sign and preserves the weight except in the cases (I) $B = \{\alpha b \beta a\}$ or (II) $B' = \{\alpha b \beta\}$ and $B = \{a\}$, where B' denotes the penultimate block of $\mathcal{B}$, $a < b$ are the two smallest elements of $[n]$ contained within the blocks of $\mathcal{B}$, and α and β are (possibly empty) sequences of elements of $[n]$. Note in this case that the weight is not preserved since moving a as indicated adds or takes away a factor of either α or β (depending on whether $\mathcal{B}$ is special or not) since both B and B' are lr min (as they correspond to the two 'smallest' blocks within $\mathcal{B}$).

If the elements a and b occur according to (I) or (II) above, then let $c < d$ denote the two smallest elements of $[n]$ contained within blocks of $\mathcal{B}$ excluding those containing a or b. (If there are less than two such elements of $[n]$ left at this point, then we stop.) We repeat, if possible, the involution above using the elements c and d just as we did a and b, where the block or blocks containing a and b are left undisturbed. If c and d cannot be moved without preserving the weight, then consider the two smallest elements in the remaining blocks and so on. We repeat this process until the operation of breaking or fusing blocks as described can be performed or until all of the blocks of $\mathcal{B}$ have been considered (except for possibly an initial singleton block). We then consider the next superblock $\mathcal{C}$ from left to right within ρ' containing at least two elements of $[n]$ within its blocks and repeat. We continue until it is possible to perform the involution on adjacent blocks as described within some superblock or all of the superblocks have been exhausted.

The set of survivors of this involution, which is denoted by $\mathcal{U}_n^*$, are those configurations in $\mathcal{U}_n$ in which each superblock $\mathcal{B}$ is comprised of a sequence of blocks where those containing a and b are of the form (I) or (II) above and occur at the end of $\mathcal{B}$, with the remaining blocks of $\mathcal{B}$ arranged in the same manner inductively from right to left upon considering the two smallest remaining elements. Note that each block within any superblock of a member of $\mathcal{U}_n^*$ is seen to be an lr min. Thus since each block receives a weight $-\alpha$ or $-\beta$, and the sign is -1 to the number of blocks, one may thus assign a weight of α or β to each block (depending on whether or not the superblock to which it belongs is special). Therefore, to complete the proof, it suffices to define a weight-preserving bijection between $\mathcal{U}_n^*$ and $\mathcal{P}_n$, where the associated weights for either set are α and β.

Let $\lambda \in \mathcal{U}_n^*$ and we replace the blocks within each superblock $\mathcal{B}$ of λ with a sequence of elements in $[n]$. If $\mathcal{B}$ is empty (in which case it is special), then it is left empty. If $\mathcal{B}$ contains one block that is a singleton, then simply remove the brackets enclosing the element contained therein. So assume $\mathcal{B}$ contains at least two elements of $[n]$ altogether in its blocks. If the last block within $\mathcal{B}$ is of the form (I) above, then form the word $a\alpha b\beta$, whereas if it is of form (II), then put $b\alpha a\beta$. Then consider the relative positions of the elements c and d defined above. If c and d occur together in a block as $\{\gamma d\delta c\}$, then write $c\gamma d\delta$ directly prior to the current word, whereas if they occur as $\{\gamma d\delta\}, \{c\}$, then put $d\gamma c\delta$ instead. We continue in this manner working from right to left within the blocks of $\mathcal{B}$, forming a sequence in $[n]$ with distinct letters. If $\mathcal{B}$ has an initial singleton block that is left after all other blocks have been used as described, then we simply write its entry at the beginning of the current word.

Thus, the sequence of blocks within $\mathcal{B}$ has been replaced by a word w such that its length equals the sum of the cardinalities of the blocks. Note that the number of lr min in w is seen to be the same as the number of blocks within β. This follows from the inductive manner in which w was formed. Note for example that if (I) holds, then only a out of $\{a, b\}$ corresponds to an lr min in w, whereas if (II) holds, then both a and b do so. Since each block within $\mathcal{B}$ is an lr min, it follows that the weight associated with $\mathcal{B}$ is the same as the weight of the word w. Forming a word for each of the superblocks of λ as described then results in a member $\widetilde{\lambda} \in \mathcal{P}_n$ having the same weight. Furthermore, one may verify that the mapping $\lambda \mapsto \widetilde{\lambda}$ is a bijection. It follows that the sum of the weights of all members of $\mathcal{U}_n^*$ is $L_{n,r}^{(\alpha,\beta)}(x)$, which completes the proof. □

Equivalent versions of the following identities involving transformations of the α and β parameters occur in [17] and were shown using generating function techniques. Their elegant form suggests trying to find some sort of a combinatorial explanation as to why.

