Journal of Integer Sequences, Vol. 24 (2021),
Article 21.8.3

The Asymptotics of Factorials, Binomial Coefficients and Catalan Numbers

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Abstract

We present a variety of not-well-known asymptotic series for factorials, binomial coefficients and Catalan numbers, all having only even or odd powers. We discuss the significance of this property in terms of the asymptotic evenness or oddness of the underlying quantities.

1 Introduction and statement of results

Probably the best-known asymptotic series in existence is Stirling's series for $n!$ [1, Equation 5.11.3]:

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \cdots\right). \tag{1}$$

There is no simple explicit formula for the coefficients in this series, but the more fundamental object is the corresponding series for $\ln n!$ [1, Equation 5.11.1], which takes the form

$$\ln n! \sim \ln\left(\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}\right) + \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)n^{2i-1}}, \tag{2}$$

where B_i denotes the ith Bernoulli number. Note the sum in Eq. (2) involves only odd powers of n, making it easier to use than Eq. (1). Stirling's series can be used to derive asymptotic series for many functions related to the factorial, such as the central binomial coefficients

$$\mathrm{CBC}(n) = \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \frac{21}{32768n^4} + \cdots\right), \tag{3}$$

and the Catalan numbers [2, Section 7.2.1.6, Formula (16)],

$$\mathrm{Cat}(n) = \frac{(2n)!}{n!(n+1)!} \sim \frac{4^n}{\sqrt{\pi n^3}}\left(1 - \frac{9}{8n} + \frac{145}{128n^2} - \frac{1155}{1024n^3} + \frac{36939}{32768n^4} + \cdots\right). \tag{4}$$

We note that as a direct result of the "odd powers only" series for $\ln n!$ there is also an "odd powers only" series for $\ln \mathrm{CBC}(n)$

$$\ln \mathrm{CBC}(n) \sim \ln\left(\frac{4^n}{\sqrt{\pi n}}\right) + \sum_{i=1}^{\infty} \frac{B_{2i}}{i(2i-1)n^{2i-1}}\left(\frac{1}{2^{2i}} - 1\right). \tag{5}$$

The aim of this paper is to present a variety of alternative asymptotic series to the standard ones just presented, and some generalizations, including some very surprising results which have barely appeared in the literature. For example, it turns out that when the central binomial coefficients are expanded in powers of $n + \frac{1}{4}$, and when the Catalan numbers are expanded in powers of $n + \frac{3}{4}$, they have asymptotic expansions involving only even powers, viz.:

$$\mathrm{CBC}(n) \sim \frac{4^n}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}\left(1 - \frac{1}{64\left(n+\frac{1}{4}\right)^2} + \frac{21}{8192\left(n+\frac{1}{4}\right)^4} - \frac{671}{524288\left(n+\frac{1}{4}\right)^6} + \cdots\right) \tag{6}$$

$$\mathrm{Cat}(n) \sim \frac{4^n}{\sqrt{\pi\left(n+\frac{3}{4}\right)^3}}\left(1 + \frac{5}{64\left(n+\frac{3}{4}\right)^2} + \frac{21}{8192\left(n+\frac{3}{4}\right)^4} + \frac{715}{524288\left(n+\frac{3}{4}\right)^6} + \cdots\right) \tag{7}$$

Each of these results is remarkable in its own right: There would seem, *ab initio*, to be no good reason to expand the central binomial coefficients in terms of $n + \frac{1}{4}$ and the Catalan numbers in terms of $n+\frac{3}{4}$. But the results are even more outlandish in juxtaposition: the nth Catalan number is just the nth central binomial coefficient divided by $n+1$. But, somehow, this act of division morphs a series involving only even powers of $n + \frac{1}{4}$ into one involving only even powers of $n + \frac{3}{4}$.

Using the formulae

$$\mathrm{CBC}(n) = \frac{4^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+2)}, \qquad \mathrm{Cat}(n) = \frac{4^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)},$$

the series (6) and (7) can be obtained as special cases of an asymptotic expansion in even powers of the ratio of two gamma functions obtained by Fields [3] and mentioned in the book of Luke [4, Section 2.11, Equation (14)] and in the DLMF [1, Subsection 5.11]. However, given the importance of central binomial coefficients, and, in particular, the Catalan numbers [5], the series (6) and (7) merit being better known in their own right. We thank P. Luschny for the observations in this paragraph, bringing references [3] and [4] to our attention, and publicizing the series (6) and (7) on the website [6].