Theorem 14. *If $n \geq 0$, $r \geq 1$ and $\beta' \neq 0$, then*

$$(\alpha r + \beta x)^{\overline{n}} = \sum_{j=0}^{n} (-1)^j L_r^{\left(\alpha + \frac{\alpha'\beta}{r\beta'}, \frac{\beta}{\beta'}\right)}(n, j)(\alpha' - \beta' x)^{\overline{j}} \tag{27}$$

and

$$L_r^{(\alpha,\beta)}(n, k) = \sum_{j=k}^{n} (-1)^j L_r^{\left(\alpha + \frac{\alpha'\beta}{r\beta'}, \frac{\beta}{\beta'}\right)}(n, j) L_1^{(\alpha', -\beta')}(j, k), \qquad 0 \leq k \leq n. \tag{28}$$

Proof. We first describe a set of configurations, the members of which have sum of weights given by the right-hand side of (27). Given $0 \leq j \leq n$, let $\mathcal{K}_j$ denote the set of ordered pairs (π, ρ) such that $\pi \in \mathcal{P}_{n,j}^{(r+1)}$ and ρ is a permutation of the non-special blocks of π wherein blocks corresponding to lr min may be marked (where it is understood that blocks of π are ordered by the relative sizes of their smallest elements). Let $\mathcal{K} = \cup_{j=0}^{n} \mathcal{K}_j$ and members of $\mathcal{K}_j$ have sign $(-1)^j$. We define a weighting of the members of $\mathcal{K}$ as follows. Let $\gamma_1(\pi)$ denote the number of lr min in the first r special blocks of π, where lr min for both special and non-special blocks are defined as before. Let $\gamma_2(\pi)$ denote the number of lr min in just the $(r+1)$-st special block of π and $\gamma_3(\pi)$ the number of lr min in its non-special blocks. Let $\delta_1(\rho)$ and $\delta_2(\rho)$ denote the number of marked and unmarked lr min in ρ, respectively. Note that an lr min of ρ corresponds to a block B of π whose smallest element is less than the smallest element of any block occurring to the left of B within ρ. Define the weight of the ordered pair $(\pi, \rho) \in \mathcal{K}$ by

$$\alpha^{\gamma_1(\pi)} \left(\frac{\alpha'\beta}{\beta'}\right)^{\gamma_2(\pi)} \left(\frac{\beta}{\beta'}\right)^{\gamma_3(\pi)} (\alpha')^{\delta_1(\rho)}(-\beta' x)^{\delta_2(\rho)}.$$

Then, upon considering the product of terms in $(\alpha' - \beta' x)^{\bar{j}}$ and recalling the definition of $L_r(n, j)$, it is seen that the right side of (27) gives the sum of the (signed) weights of all members of $\mathcal{K}$.

To define an involution on $\mathcal{K}$, the following definition will be useful. Suppose that the sequence of elements within a non-special block C of π is expressed as $s = a_1\tau_1a_2\tau_2\cdots a_p\tau_p$, where $a_1, \ldots, a_p$ denotes the complete set of lr min of s and $\tau_1, \ldots, \tau_p$ are possibly empty. Then we refer to a subsequence of elements of the from $a_i\tau_i$ for some $1 \leq i \leq p$ as a *section* of C. Similar terminology is applied to the special blocks of π, where it is understood in this case that s does not include the special element and hence can be empty. Let $\lambda = (\pi, \rho) \in \mathcal{K}$ and we pair λ with some $\lambda' \in \mathcal{K}$ of opposite weight. To define the mapping $\lambda \mapsto \lambda'$, first suppose that ρ has exactly t lr min which we denote by $Q_1, \ldots, Q_t$. Let $\mathcal{Q}_i$ be the (possibly empty) sequence of blocks of π occurring between Q_i and Q_{i+1} if $1 \leq i \leq t-1$ and to the right of Q_t if $i = t$.

We first consider the case when $\mathcal{Q}_t$ is non-empty with the blocks comprising $\mathcal{Q}_t$ containing at least two sections altogether. Let D denote the last block of $\mathcal{Q}_t$. If D contains two or more sections, then we remove the first section and form a separate block of π that is to directly follow D. We reverse this operation if D contains a single section whose first element is greater than the first element of the block directly preceding it. Thus, we may assume that D contains a single section whose first element is less than the first element of its predecessor. In this case, we consider the penultimate block of $\mathcal{Q}_t$ and apply one of the above operations if possible. If not, continue considering blocks of $\mathcal{Q}_t$ from right to left until it is the case that a block having two or more sections is encountered or one having a single section whose first element is greater than the first element of its predecessor in $\mathcal{Q}_t$. Note that if no such block is ever encountered, then it must be the case that the blocks in $\mathcal{Q}_t$ each have one section and occur in decreasing order of their smallest elements.

We now extend our involution to the last aforementioned case concerning $\mathcal{Q}_t$. Here, we also allow for $\mathcal{Q}_t$ to consist of a single block containing one section or for $\mathcal{Q}_t$ to be empty, with Q_t containing more than one section if the latter applies. Let E denote the first block of $\mathcal{Q}_t$. If the first element of E is greater than the first element of Q_t, then we write the elements comprising the section E directly prior to the current sequence of elements in Q_t. If the first element of Q_t is greater than that of E (or if $\mathcal{Q}_t$ is empty), then we break off the first section of Q_t and form a new block to directly follow. Observe that since the blocks of $\mathcal{Q}_t$ occur in decreasing order of first elements, these operations are inverses. Note further that in the latter case, the block Q_t must contain more than one section since $\min(Q_t) < \min(E)$ if $\mathcal{Q}_t$ is non-empty. Let λ' denote the member of $\mathcal{K}$ obtained by applying any of the preceding operations. In all cases, the magnitude of the weight of λ and λ' is the same since moving a section as described does not change the weighting of π with respect to its non-special blocks and neither does it affect the weighting of ρ since no lr min are introduced or removed. On the other hand, since the number j of non-special blocks of π changes by one in all cases, the sign of the weight is always reversed. Finally, one may verify that the mapping $\lambda \mapsto \lambda'$ is indeed an involution where defined.

So we may assume henceforth that Q_t contains a single section and is the final block of ρ (i.e., $\mathcal{Q}_t$ is empty). Now let i_0 denote the largest index $i < t$, if it exists, such that either $\mathcal{Q}_i$ is non-empty or $\mathcal{Q}_i$ is empty, but Q_i contains two or more sections. We then apply in this case the previous involution $\lambda \mapsto \lambda'$ to the blocks belonging to $\mathcal{Q}_{i_0} \cup \{Q_{i_0}\}$, leaving the remainder of ρ undisturbed. The set of survivors in $\mathcal{K}$ of this extended involution are those members (π, ρ) where i_0 fails to exist, i.e., those in which the non-special blocks of π all contain a single section and are arranged within ρ in decreasing order from left to right. In particular, every block of ρ corresponds to an lr min.

In this case, we let $\mathcal{U}$ denote the set consisting of the sections contained either within the $(r+1)$-st special block of π or within the non-special blocks of π corresponding to any marked lr min of ρ. Assume for now $\mathcal{U} \neq \varnothing$. We compare the various sections in $\mathcal{U}$ by comparing the sizes of their first elements. If the smallest section F in $\mathcal{U}$ belongs to a non-special block, then delete the block containing F from ρ and move F to the $(r+1)$-st special block, writing its sequence of elements so that they directly follow any elements already present. If F belongs to the $(r+1)$-st special block of π, necessarily as the final section, then we form a new (non-special) block containing only the elements in F, which we mark and insert into ρ so that ρ is still decreasing after the insertion. Note that this uniquely determines the position of F within ρ and implies that these operations are inverses of one another. Furthermore, the magnitude of the weight is preserved since a section within the $(r+1)$-st special block of π or within a non-special block of π corresponding to a marked lr min of ρ both contribute a factor of $\frac{\alpha'\beta}{\beta'}$ towards the overall weight. On the other hand, the sign is again reversed since a non-special block is either created or removed.