There are corresponding series for $\ln \mathrm{CBC}(n)$ and $\ln \mathrm{Cat}(n)$ for which there are explicit expressions for the coefficients, viz.:

$$\begin{aligned}\ln \mathrm{CBC}(n) &\sim \ln\left(\frac{4^n}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}\right) + \sum_{i=1}^{\infty} \frac{E_{2i}}{4^{2i+1} i \left(n+\frac{1}{4}\right)^{2i}} \qquad (8)\\ &= \ln\left(\frac{4^n}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}\right) - \frac{1}{64\left(n+\frac{1}{4}\right)^2} + \frac{5}{2048\left(n+\frac{1}{4}\right)^4} - \frac{61}{49152\left(n+\frac{1}{4}\right)^6} + \cdots;\\ \ln \mathrm{Cat}(n) &\sim \ln\left(\frac{4^n\sqrt{n+\frac{3}{4}}}{\sqrt{\pi}\left(n+\frac{1}{2}\right)(n+1)}\right) - \sum_{i=1}^{\infty} \frac{E_{2i}}{4^{2i+1} i \left(n+\frac{3}{4}\right)^{2i}} \qquad (9)\\ &= \ln\left(\frac{4^n\sqrt{n+\frac{3}{4}}}{\sqrt{\pi}\left(n+\frac{1}{2}\right)(n+1)}\right) + \frac{1}{64\left(n+\frac{3}{4}\right)^2} - \frac{5}{2048\left(n+\frac{3}{4}\right)^4}\\ &\quad + \frac{61}{49152\left(n+\frac{3}{4}\right)^6} + \cdots,\end{aligned}$$

where E_i denotes the ith Euler number.

There are several well-known variations on the original Stirling series, for example the convergent version of Stirling's series [7] and the Lanczos approximation [8, 9, 10]. Another often overlooked series is the expansion of $\ln n!$ in powers of $n+\frac{1}{2}$. Of course, any asymptotic expansion in (negative) powers of n can be rewritten as an asymptotic expansion in powers of $n+a$ for any constant a. The remarkable fact about the expansion of $\ln n!$ in powers of $n+\frac{1}{2}$ is that like the standard expansion (2), it only contains odd powers of $n+\frac{1}{2}$. Explicitly, we have

$$\ln n! \sim \ln\left(\sqrt{2\pi}\left(n+\frac{1}{2}\right)^{n+\frac{1}{2}} e^{-n-\frac{1}{2}}\right) + \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)\left(n+\frac{1}{2}\right)^{2i-1}}\left(\frac{1}{2^{2i-1}} - 1\right). \qquad (10)$$

Writing

$$\ln \mathrm{CBC}(n) = \ln\left((2n+1)!\right) - \ln(2n+1) - 2\ln\left(n!\right),$$

and using the series (2) to expand the $\ln\left((2n+1)!\right)$ factor and the series (10) to expand the $\ln\left(n!\right)$ factor, gives an "odd powers only" series for $\ln \mathrm{CBC}(n)$ in powers of $n+\frac{1}{2}$:

$$\ln \mathrm{CBC}(n) \sim \ln\left(\frac{4^n}{\sqrt{\pi\left(n+\frac{1}{2}\right)}}\right) + \sum_{i=1}^{\infty} \frac{B_{2i}}{i(2i-1)\left(n+\frac{1}{2}\right)^{2i-1}}\left(1-\frac{1}{2^{2i}}\right). \tag{11}$$

This is the third different series we have seen for $\ln \mathrm{CBC}(n)$; Eq. (5) and Eq. (11) have only odd powers and Eq. (8) has only even powers. As we shall see in the sequel, the fact the coefficients in the two "odd powers only" series (5) and (11) are "opposite and equal" gives rise to the existence of the "even powers only" series (8). Other binomial coefficients also have asymptotic expansions with only odd powers. We will show that for any integer m

$$\ln\binom{2n}{n+m} \sim \ln\left(\frac{4^n}{\sqrt{\pi\left(n+\frac{1}{2}\right)}}\right) + \sum_{i=1}^{\infty} \frac{2^{-2i}B_{2i} + B_{2i}(m) - 2^{1-2i}B_{2i}(2m)}{i(2i-1)\left(n+\frac{1}{2}\right)^{2i-1}} \tag{12}$$

and

$$\ln\binom{2n-1}{n+m} \sim \ln\left(\frac{2^{2n-1}(n-m)}{\sqrt{\pi n}(n+m)}\right) + \sum_{i=1}^{\infty} \frac{2^{-2i}B_{2i} - B_{2i}(m)}{i(2i-1)n^{2i-1}}, \tag{13}$$

where $B_j(x)$ denotes the jth Bernoulli polynomial [1, Section 24]. The series (11) is obtained from the case $m=0$ of (12) using the result $B_j(0) = B_j$.