Thus, the set $\mathcal{K}'$ of survivors of all the preceding involutions are those (π, ρ) in which (i) the $(r+1)$-st special block of π contains only $r+1$, (ii) each non-special block of π contains one section, and (iii) ρ is the decreasing permutation, with no blocks in ρ marked. Within members of $\mathcal{K}'$, each new lr min in one of the first r special blocks of π contributes α and each non-special block of π within ρ contributes $-\frac{\beta}{\beta'}(-\beta' x) = \beta x$ towards the signed weight. Thus, there are $\alpha r + \beta x$ possibilities for each section starter. All other members of $[r+2, r+n+1]$ fail to be lr min and thus contribute one towards the weight. Upon successively considering each element of $[r+2, r+n+1]$ starting with $r+2$, it is seen that the sum of the weights of all the members of $\mathcal{K}'$ is given by $(\alpha r + \beta x)^{\overline{n}}$, which completes the proof of (27).

To show (28), we extend the proof of (27) and let $\mathcal{L}_j$ for $k \leq j \leq n$ denote the set of ordered pairs (π, τ) such that π is as before and τ is a partition of the non-special blocks of π arranged according to a member of $\mathcal{P}_{j,k}^{(1)}$. Let $\mathcal{L} = \cup_{j=k}^{n} \mathcal{L}_j$ and members of $\mathcal{L}_j$ have sign $(-1)^j$. Define the weight of $(\pi, \tau) \in \mathcal{L}$ by

$$\alpha^{\gamma_1(\pi)} \left(\frac{\alpha'\beta}{\beta'}\right)^{\gamma_2(\pi)} \left(\frac{\beta}{\beta'}\right)^{\gamma_3(\pi)} (\alpha')^{\nu_1(\tau)}(-\beta')^{\nu_2(\tau)}.$$

Here, the statistics γ_i are as in the proof of (27) and the ν_i are as in the second section above, with the latter now applied to the partition τ whose "elements" are blocks of π. (We refer to the blocks of τ as *superblocks*.) From the various definitions, it is seen that the sum of the (signed) weights of all members of $\mathcal{L}$ is given by the right side of (28).

We now define a sign-reversing, weight-preserving involution on $\mathcal{L}$. We first apply the involution defined above in the third and fourth paragraphs of this proof to the sequence of blocks of π contained in the special and each of the non-special superblocks of τ. Thus, we may assume that all non-special blocks of π contain one section and are arranged in decreasing order within each superblock of τ. Now let $\mathcal{V}$ denote the set consisting of sections that belong either to the $(r+1)$-st special block of π or to non-special blocks of π going in the special superblock of τ (necessarily in decreasing order). Assume $\mathcal{V} \neq \varnothing$ and let H be the smallest section in $\mathcal{V}$. We now apply the same type of involution as we did in conjunction with the set $\mathcal{U}$ above, moving the section H from a special to a non-special block, or vice versa. Combining this mapping with the previous, one may verify that this yields the desired involution on $\mathcal{L}$.

The set $\mathcal{L}'$ of survivors of this involution consists of those (π, τ) such that (i) the $(r+1)$-st special block of π contains only $r+1$, (ii) the special superblock of τ contains no blocks of π, and (iii) non-special blocks of π each have a single section and occur in decreasing order within the non-special superblocks of τ. Note that within members of $\mathcal{L}'$, each non-special block of π contributes $-\frac{\beta}{\beta'}(-\beta') = \beta$ towards the weight and may be viewed itself as some section of a larger contents-ordered block, upon removing the enclosing parentheses. Thus, combining the k non-special superblocks of τ with the first r special blocks of π, it is seen that the sum of the weights of all members of $\mathcal{L}'$ is given by $L_r^{(\alpha,\beta)}(n,k)$, which completes the proof of (28). □

Remark 15. Equivalent forms of (27) and (28) involving $S_{\alpha,\beta}(n,k)$ occur as [17, Corollary 18] and are given by

$$(-\alpha - \beta x)^{\overline{n}} = \sum_{j=0}^{n} (-1)^j S_{\alpha - \frac{\alpha'\beta}{\beta'}, \frac{\beta}{\beta'}}(n,j)(-\alpha' - \beta' x)^{\overline{j}}$$

and

$$S_{\alpha,\beta}(n,k) = \sum_{j=k}^{n} (-1)^j S_{\alpha - \frac{\alpha'\beta}{\beta'}, \frac{\beta}{\beta'}}(n,j) S_{\alpha',\beta'}(j,k).$$

Note that the first identity above can be obtained from (27) by replacing α, β and α' with $-\alpha/r$, $-\beta$ and $-\alpha'$, respectively, and using $L_r^{(\alpha,\beta)}(n,k) = S_{-r\alpha,-\beta}(n,k)$. The second identity may be obtained similarly from (28).

5 A further polynomial extension

In this concluding section, we derive a (p,q)-analogue of $L_r(n,k)$ and $L_{n,r}(x)$ by considering two further statistics on $\mathcal{P}_{n,k}$. Suppose that the special blocks of $\pi \in \mathcal{P}_{n,k}$ are labeled $0,1,\ldots,r-1$ from left to right. For each special block lr min x of π, let p_x denote the position number of the block containing x. Define $\sigma_1(\pi) = \sum_x p_x$, where the sum is taken over all possible x in π. Furthermore, suppose that the non-special blocks of π are arranged from left to right in increasing order of minimal elements and then labeled $0,1,\ldots,k-1$. Similarly, let $\sigma_2(\pi)$ be the sum of the block position numbers corresponding to the non-special lr min of π. Let $\mathbf{L}_r(n,k) = \mathbf{L}_r^{(\alpha,\beta)}(n,k;p,q)$ for $0 \le k \le n$ be given as the distribution

$$\mathbf{L}_r(n,k) = \sum_{\pi \in \mathcal{P}_{n,k}^{(r)}} \alpha^{\nu_1(\pi)}\beta^{\nu_2(\pi)}p^{\sigma_1(\pi)}q^{\sigma_2(\pi)}.$$

Define $\mathbf{L}_{n,r}(x) = \mathbf{L}_{n,r}^{(\alpha,\beta)}(x;p,q)$ by $\mathbf{L}_{n,r}(x) = \sum_{k=0}^n \mathbf{L}_r(n,k)x^k$. Note that $\mathbf{L}_r(n,k)$ and $\mathbf{L}_{n,r}(x)$ reduce to $L_r(n,k)$ and $L_{n,r}(x)$ when $p = q = 1$. Letting $\alpha = p$ and $\beta = q$, one gets the joint distribution on $\mathcal{P}_{n,k}$ for variants of the σ_1 and σ_2 statistics obtained by using instead the labelings $1,\ldots,r$ and $1,\ldots,k$ for the special and non-special blocks, respectively. We remark that σ_1 and σ_2 are related to a family of statistics on set partitions considered originally in [3] and later studied (see, e.g., [25]).

Let $m_q = 1 + q + \cdots + q^{m-1}$ for $m \ge 1$, with $0_q = 0$. By an argument similar to the proof of Theorem 1 above, we have the recurrence

$$\mathbf{L}_r(n,k) = \beta q^{k-1}\mathbf{L}_r(n-1,k-1) + (\alpha r_p + \beta k_q + n - 1)\mathbf{L}_r(n-1,k), \qquad n,k \ge 1, \tag{29}$$

with initial conditions $\mathbf{L}_r(n,0) = (\alpha r_p)^{\overline{n}}$ and $\mathbf{L}_r(0,k) = \delta_{k,0}$ for all $n,k \ge 0$.

To find an explicit formula for $\mathbf{L}_r(n,k)$, we make use of a previous result. Let $u(n,k)$ be an array defined recursively by

$$u(n,k) = u(n-1,k-1) + (a_{n-1} + b_k)u(n-1,k), \qquad n,k \ge 1, \tag{30}$$

and satisfying the initial conditions $u(n,0) = \prod_{i=0}^{n-1}(a_i + b_0)$ and $u(0,k) = \delta_{k,0}$ for $n,k \ge 0$, where the a_i and b_i denote arbitrary sequences and it is assumed that the b_i are distinct. Then there is the formula [12, Theorem 1.1]:

$$u(n,k) = \sum_{j=0}^{k} \left(\frac{\prod_{i=0}^{n-1}(b_j + a_i)}{\prod_{\substack{i=0\\ i\ne j}}^{k}(b_j - b_i)} \right), \qquad n,k \ge 0. \tag{31}$$