Having stated our main results (the series (6)–(13)), the rest of this paper proceeds as follows: in Section 2 we discuss what it means when an asymptotic series has only odd or only even terms, and show how to prove the existence of such series. In Section 3 we present proofs of the explicit forms of the various "odd powers only" results listed above. In Section 4 we do the same for the "even powers only" series, including a generalization. Throughout the continuation of this paper we extend the factorial, CBC and Cat functions beyond integer values by replacing $n!$ by $\Gamma(n+1)$, and defining

$$\mathrm{CBC}(n) = \frac{\Gamma(2n+1)}{\Gamma(n+1)^2}, \qquad \mathrm{Cat}(n) = \frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+2)}.$$

Whenever necessary (for the definition of ln and fractional powers) we use a branch cut along the negative real axis in the complex n-plane. All the series we have given above are valid in any sector of the complex n plane with $\arg(n)$ bounded away from π.

2 What does it mean for an asymptotic series to have only odd or only even powers?

The fact that the standard series (5) for $\ln\Gamma(n+1)$ has only odd powers is usually thought of as related to the fact that all the odd Bernoulli numbers vanish except $B_1 = -\frac{1}{2}$. But in fact it is a statement about the function $\ln\Gamma(n+1)$. If the Taylor or Laurent series of a function consists of only even or only odd powers, then the function must be even or odd. Similarly, if the asymptotic series of a function consists of only even or only odd powers, then the function must be even or odd, modulo exponentially small terms. Thus the absence of even powers in the series in Eq. (2) indicates that the function

$$f(n) = \ln\left(\frac{\Gamma(n+1)e^n}{\sqrt{2\pi}n^{n+\frac{1}{2}}}\right)$$

is odd modulo exponentially small terms, i.e., that $f(n)+f(-n)$ is exponentially small, at least whenever the asymptotic series for $f(n)$ and $f(-n)$ are valid (which in this case means in any sector of the complex plane with $\arg(n)$ bounded away from 0 and π). To check this is straightforward: because of the choice of branch cut along the negative real axis, for $\operatorname{Im}(n)>0$ we have $(-n)^{-n+\frac{1}{2}} = e^{i\pi\left(n-\frac{1}{2}\right)}n^{-n+\frac{1}{2}}$, and thus

$$\begin{aligned} f(n)+f(-n) &= \ln\left(\frac{\Gamma(n+1)\Gamma(1-n)}{2\pi n e^{i\pi\left(n-\frac{1}{2}\right)}}\right) \\ &= \ln\left(\frac{\Gamma(n)\Gamma(1-n)}{2\pi e^{i\pi\left(n-\frac{1}{2}\right)}}\right) \qquad (\text{using } \Gamma(n+1)=n\Gamma(n)) \\ &= -\ln\left(2e^{i\pi\left(n-\frac{1}{2}\right)}\sin\pi n\right) \quad (\text{using } \Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin\pi n}) \\ &= -\ln\left(1-e^{2\pi i n}\right), \end{aligned}$$

which is exponentially small if $\operatorname{Im}(n)>0$.

As further examples of this technique we have the following:

Theorem 1.

(a) For every integer m, the quantity $\ln\left(\frac{\sqrt{\pi\left(n+\frac{1}{2}\right)}}{4^n}\binom{2n}{n+m}\right)$ has an asymptotic expansion involving only odd powers of $n+\frac{1}{2}$. (See (12).)

(b) For every integer m, the quantity $\ln\left(\frac{\sqrt{\pi n}}{2^{2n-1}}\frac{n+m}{n-m}\binom{2n-1}{n+m}\right)$ has an asymptotic expansion involving only odd powers of n. (See (13).)

(c) For every integer k, the quantity $\ln\left(\frac{\sqrt{\pi}\left(n+\frac{1}{4}+\frac{k}{2}\right)^{k+\frac{1}{2}}}{4^n}\frac{(2n)!}{n!(n+k)!}\right)$ has an asymptotic expansion involving only even powers of $n+\frac{1}{4}+\frac{k}{2}$. (The special case $k=0$ gives the series (8). The special case $k=1$ gives the series (9). Since the property of being an even series is preserved under exponentiation, these in turn give rise to the even series (6) and (7).)

Proof. The proofs of (a) and (b) are similar so we omit (b).

(a) Here we want to show that

$$f(n) = \ln\left(\frac{\sqrt{\pi\left(n+\frac{1}{2}\right)}}{4^n}\frac{\Gamma(2n+1)}{\Gamma(n+m+1)\Gamma(n-m+1)}\right)$$

is "almost odd" as a function of $n+\frac{1}{2}$. Define

$$g(n) = f\left(n-\frac{1}{2}\right) = \ln\left(\frac{\sqrt{\pi n}}{4^{n-\frac{1}{2}}}\frac{\Gamma(2n)}{\Gamma\left(n+m+\frac{1}{2}\right)\Gamma\left(n-m+\frac{1}{2}\right)}\right).$$

For $\operatorname{Im}(n) > 0$ we then have

$$\begin{aligned}
g(n)+g(-n) &= \ln\left(\frac{-4i\pi n\Gamma(2n)\Gamma(-2n)}{\Gamma\left(n+m+\frac{1}{2}\right)\Gamma\left(n-m+\frac{1}{2}\right)\Gamma\left(-n+m+\frac{1}{2}\right)\Gamma\left(-n-m+\frac{1}{2}\right)}\right) \\
&= \ln\left(\frac{2i\sin\left(\left(n+m+\frac{1}{2}\right)\pi\right)\sin\left(\left(n-m+\frac{1}{2}\right)\pi\right)}{\sin 2n\pi}\right) \\
&\quad \text{(using the reflection formula 3 times)} \\
&= \ln\left(\frac{2i\cos^2(n\pi)}{\sin 2n\pi}\right) \\
&= \ln\left(\frac{1+e^{2in\pi}}{1-e^{2in\pi}}\right),
\end{aligned}$$

and the latter is exponentially small.

(c) Here we want to show that

$$f(n) = \ln\left(\frac{\sqrt{\pi}\left(n+\frac{1}{4}+\frac{k}{2}\right)^{k+\frac{1}{2}}}{4^n}\frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+k+1)}\right)$$

is "almost even" as a function of $n+\frac{1}{4}+\frac{k}{2}$. The calculation is simplified if we first exploit the duplication formula for the gamma function $\Gamma(2z) = \frac{1}{\sqrt{\pi}}2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$ to write $\frac{\sqrt{\pi}}{4^n}\frac{\Gamma(2n+1)}{\Gamma(n+1)} = \Gamma\left(n+\frac{1}{2}\right)$. Then we have the simplified formula

$$f(n) = \ln\left(\left(n+\frac{1}{4}+\frac{k}{2}\right)^{k+\frac{1}{2}}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+k+1)}\right).$$

Define

$$g(n) = f\left(n-\frac{1}{4}-\frac{k}{2}\right) = \ln\left(n^{k+\frac{1}{2}}\frac{\Gamma\left(n+\frac{1}{4}-\frac{k}{2}\right)}{\Gamma\left(n+\frac{3}{4}+\frac{k}{2}\right)}\right).$$

For $\mathrm{Im}(n) > 0$ we have

$$g(n) - g(-n) = \ln\left(e^{i\pi\left(k+\frac{1}{2}\right)} \frac{\Gamma\left(n+\frac{1}{4}-\frac{k}{2}\right)\Gamma\left(-n+\frac{3}{4}+\frac{k}{2}\right)}{\Gamma\left(n+\frac{3}{4}+\frac{k}{2}\right)\Gamma\left(-n+\frac{1}{4}-\frac{k}{2}\right)}\right)$$

$$= \ln\left(e^{i\pi\left(k+\frac{1}{2}\right)} \frac{\sin\left(\pi\left(n+\frac{3}{4}+\frac{k}{2}\right)\right)}{\sin\left(\pi\left(n+\frac{1}{4}-\frac{k}{2}\right)\right)}\right) \quad \text{(using the reflection formula twice)}$$

$$= \ln\left(\frac{1-qe^{2\pi in}}{1-q^{-1}e^{2\pi in}}\right),$$

where $q = e^{i\pi\left(k+\frac{3}{2}\right)}$. The answer is clearly exponentially small.

□

It should be emphasized that in all the calculations above we assume that $\arg(n)$ is bounded away from 0 and π. As the real axis is approached the functions will no longer exhibit "almost odd" or "almost even" behavior (there are singularities on the negative real axis).

The technique we have used in the theorem is sufficient to prove the absence of odd or even terms in all of the series given in the introduction. But the technique does not give explicit expressions for the coefficients. This requires some further calculations and we now turn to these.

3 Some series with only odd powers

Proof of (10). The proof of the alternative series (10) for $\ln n!$ is very simple. From the duplication formula for the gamma function we have

$$\Gamma(n+1) = \frac{\sqrt{\pi}}{2^{2n+1}} \frac{\Gamma\left(2\left(n+\frac{1}{2}\right)+1\right)}{\Gamma\left(\left(n+\frac{1}{2}\right)+1\right)}.$$

Applying the logarithm to both sides and using the standard series for $\ln\Gamma(z+1)$ twice on the right clearly yields a series for $\Gamma(n+1)$ in powers of $n+\frac{1}{2}$, which is precisely (10).

The series can also be obtained directly. We recall that the standard series for $\ln n!$ can be obtained from the Euler-Maclaurin summation formula [11, Section 14, Equation (18)]

$$\sum_{i=1}^{n} f(i) \sim \int^{n} f(x)dx + C + \frac{1}{2}f(n) + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(n),$$

by setting $f(x) = \ln x$. The alternative series can be obtained from the "midpoint version" of the Euler-Maclaurin summation formula [11, Section 14, Equation (19)]

$$\sum_{i=1}^{n} f(i) \sim \int^{n+\frac{1}{2}} f(x)dx + C' + \sum_{j=1}^{\infty} \frac{B_{2j}\left(\frac{1}{2}\right)}{(2j)!} f^{(2j-1)}\left(n+\frac{1}{2}\right).$$

Note that $B_{2j}\left(\frac{1}{2}\right) = B_{2j}\left(2^{1-2j}-1\right)$ [1, Equation 24.4.27]. □

Proof of (12) and (13). The proofs of (12) and (13) are similar, so we give full details just for the latter, and an outline for the former.