Dividing both sides of (29) by $\beta^k q^{\binom{k}{2}}$, and letting

$$\mathbf{L}_r^*(n,k) = \frac{\mathbf{L}_r(n,k)}{\beta^k q^{\binom{k}{2}}},$$

we have that $\mathbf{L}_r^*(n,k)$ satisfies recurrence (30) with $a_n = \alpha r_p + n$ and $b_k = \beta k_q$, where $\mathbf{L}_r^*(n,0) = (\alpha r_p)^{\overline{n}} = \prod_{i=0}^{n-1}(a_i + b_0)$ and $\mathbf{L}_r^*(0,k) = \delta_{k,0}$ for $n,k \geq 0$. Applying (31) gives

$$\mathbf{L}_r^*(n,k) = \sum_{j=0}^{k} \left(\frac{\prod_{i=0}^{n-1}(\beta j_q + \alpha r_p + i)}{\prod_{\substack{i=0\\ i\neq j}}^{k}(\beta j_q - \beta i_q)} \right) = \sum_{j=0}^{k} \frac{\prod_{i=0}^{n-1}(\alpha r_p + \beta j_q + i)}{(-1)^{k-j}\beta^k q^{\binom{j}{2}+j(k-j)} j_q^! (k-j)_q^!},$$

where $j_q^! = 1_q 2_q \cdots j_q$ denotes the q-factorial. Let $\binom{k}{j}_q = \frac{k_q^!}{j_q^!(k-j)_q^!}$ for $0 \leq j \leq k$ denote the q-binomial coefficient. By the preceding, we have the following explicit formula for $\mathbf{L}_r(n,k)$.

Theorem 16. *If $n,k,r \geq 0$, then*

$$\mathbf{L}_r(n,k) = \frac{1}{k_q^!} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \binom{k}{j}_q (\alpha r_p + \beta j_q)^{\overline{n}}. \tag{32}$$

Note that (32) reduces to (13) when $p = q = 1$. We now establish the log-concavity of each row within the array $\mathbf{L}_r(n,k)$. Recall that a sequence d_n of non-negative real numbers is said to be log-concave if $d_n^2 \geq d_{n-1}d_{n+1}$ for all n.

Theorem 17. *Let $n \geq 2$ be fixed. If $\alpha, \beta, p, r \geq 0$ and $0 \leq q \leq 1$, then the sequence $\mathbf{L}_r(n,k)$ for $0 \leq k \leq n$ is log-concave.*

Proof. Let $\mathbf{M}_r(n,k) = q^{-\binom{k}{2}}\mathbf{L}_r(n,k)$, where we may assume $q \neq 0$. Then $\mathbf{L}_r(n,k)^2 \geq \mathbf{L}_r(n,k-1)\mathbf{L}_r(n,k+1)$ if and only if $q^{-1}\mathbf{M}_r(n,k)^2 \geq \mathbf{M}_r(n,k-1)\mathbf{M}_r(n,k+1)$. Since $0 < q \leq 1$, it suffices to establish the log-concavity of $\mathbf{M}_r(n,k)$. To do so, we proceed by induction on n, the $n = 2$ case following from a direct calculation. Let $n \geq 3$ and $1 \leq k \leq n-1$. If $\beta = 0$, then the result is trivial, so assume further $\beta \neq 0$, in which case $\mathbf{M}_r(n,k) > 0$ for $1 \leq k \leq n$. Then $\mathbf{M}_r(n,k)^2 \geq \mathbf{M}_r(n,k-1)\mathbf{M}_r(n,k+1)$ if and only if

$$\begin{aligned}
&\big(\beta\mathbf{M}_r(n-1,k-1) + (\alpha r_p + \beta k_q + n - 1)\mathbf{M}_r(n-1,k)\big)^2 \\
&\quad \geq \big(\beta\mathbf{M}_r(n-1,k-2) + (\alpha r_p + \beta(k-1)_q + n - 1)\mathbf{M}_r(n-1,k-1)\big) \cdot \big(\beta\mathbf{M}_r(n-1,k) \\
&\qquad + (\alpha r_p + \beta(k+1)_q + n - 1)\mathbf{M}_r(n-1,k+1)\big).
\end{aligned} \tag{33}$$