We start the proof of (13) by writing

$$\ln\binom{2n-1}{n+m} = \ln\left(\frac{n-m}{2n}\frac{(2n)!}{(n+m)!(n-m)!}\right). \tag{14}$$

We now apply the standard expansion (2) of $\ln z!$ to this expression 3 times. Each application gives a ln term (the leading order term) and an infinite series. Ignoring the three infinite series for now gives the leading order term

$$\begin{aligned}
&\ln\left(\frac{1}{\sqrt{2\pi}}\frac{n-m}{2n}\frac{(2n)^{2n+\frac{1}{2}}}{(n+m)^{n+m+\frac{1}{2}}(n-m)^{n-m+\frac{1}{2}}}\right)\\
&= \ln\left(\frac{2^{2n-1}}{\sqrt{\pi n}}\frac{n-m}{n+m}\right) + \ln\left(\frac{1}{\left(1+\frac{m}{n}\right)^{n+m-\frac{1}{2}}\left(1-\frac{m}{n}\right)^{n-m+\frac{1}{2}}}\right)\\
&= \ln\left(\frac{2^{2n-1}}{\sqrt{\pi n}}\right) - \left(n+m-\frac{1}{2}\right)\ln\left(1+\frac{m}{n}\right) - \left(n-m+\frac{1}{2}\right)\ln\left(1-\frac{m}{n}\right).
\end{aligned}$$

Using the Taylor series $\ln(1+x) = \sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}x^i$ to expand the logarithms in the second and third terms, we find the leading order term of (14) is

$$\ln\left(\frac{2^{2n-1}}{\sqrt{\pi n}}\frac{n-m}{n+m}\right) + \sum_{i=1}^{\infty}\frac{m^{2i-1}(i-m)}{i(2i-1)n^{2i-1}}. \tag{15}$$

We now have to incorporate the correction terms (the infinite series coming from the 3 applications of (2) to (14)). These are

$$\begin{aligned}
&\sum_{i=1}^{\infty}\frac{B_{2i}}{2i(2i-1)}\left(\frac{1}{(2n)^{2i-1}} - \frac{1}{(n+m)^{2i-1}} - \frac{1}{(n-m)^{2i-1}}\right)\\
&= \sum_{i=1}^{\infty}\frac{B_{2i}}{2i(2i-1)n^{2i-1}}\left(\frac{1}{2^{2i-1}} - \left(1+\frac{m}{n}\right)^{1-2i} - \left(1-\frac{m}{n}\right)^{1-2i}\right).
\end{aligned}$$

Using the binomial theorem $(1+x)^{1-2i} = \sum_{r=0}^{\infty}\binom{2i-2+r}{r}(-x)^r$ twice and rearranging the sums, this can be written

$$
\begin{aligned}
&\sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)(2n)^{2i-1}} - \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)n^{2i-1}} \sum_{r=0}^{\infty} \binom{2i-2+r}{r} \left(\left(-\frac{m}{n}\right)^r + \left(\frac{m}{n}\right)^r \right) \\
&= \sum_{i=1}^{\infty} \frac{B_{2i}}{i(2i-1)(2n)^{2i-1}} - \sum_{i=1}^{\infty} \sum_{s=0}^{\infty} \frac{B_{2i}}{i(2i-1)n^{2i-1}} \frac{(2i-2+2s)!}{(2i-2)!(2s)!} \left(\frac{m}{n}\right)^{2s} \\
&= \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)(2n)^{2i-1}} - \sum_{j=1}^{\infty} \sum_{i=0}^{j} \frac{2(2j-2)! B_{2i} m^{2(j-i)}}{(2i)!(2(j-i))! n^{2j-1}} \\
&= \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)(2n)^{2i-1}} - \sum_{i=1}^{\infty} \sum_{j=0}^{i} \frac{(2i)! B_{2j} m^{2(i-j)}}{i(2i-1)(2j)!(2(i-j))! n^{2i-1}} \\
&= \sum_{i=1}^{\infty} \frac{1}{i(2i-1)n^{2i-1}} \left(\frac{B_{2i}}{2^{2i}} - \sum_{j=0}^{i} \binom{2i}{2j} B_{2j} m^{2(i-j)} \right). \qquad (16)
\end{aligned}
$$

Here between the first and second line in the double sum we have replaced r by $2s$ as only even values of r contribute; between the second and the third line we have replaced the index s with a new index $j = i + s$; and between the third and fourth lines we have switched the indices i and j. Combining the dominant terms (15) and the correction terms (16) we have

$$
\begin{aligned}
\ln \binom{2n-1}{n+m} \sim \ln &\left(\frac{2^{2n-1}}{\sqrt{\pi n}} \frac{n-m}{n+m} \right) \\
&+ \sum_{i=1}^{\infty} \frac{1}{i(2i-1)n^{2i-1}} \left((i-m)m^{2i-1} + \frac{B_{2i}}{2^{2i}} - \sum_{j=1}^{i} B_{2j} \binom{2i}{2j} m^{2(i-j)} \right).
\end{aligned}
$$

At this stage we observe that since $B_0 = 1$, $B_1 = -\frac{1}{2}$ and all the other odd Bernoulli numbers vanish,

$$
\sum_{k=0}^{2i} B_k \binom{2i}{k} m^{2i-k} = m^{2i} - i m^{2i-1} + \sum_{j=1}^{i} B_{2j} \binom{2i}{2j} m^{2(i-j)},
$$

and thus our result so far can be written in the simpler form

$$
\ln \binom{2n-1}{n+m} \sim \ln \left(\frac{2^{2n-1}}{\sqrt{\pi n}} \frac{n-m}{n+m} \right) + \sum_{i=1}^{\infty} \frac{1}{i(2i-1)n^{2i-1}} \left(\frac{B_{2i}}{2^{2i}} - \sum_{k=0}^{2i} B_k \binom{2i}{k} m^{2i-k} \right).
$$

To obtain the final result (13) it just remains to use the standard fact about the Bernoulli polynomials [1, Equation 24.2.5]

$$
B_s(x) = \sum_{k=0}^{s} \binom{s}{k} B_k x^{s-k}.
$$

For the proof of (12) we start by writing

$$\ln\binom{2n}{n+m} = \ln\left(\frac{1}{2n+1}\frac{\left(2\left(n+\frac{1}{2}\right)\right)!}{(n+m)!(n-m)!}\right),$$

but we now expand the factorial in the numerator using the standard series (2) and the factorials in the denominator using the alternative series (10). The leading order terms become

$$\ln\left(\frac{4^n}{\sqrt{\pi\left(n+\frac{1}{2}\right)}}\right) + \ln\left(\frac{1}{\left(1+\frac{m}{n+\frac{1}{2}}\right)^{n+\frac{1}{2}+m}\left(1-\frac{m}{n+\frac{1}{2}}\right)^{n+\frac{1}{2}-m}}\right),$$

and the correction terms become

$$\sum_{i=1}^{\infty}\frac{B_{2i}}{2i(2i-1)\left(n+\frac{1}{2}\right)^{2i-1}}\left(\frac{1}{2^{2i-1}} - \frac{\frac{1}{2^{2i-1}}-1}{(1+\frac{m}{n+\frac{1}{2}})^{2i-1}} - \frac{\frac{1}{2^{2i-1}}-1}{(1-\frac{m}{n+\frac{1}{2}})^{2i-1}}\right).$$

Both of these expressions are easily expanded in inverse powers of $n+\frac{1}{2}$ and combined to give (12). □

A final comment in this section concerns the existence of two odd-power expansions for $\ln \mathrm{CBC}(n)$, Equations (5) and (11). The coefficients in the series are "opposite and equal". Denoting the series (of odd powers) in (5) by $s(n)$ we have

$$\ln \mathrm{CBC}(n) \sim \ln\left(\frac{4^n}{\sqrt{\pi n}}\right) + s(n), \quad \ln \mathrm{CBC}(n) \sim \ln\left(\frac{4^n}{\sqrt{\pi\left(n+\frac{1}{2}\right)}}\right) - s\left(n+\frac{1}{2}\right).$$

Averaging these two results gives

$$\begin{aligned}
\ln \mathrm{CBC}(n) &\sim \ln\left(\frac{4^n}{\sqrt{\pi}\left(n\left(n+\frac{1}{2}\right)\right)^{\frac{1}{4}}}\right) + \frac{1}{2}\left(s(n) - s\left(n+\frac{1}{2}\right)\right) \\
&\sim \ln\left(\frac{4^n}{\sqrt{\pi}\left(n\left(n+\frac{1}{2}\right)\right)^{\frac{1}{4}}}\right) - \frac{1}{2}\left(s(-n) + s\left(n+\frac{1}{2}\right)\right) \\
&\sim \ln\left(\frac{4^n}{\sqrt{\pi}\left(n\left(n+\frac{1}{2}\right)\right)^{\frac{1}{4}}}\right) - \frac{1}{2}\left(s\left(\frac{1}{4}-\left(n+\frac{1}{4}\right)\right) + s\left(\frac{1}{4}+\left(n+\frac{1}{4}\right)\right)\right).
\end{aligned}$$

The last expression is evidently an even function of $n+\frac{1}{4}$. Thus we see there is a direct connection between the existence of two "opposite and equal" odd power series for $\ln \mathrm{CBC}(n)$ and the even power series for $\ln \mathrm{CBC}(n)$.