Upon expanding both sides and comparing the various terms, it suffices to show

$$(\alpha r_p + \beta k_q + n - 1)^2 \geq (\alpha r_p + \beta(k-1)_q + n - 1)\,(\alpha r_p + \beta(k+1)_q + n - 1) \tag{34}$$

and

$$(\alpha r_p + \beta(2k_q - (k-1)_q) + n - 1)\,\mathbf{M}_r(n-1,k-1)\mathbf{M}_r(n-1,k) \\ \geq (\alpha r_p + \beta(k+1)_q + n - 1)\,\mathbf{M}_r(n-1,k-2)\mathbf{M}_r(n-1,k+1), \tag{35}$$

from which (33) would follow by induction. Note that (34) may be simplified to

$$2(\alpha r_p + n - 1)k_q + \beta k_q^2 \geq (\alpha r_p + n - 1)((k-1)_q + (k+1)_q) + \beta(k-1)_q(k+1)_q.$$

The last inequality follows from observing that $2k_q \geq (k-1)_q + (k+1)_q$ for $0 \leq q \leq 1$ and $k_q^2 \geq (k-1)_q(k+1)_q$ for all $q \geq 0$. Inequality (35) follows from observing that $\mathbf{M}_r(n-1,k-1)\mathbf{M}_r(n-1,k) \geq \mathbf{M}_r(n-1,k-2)\mathbf{M}_r(n-1,k+1)$ since the ratio $\frac{\mathbf{M}_r(n-1,\ell)}{\mathbf{M}_r(n-1,\ell+1)}$ is increasing by the induction hypothesis and since $2k_q - (k-1)_q \geq (k+1)_q$, which completes the proof. □

Remark 18. The $p = q = r = 1$ case of Theorem 17 is equivalent to [17, Corollary 13], which was shown by a different method making use of Newton's inequality.

Extending prior proofs yields generalizations of several of the identities given above for $L_r(n,k)$. For example, extending the combinatorial proof of (16) to account for the σ_1 and σ_2 statistics gives

$$\mathbf{L}_r(n+m,k) = \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\left(\alpha p^s(r-s)_p\right)^{\overline{i+j}}\,\mathbf{L}_s(n+m-i-j,k), \qquad 0 \leq s \leq r, \tag{36}$$

the $m = 0$ case of which is

$$\mathbf{L}_r(n+m,k) = \sum_{j=k}^{n}\binom{n}{j}(\alpha p^s(r-s)_p)^{\overline{n-j}}\mathbf{L}_s(j,k). \tag{37}$$

Note that (37) may also be shown by an algebraic argument comparable to the one given above for (14), using (32) in place of (13).

The proof given above for (19) may be generalized to give

$$\mathbf{L}_r^{(\alpha,\beta)}(n+1,m+j+1;p,q) = \beta q^m \sum_{k=m}^{n}\sum_{\ell=j}^{n-k}\binom{n-k}{\ell}\left(\alpha r_p + \beta(m+1)_q + k + 1\right)^{\overline{n-k-\ell}} \\ \times \mathbf{L}_r^{(\alpha,\beta)}(k,m;p,q)\mathbf{L}_0^{(\alpha,\beta q^{m+1})}(\ell,j;p,q). \tag{38}$$

Note that the q^m factor at the beginning accounts for the lr min created when $m+1$ is initially placed in the $(k+1)$-st non-special block. Also, the β parameter must be replaced by βq^{m+1} in the final factor within the summand since each lr min within the final j non-special blocks contributes an extra $m+1$ (more than it ordinarily would) towards the σ_2 statistic value.

Finally, extending prior arguments yields the following recurrence for $\mathbf{L}_{n,r}(x)$:

$$\mathbf{L}_{m+n,r}^{(\alpha,\beta)}(x;p,q) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{n}{j}x^i\left(\alpha p^{r-s}s_p + \beta i_q + m\right)^{\overline{n-j}}\mathbf{L}_{r}^{(\alpha,\beta)}(m,i;p,q)\mathbf{L}_{j,r-s}^{(\alpha,\beta q^i)}(x;p,q), \tag{39}$$

which provides a (p,q)-analogue of (20).

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