4 Some series with only even powers

The remaining results from the introduction that need to be explained are the explicit forms of the coefficients in the even power series for $\ln \mathrm{CBC}(n)$ and $\ln \mathrm{Cat}(n)$, (8) and (9). In greater generality, part (c) of Theorem 1 stated that for any integer k the quantity

$$\ln\left(\frac{\sqrt{\pi}\left(n+\frac{1}{4}+\frac{k}{2}\right)^{k+\frac{1}{2}}}{4^n}\frac{(2n)!}{n!(n+k)!}\right)$$

has an asymptotic expansion involving only even powers of $n+\frac{1}{4}+\frac{k}{2}$. We now show the following:

Theorem 2. *For any integer k*

$$\ln\left(\frac{(2n)!}{n!(n+k)!}\right)\sim\ln\left(\frac{4^n}{\sqrt{\pi}\left(n+\frac{1}{4}+\frac{k}{2}\right)^{k+\frac{1}{2}}}\right)-\sum_{i=1}^{\infty}\frac{1}{i(2i+1)\left(n+\frac{1}{4}+\frac{k}{2}\right)^{2i}}B_{2i+1}\left(\frac{1}{4}-\frac{k}{2}\right), \tag{17}$$

where $B_j(x)$ denotes the jth Bernoulli polynomial. The series (8) is obtained from the case $k=0$ using the result [1, Equations 24.4.31 and 24.2.7]

$$B_{2i+1}\left(\frac{1}{4}\right)=-\frac{(2i+1)E_{2i}}{4^{2i+1}},\qquad i=1,2,\ldots.$$

The series (9) is obtained from the case $k=1$ using the result [1, Equations 24.4.3 and 24.4.31]

$$B_{2i+1}\left(-\frac{1}{4}\right)=\frac{(2i+1)(E_{2i}-4)}{4^{2i+1}},\qquad i=1,2,\ldots$$

and the Taylor series for $\ln\left(\frac{1+x}{1-x}\right)$.

Proof. As in the proof of part (c) of Theorem 1 we write

$$f(n)=\ln\left(\frac{\sqrt{\pi}\left(n+\frac{1}{4}+\frac{k}{2}\right)^{k+\frac{1}{2}}}{4^n}\frac{(2n)!}{n!(n+k)!}\right)$$

and $g(n)=f\left(n-\frac{1}{4}-\frac{k}{2}\right)$. Our task is to compute the asymptotic expansion of $g(n)$ in inverse powers of n. As before we obtain

$$g(n)=\ln\left(n^{k+\frac{1}{2}}\frac{\Gamma\left(n+\frac{1}{4}-\frac{k}{2}\right)}{\Gamma\left(n+\frac{3}{4}+\frac{k}{2}\right)}\right).$$

Writing $x = \frac{1}{4} - \frac{k}{2}$ we now proceed as in the proofs given in Section 3:

$$
\begin{aligned}
g(n) &= (1-2x)\ln n + \ln\Gamma(n+x) - \ln\Gamma(n+1-x) \\
&\sim (1-2x)\ln n + \left(n+x-\frac{1}{2}\right)\ln(n+x-1) - \left(n+\frac{1}{2}-x\right)\ln(n-x) \\
&\quad + (1-2x) + \sum_{j=1}^{\infty}\frac{B_{2j}}{2j(2j-1)}\left(\frac{1}{(n+x-1)^{2j-1}} - \frac{1}{(n-x)^{2j-1}}\right) \\
&= n\left(\ln\left(1+\frac{x-1}{n}\right) - \ln\left(1-\frac{x}{n}\right)\right) \\
&\quad + \left(x-\frac{1}{2}\right)\left(\ln\left(1+\frac{x-1}{n}\right) + \ln\left(1-\frac{x}{n}\right)\right) + (1-2x) \\
&\quad + \sum_{j=1}^{\infty}\frac{B_{2j}}{2j(2j-1)n^{2j-1}}\left(\left(1+\frac{x-1}{n}\right)^{-2j+1} - \left(1-\frac{x}{n}\right)^{-2j+1}\right) \\
&= n\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}\left(\left(\frac{x-1}{n}\right)^{r} - \left(\frac{-x}{n}\right)^{r}\right) \\
&\quad + \left(x-\frac{1}{2}\right)\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}\left(\left(\frac{x-1}{n}\right)^{r} + \left(\frac{-x}{n}\right)^{r}\right) + (1-2x) \\
&\quad + \sum_{j=1}^{\infty}\frac{B_{2j}}{2j(2j-1)n^{2j-1}}\sum_{r=0}^{\infty}(-1)^{r}\frac{(2j+r-2)!}{(2j-2)!r!}\left(\left(\frac{x-1}{n}\right)^{r} - \left(\frac{-x}{n}\right)^{r}\right) \\
&= \sum_{r=2}^{\infty}\frac{(-1)^{r+1}}{rn^{r-1}}\left((x-1)^r - (-x)^r\right) + \left(x-\frac{1}{2}\right)\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{rn^{r}}\left((x-1)^r + (-x)^r\right) \\
&\quad + \sum_{j=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{r}B_{2j}}{n^{2j+r-1}}\frac{(2j+r-2)!}{(2j)!r!}\left((x-1)^r - (-x)^r\right).
\end{aligned}
$$

Since $B_i = 0$ for i odd and greater than 1, we can rewrite the sum on the last line as

$$
\begin{aligned}
&\sum_{k=2}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{r}B_{k}}{n^{k+r-1}}\frac{(k+r-2)!}{k!r!}\left((x-1)^r - (-x)^r\right) \\
&= \sum_{s=1}^{\infty}\sum_{k=2}^{s+1}\frac{B_k}{n^s}\frac{(s-1)!}{k!(s+1-k)!}\left((1-x)^{s+1-k} - x^{s+1-k}\right). \qquad (18)
\end{aligned}
$$

where in the double sum we have replaced summation over r by summation over $s = k + r - 1$. Furthermore, recalling that $B_0 = 1$, $B_1 = -\frac{1}{2}$, a straightforward but rather lengthy manipulation of the terms on the penultimate line of the calculation of $g(n)$ shows that they are exactly the terms required to increase the range of summation over k in (18) to start from 0. Thus we obtain

$$g(n) \sim \sum_{s=1}^{\infty} \frac{1}{s(s+1)n^s} \sum_{k=0}^{s+1} \binom{s+1}{k} B_k \left((1-x)^{s+1-k} - x^{s+1-k}\right)$$
$$= \sum_{s=1}^{\infty} \frac{1}{s(s+1)n^s} (B_{s+1}(1-x) - B_{s+1}(x)).$$

Finally, using the symmetry property of the Bernoulli polynomials [1, Equation 24.4.3]

$$B_n(1-x) = (-1)^n B_n(x)$$

we see that the contribution to this sum from odd values of s vanishes, and writing $s = 2i$ we have

$$g(n) \sim -\sum_{i=1}^{\infty} \frac{1}{2i(2i+1)n^{2i}} B_{2i+1}(x),$$

from which the result in the theorem follows. □

Notes

1. The series appearing in the $k = 0$ and $k = 1$ cases, i.e., the expansions (8) and (9), have identical coefficients. As mentioned in the introduction, there is an obvious relation between the CBC and Cat functions, namely

$$\mathrm{Cat}(n) = \frac{\mathrm{CBC}(n)}{n+1}.$$

This does not make the passage between the expansions (8) and (9) obvious. There is, however, a second relation between the functions

$$\mathrm{CBC}\left(n + \frac{1}{2}\right) \mathrm{Cat}(n) = \frac{2^{4n+1}}{\pi \left(n + \frac{1}{2}\right)(n+1)},$$

which can easily be established using the duplication formula for the gamma function. Using this, it is easy to pass between the series (8) and (9).

2. The previous note concerned the relation between the cases $k = 0$ and $k = 1$ in the theorem. Other cases can also be related. For example we now explain how to pass from the case $k = 0$ to the case $k = 2$. The $k = 0$ result tells us about the expansion of

$$\frac{(2n)!}{(n!)^2}$$

in powers of $n + \frac{1}{4}$. Shifting n by 1, this tells us about the expansion of

$$\frac{(2n+2)!}{((n+1)!)^2} = 2\frac{(2n+1)!}{n!(n+1)!} = 2\binom{2n+1}{n}$$

in powers of $n + \frac{5}{4}$. Dividing by $4(n + \frac{1}{2})(n + 2) = 4\left(n + \frac{5}{4}\right)^2 - \frac{9}{4}$, which is an even function of $n + \frac{5}{4}$, this tells us about the expansion of $\frac{(2n)!}{n!(n+2)!}$ in powers of $n + \frac{5}{4}$, i.e., we obtain the result of the theorem for $k = 2$.

5 Postscript

This paper is a slightly updated version of a paper originally posted by one of the authors on a personal website in 2006. It is being submitted for publication in 2021 as the aforementioned website is about to be closed. The original paper has been cited by Luschny on the website [6] and in entries A220002, A220422 and A239739 in the *On-Line Encyclopedia of Integer Sequences*, by Elezovíc in [12], and by others.

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2010 *Mathematics Subject Classification*: Primary 05A10; Secondary 30E15, 41A60.
Keywords: factorial, binomial coefficient, Catalan number, asymptotic expansion, Stirling's formula.

(Concerned with sequences A220002, A220422, and A239739.)

Received April 11 2021; revised versions received August 9 2021; August 13 2021. Published in *Journal of Integer Sequences*, August 14 2021.

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